# 2

# Central charges in superalgebras

In this Section we will briefly review general issues related to central charges (CC) in superalgebras.

## 2.1 History

The first superalgebra in four-dimensional field theory was derived by Golfand and Likhtman [18] in the form

$$\{\bar{Q}_{\dot{\alpha}} Q_\beta\} = 2P_\mu \left(\sigma^\mu\right)_{\alpha\beta}, \quad \{\bar{Q}_\alpha \bar{Q}_\beta\} = \{Q_\alpha Q_\beta\} = 0, \tag{2.1.1}$$

i.e. with no central charges. Possible occurrence of CC (elements of superalgebra commuting with all other operators) was first mentioned in an unpublished paper of Lopuszanski and Sohnius [19] where the last two anticommutators were modified as

$$\{Q^I_\alpha Q^G_\beta\} = Z^{IG}_{\alpha\beta}. \tag{2.1.2}$$

The superscripts $I, G$ mark extended supersymmetry. A more complete description of superalgebras with CC in quantum field theory was worked out in [20]. The only central charges analyzed in this paper were Lorentz scalars (in four dimensions), $Z_{\alpha\beta} \sim \varepsilon_{\alpha\beta}$. Thus, by construction, they could be relevant only to extended supersymmetries.

A few years later, Witten and Olive [1] showed that in supersymmetric theories with solitons, central extension of superalgebras is typical; topological quantum numbers play the role of central charges.

It was generally understood that superalgebras with (Lorentz-scalar) central charges can be obtained from superalgebras without central charges in higher-dimensional space-time by interpreting some of the extra components of the momentum as CC's (see e.g. [21]). When one compactifies extra dimensions one obtains an extended supersymmetry; the extra components of the momentum act as scalar central charges.



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Algebraic analysis extending that of [20] carried out in the early 1980s (see e.g. [22]) indicated that the super-Poincaré algebra admits CC's of a more general form, but the dynamical role of additional tensorial charges was not recognized until much later. Now it is common knowledge that central charges that originate from operators other than the energy-momentum operator in higher dimensions can play a crucial role. These tensorial central charges take non-vanishing values on extended objects such as strings and membranes.

Central charges that are antisymmetric tensors in various dimensions were introduced (in the supergravity context, in the presence of $p$-branes) in Ref. [23] (see also [24, 25]). These CC's are relevant to extended objects of the domain wall type (membranes). Their occurrence in four-dimensional super-Yang–Mills theory (as a quantum anomaly) was first observed in [11]. A general theory of central extensions of superalgebras in three and four dimensions was discussed in Ref. [26]. It is worth noting that those central charges that have the Lorentz structure of Lorentz vectors were not considered in [26]. The gap was closed in [27].

## 2.2 Minimal supersymmetry

The minimal number of supercharges $\nu_Q$ in various dimensions is given in Table 2.1. Two-dimensional theories with a single supercharge, although algebraically possible, are quite exotic. In "conventional" models in $D = 2$ with local interactions the minimal number of supercharges is two.

The minimal number of supercharges in Table 2.1 is given for a real representation. Then, it is clear that, generally speaking, the maximal possible number of CC's is determined by the dimension of the symmetric matrix $\{Q_i Q_j\}$ of the size $\nu_Q \times \nu_Q$, namely,

$$\nu_{\rm CC} = \frac{\nu_Q(\nu_Q + 1)}{2}. \tag{2.2.1}$$

In fact, $D$ anticommutators have the Lorentz structure of the energy-momentum operator $P_\mu$. Therefore, up to $D$ central charges could be absorbed in $P_\mu$, generally speaking. In particular situations this number can be smaller, since although algebraically the corresponding CC's have the same structure as $P_\mu$, they are dynamically distinguishable. The point is that $P_\mu$ is uniquely defined through the conserved and symmetric energy-momentum tensor of the theory.

Additional dynamical and symmetry constraints can further diminish the number of independent central charges, see e.g. Section 2.2.1.

The total set of CC's can be arranged by classifying CC's with respect to their Lorentz structure. Below we will present this classification for $D = 2, 3$ and 4, with

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Table 2.1. *The minimal number of supercharges, the complex dimension of the spinorial representation and the number of additional conditions (i.e. the Majorana and/or Weyl conditions).*

| $D$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $\nu_Q$ | (1*) 2 | 2 | 4 | 8 | 8 | 8 | 16 | 16 | 16 |
| $\mathrm{Dim}(\psi)_C$ | 2 | 2 | 4 | 4 | 8 | 8 | 16 | 16 | 32 |
| # cond. | 2 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 2 |

special emphasis on the four-dimensional case. In Section 2.3 we will deal with $\mathcal{N}=2$ superalgebras.

### 2.2.1 $D = 2$

Consider two-dimensional non-chiral theories with two supercharges. From the discussion above, on purely algebraic grounds, three CC's are possible: one Lorentz-scalar and a two-component vector,

$$\{Q_\alpha,\, Q_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}(P_\mu + Z_\mu) + i(\gamma^5\gamma_0)_{\alpha\beta} Z\,. \tag{2.2.2}$$

We refer to Appendix A for our conventions regarding gamma matrices. $Z^\mu \neq 0$ would require existence of a vector order parameter taking distinct values in different vacua. Indeed, if this central charge existed, its current would have the form

$$\zeta_\nu^{\ \mu} = \varepsilon_{\nu\rho}\,\partial^\rho A^\mu, \quad Z^\mu = \int \zeta_0^{\ \mu}\, dz,$$

where $A^\mu$ is the above-mentioned order parameter. However, $\langle A^\mu\rangle \neq 0$ will break Lorentz invariance and supersymmetry of the vacuum state. This option will not be considered. Limiting ourselves to supersymmetric vacua we conclude that a single (real) Lorentz-scalar central charge $Z$ is possible in $\mathcal{N}=1$ theories. This central charge is saturated by kinks.

### 2.2.2 $D = 3$

The central charge allowed in this case is a Lorentz-vector $Z_\mu$, i.e.

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}(P_\mu + Z_\mu). \tag{2.2.3}$$

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One should arrange $Z_\mu$ to be orthogonal to $P_\mu$. In fact, this is the scalar central charge of Section 2.2.1 elevated by one dimension. Its topological current can be written as

$$\zeta_{\mu\nu} = \varepsilon_{\mu\nu\rho}\,\partial^\rho A, \quad Z_\mu = \int d^2x\,\zeta_{\mu 0}. \tag{2.2.4}$$

By an appropriate choice of the reference frame $Z_\mu$ can always be reduced to a real number times $(0, 0, 1)$. This central charge is associated with a domain line oriented along the second axis.

Although from the general relation (2.2.3) it is pretty clear why BPS vortices cannot appear in theories with two supercharges, it is instructive to discuss this question from a slightly different standpoint. Vortices in three-dimensional theories are localized objects, particles (BPS vortices in $2+1$ dimensions were previously considered in [28]; see also references therein). The number of broken translational generators is $d$, where $d$ is the soliton's co-dimension, $d = 2$ in the case at hand. Then *at least d* supercharges are broken. Since we have only two supercharges in the problem at hand, both must be broken. This simple argument tells us that for a 1/2-BPS vortex the minimal matching between bosonic and fermionic zero modes in the (super) translational sector is one-to-one.

Consider now a putative BPS vortex in a theory with minimal $\mathcal{N} = 1$ supersymmetry (SUSY) in $2+1$D. Such a configuration would require a world volume description with two bosonic zero modes, but only one fermionic mode. This is not permitted by the argument above, and indeed no configurations of this type are known. Vortices always exhibit at least two fermionic zero modes and can be BPS-saturated only in $\mathcal{N} = 2$ theories.

### 2.2.3 D = 4

Maximally one can have 10 CC's which are decomposed into Lorentz representations as $(0, 1) + (1, 0) + (1/2, 1/2)$:

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2(\gamma^\mu)_{\alpha\dot\alpha}(P_\mu + Z_\mu), \tag{2.2.5}$$

$$\{Q_\alpha, Q_\beta\} = (\Sigma^{\mu\nu})_{\alpha\beta} Z_{[\mu\nu]}, \tag{2.2.6}$$

$$\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = (\bar{\Sigma}^{\mu\nu})_{\dot\alpha\dot\beta} \bar{Z}_{[\mu\nu]}, \tag{2.2.7}$$

where $(\Sigma^{\mu\nu})_{\alpha\beta} = (\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha}_\beta$ is a chiral version of $\sigma^{\mu\nu}$ (see e.g. [29]). The antisymmetric tensors $Z_{[\mu\nu]}$ and $\bar{Z}_{[\mu\nu]}$ are associated with domain walls, and reduce to a complex number and a spatial vector orthogonal to the domain wall. The (1/2, 1/2) CC $Z_\mu$ is a Lorentz vector orthogonal to $P_\mu$. It is associated with strings (flux tubes), and reduces to one real number and a three-dimensional unit spatial vector parallel to the string.

## 2.3 Extended SUSY

In four dimensions one can extend superalgebra up to $\mathcal{N} = 4$, which corresponds to sixteen supercharges. Reducing this to lower dimensions we get a rich variety of extended superalgebras in $D = 3$ and 2. In fact, in two dimensions the Lorentz invariance provides a much weaker constraint than in higher dimensions, and one can consider a wider set of $(p, q)$ superalgebras comprising $p + q = 4$, 8, or 16 supercharges. We will not pursue a general solution; instead, we will limit our task to (i) analysis of central charges in $\mathcal{N} = 2$ in four dimensions; (ii) reduction of the minimal SUSY algebra in $D = 4$ to $D = 2$ and 3, namely the $\mathcal{N} = 2$ SUSY algebra in those dimensions. Thus, in two dimensions we will consider only the non-chiral $\mathcal{N} = (2, 2)$ case. As should be clear from the discussion above, in the dimensional reduction the maximal number of CC's stays intact. What changes is the decomposition in Lorentz and $R$-symmetry irreducible representations.

### 2.3.1 $\mathcal{N} = 2$ in $D = 2$

Let us focus on the non-chiral $\mathcal{N} = (2, 2)$ case corresponding to dimensional reduction of the $\mathcal{N} = 1$, $D = 4$ algebra. The tensorial decomposition is as follows:

$$\begin{aligned}\{Q^I_\alpha, Q^J_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}\left[(P_\mu + Z_\mu)\delta^{IJ} + Z^{(IJ)}_\mu\right] + 2i\,(\gamma^5\gamma^0)_{\alpha\beta}\,Z^{\{IJ\}}\\ + 2i\,\gamma^0_{\alpha\beta}Z^{[IJ]}, \quad I, J = 1, 2.\end{aligned} \tag{2.3.1}$$

Here $Z^{[IJ]}$ is antisymmetric in $I, J$; $Z^{\{IJ\}}$ is symmetric while $Z^{(IJ)}$ is symmetric and traceless. We can discard all vectorial central charges $Z^{IJ}_\mu$ for the same reasons as in Section 2.2.1. Then we are left with two Lorentz singlets $Z^{(IJ)}$, which represent the reduction of the domain wall charges in $D = 4$ and two Lorentz singlets $\mathrm{Tr}Z^{\{IJ\}}$ and $Z^{[IJ]}$, arising from $P_2$ and the vortex charge in $D = 3$ (see Section 2.3.2). These central charges are saturated by kinks.

Summarizing, the $(2, 2)$ superalgebra in $D = 2$ is

$$\{Q^I_\alpha, Q^J_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}\,P_\mu\,\delta^{IJ} + 2i(\gamma^5\gamma^0)_{\alpha\beta}\,Z^{\{IJ\}} + 2i\,\gamma^0_{\alpha\beta}Z^{[IJ]}. \tag{2.3.2}$$

It is instructive to rewrite Eq. (2.3.2) in terms of complex supercharges $Q_\alpha$ and $Q^\dagger_\beta$ corresponding to four-dimensional $Q_\alpha$, $\bar{Q}_{\dot\alpha}$, see Section 2.2.3. Then

$$\begin{aligned}&\left\{Q_\alpha, Q^\dagger_\beta\right\}(\gamma^0)_{\beta\gamma} = 2\left[P_\mu\gamma^\mu + Z\,\frac{1-\gamma_5}{2} + Z^\dagger\,\frac{1+\gamma_5}{2}\right]_{\alpha\gamma}, \qquad (2.3.3)\\ &\left\{Q_\alpha, Q_\beta\right\}(\gamma^0)_{\beta\gamma} = -2Z'\,(\gamma_5)_{\alpha\gamma}, \quad \left\{Q^\dagger_\alpha, Q^\dagger_\beta\right\}(\gamma^0)_{\beta\gamma} = 2Z'^\dagger\,(\gamma_5)_{\alpha\gamma}.\end{aligned}$$

The algebra contains two complex central charges, $Z$ and $Z'$. In terms of components $Q_\alpha = (Q_R, Q_L)$ the nonvanishing anticommutators are

$$\begin{aligned} \{Q_L, Q_L^\dagger\} &= 2(H+P), & \{Q_R, Q_R^\dagger\} &= 2(H-P), \\ \{Q_L, Q_R^\dagger\} &= 2iZ, & \{Q_R, Q_L^\dagger\} &= -2iZ^\dagger, \\ \{Q_L, Q_R\} &= 2iZ', & \{Q_R^\dagger, Q_L^\dagger\} &= -2iZ'^\dagger. \end{aligned} \tag{2.3.4}$$

It exhibits the automorphism $Q_R \leftrightarrow Q_R^\dagger$, $Z \leftrightarrow Z'$ associated [30] with the transition to a mirror representation [31]. The complex central charges $Z$ and $Z'$ can be readily expressed in terms of real $Z^{\{IJ\}}$ and $Z^{[IJ]}$,

$$Z = Z^{[12]} + \frac{i}{2}\left(Z^{\{11\}} + Z^{\{22\}}\right), \quad Z' = \frac{Z^{\{12\}} + Z^{\{21\}}}{2} - i\,\frac{Z^{\{11\}} - Z^{\{22\}}}{2}. \tag{2.3.5}$$

Typically, in a given model either $Z$ or $Z'$ vanish. A practically important example to which we will repeatedly turn below (e.g. Sections 3.5 and 4.5.3) is provided by the so-called twisted-mass-deformed CP($N-1$) model [32]. The central charge $Z$ emerges in this model at the classical level. At the quantum level it acquires additional anomalous terms [33, 34]. Both $Z \neq 0$ and $Z' \neq 0$ simultaneously in a contrived model [33] in which the Lorentz symmetry and a part of supersymmetry are spontaneously broken.

### 2.3.2 $\mathcal{N} = 2$ in $D = 3$

The superalgebra can be decomposed into Lorentz and $R$-symmetry tensorial structures as follows:

$$\{Q_\alpha^I, Q_\beta^J\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}[(P_\mu + Z_\mu)\delta^{IJ} + Z_\mu^{(IJ)}] + 2i\,\gamma^0_{\alpha\beta} Z^{[IJ]}, \tag{2.3.6}$$

where all central charges above are real. The maximal set of 10 CC's enter as a triplet of spacetime vectors $Z_\mu^{IJ}$ and a singlet $Z^{[IJ]}$. The singlet CC is associated with vortices (or lumps), and corresponds to the reduction of the (1/2,1/2) charge or the 4th component of the momentum vector in $D = 4$. The triplet $Z_\mu^{IJ}$ is decomposed into an $R$-symmetry singlet $Z_\mu$, algebraically indistinguishable from the momentum, and a traceless symmetric combination $Z_\mu^{(IJ)}$. The former is equivalent to the vectorial charge in the $\mathcal{N} = 1$ algebra, while $Z_\mu^{(IJ)}$ can be reduced to a complex number and vectors specifying the orientation. We see that these are the direct reduction of the (0,1) and (1,0) wall charges in $D = 4$. They are saturated by domain lines.

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### 2.3.3 On extended supersymmetry (eight supercharges) in D = 4

Complete algebraic analysis of all tensorial central charges in this problem is analogous to the previous cases and is rather straightforward. With eight supercharges the maximal number of CC's is 36. Dynamical aspect is less developed – only a modest fraction of the above 36 CC's are known to be non-trivially realized in models studied in the literature. We will limit ourselves to a few remarks regarding the well-established CC's. We will use a complex (holomorphic) representation of the supercharges. Then the supercharges are labeled as follows

$$Q_\alpha^F, \quad \bar{Q}_{\dot\alpha\, G}, \quad \alpha, \dot\alpha = 1, 2, \quad F, G = 1, 2. \tag{2.3.7}$$

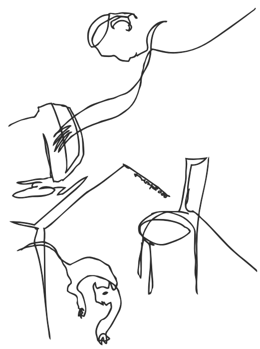
On general grounds one can write

$$\begin{aligned} \{Q_\alpha^F,\, \bar{Q}_{\dot\alpha\, G}\} &= 2\delta_G^F\, P_{\alpha\dot\alpha} + 2(Z_G^F)_{\alpha\dot\alpha}, \\ \{Q_\alpha^F,\, Q_\beta^G\} &= 2Z_{\{\alpha\beta\}}^{\{FG\}} + 2\,\varepsilon_{\alpha\beta}\,\varepsilon^{FG}\, Z, \\ \{\bar{Q}_{\dot\alpha\, F},\, \bar{Q}_{\dot\beta\, G}\} &= 2\left(\bar{Z}_{\{FG\}}\right)_{\{\dot\alpha\dot\beta\}} + 2\varepsilon_{\dot\alpha\dot\beta}\,\varepsilon_{FG}\,\bar{Z}. \end{aligned} \tag{2.3.8}$$

Here $(Z_G^F)_{\alpha\dot\alpha}$ are four vectorial central charges (1/2, 1/2), (16 components altogether) while $Z_{\{\alpha\beta\}}^{\{FG\}}$ and the complex conjugate are (1,0) and (0,1) central charges. Since the matrix $Z_{\{\alpha\beta\}}^{\{FG\}}$ is symmetric with respect to $F, G$, there are three flavor components, while the total number of components residing in (1,0) and (0,1) central charges is 18. Finally, there are two scalar central charges, $Z$ and $\bar{Z}$.

Dynamically the above central charges can be described as follows. The scalar CC's $Z$ and $\bar{Z}$ are saturated by monopoles/dyons. One vectorial central charge $Z_\mu$ (with the additional condition $P^\mu Z_\mu = 0$) is saturated [35] by Abrikosov–Nielsen–Olesen string (ANO for short) [36]. A (1,0) central charge with $F = G$ is saturated by domain walls [37].

Let us briefly discuss the Lorentz-scalar central charges in Eq. (2.3.8) that are saturated by monopoles/dyons. They will be referred to as monopole central charges. A rather dramatic story is associated with them. Historically they were the first to be introduced within the framework of an extended 4D superalgebra [19, 20]. On the dynamical side, they appeared as the first example of the "topological charge $\leftrightarrow$ central charge" relation revealed by Witten and Olive in their pioneering paper [1]. Twenty years later, the $\mathcal{N} = 2$ model where these central charges first appeared, was solved by Seiberg and Witten [2, 3], and the exact masses of the BPS-saturated monopoles/dyons found. No direct comparison with the operator expression for the central charges was carried out, however. In Ref. [38] it was noted that for the Seiberg–Witten formula to be valid, a boson-term anomaly should exist in the monopole central charges. Even before [38] a fermion-term anomaly was identified [37], which plays a crucial role [39] for the monopoles in the Higgs regime (confined monopoles).



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