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ASSOCIATIVITY OF THE TENSOR PRODUCT OF SEMILATTICES

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The tensor product of semilattices has been studied in [2], [3] and [5]. A survey of this work is given in [4]. Although a number of problems were settled completely in these papers, the question of the associativity of the tensor product was only partially answered. In the present paper we give a complete solution to this problem.

For terminology and basic results of lattice theory and universal algebra, consult Birkhoff [1] and Grätzer [6], [7]. The join and meet of elements $a_1, \ldots, a_n$ of a lattice are denoted by $\sum_{i=1}^n a_i$ and $\prod_{i=1}^n a_i$ respectively. All semilattices considered are join-semilattices. The reader is referred to [2] for definitions and results concerning the tensor product $A \otimes B$ of semilattices A and B. In fact, much of [2] is concerned with the special situation in which A and B are distributive lattices, and $A \otimes B$ is obtained by considering A and B as join-semilattices.

We will need the following results from the earlier papers [2, Theorem 2.5; 3, Theorem 1].

Theorem 1. *Let A and B be the distributive lattices and let a, $a_i \in A$ and b, $b_i \in B$ for $i = 1, \ldots, n$. Let $\mathbf{n}$ be the set $\{1, \ldots, n\}$. Then $a \otimes b \leqq \sum_{i=1}^n (a_i \otimes b_i)$ in $A \otimes B$ if and only if there exist non-empty subsets $S_1, \ldots, S_m$ of $\mathbf{n}$ such that $a \leqq \sum_{j=1}^m \prod_{i \in S_j} a_i$ and $b \leqq \prod_{j=1}^m \sum_{i \in S_j} b_i$.*

Theorem 2. *Let A and B be semilattices and let $a, a_i \in A$ and $b, b_i \in B$ for $i = 1, \ldots, n$. Then $a \otimes b \leqq \sum_{i=1}^n (a_i \otimes b_i)$ in $A \otimes B$ if and only there is an n-ary lattice polynomial p such that $a \in p((a_1), \ldots, (a_n))$ and $b \in p^*((b_1), \ldots, (b_n))$.*

Here (x) denotes the principal ideal generated by x and p^* is the polynomial obtained by interchanging the lattice operations in p.

Now the partial result on associativity of the tensor product obtained earlier is the following [2, Theorem 5.1].

Theorem 3. *Let A, B and C be finite distributive lattices. Then $(A \otimes B) \otimes C$ is isomorphic with $A \otimes (B \otimes C)$.*

We shall extend this result first to arbitrary distributive lattices and then to arbitrary semilattices.

Theorem 4. *Let A, B and C be distributive lattices. Then $(A \otimes B) \otimes C$ is isomorphic with $A \otimes (B \otimes C)$.*

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Proof. Every element of $(A\otimes B)\otimes C$ can be written in the form $\sum_{i=1}^{n}[(a_i\otimes b_i)\otimes c_i]$, where $a_i\in A$, $b_i\in B$ and $c_i\in C$ for $i=1,\ldots,n$. Let $\varphi:(A\otimes B)\otimes C\to A\otimes(B\otimes C)$ be defined by $\varphi(\sum_{i=1}^{n}[(a_i\otimes b_i)\otimes c_i])=\sum_{i=1}^{n}[a_i\otimes(b_i\otimes c_i)]$. We prove that φ is an isomorphism by showing that for all a_i, $e_j\in A$, b_i, $f_j\in B$, c_i, $g_j\in C$, $i=1,\ldots,n$, $j=1,\ldots,m$, we have $\sum_{i=1}^{n}[(a_i\otimes b_i)\otimes c_i]\leqq\sum_{j=1}^{m}[(e_j\otimes f_j)\otimes g_j]$ if and only if $\sum_{i=1}^{n}[a_i\otimes(b_i\otimes c_i)]\leqq\sum_{j=1}^{m}[e_j\otimes(f_j\otimes g_j)]$. Clearly it suffices to prove that $(a\otimes b)\otimes c\leqq\sum_1^n[(a_i\otimes b_i)\otimes c_i]$ if and only if $a\otimes(b\otimes c)\leqq\sum_1^n[a_i\otimes(b_i\otimes c_i)]$. In view of the symmetry of this assertion, it is enough to prove it in one direction.

Suppose that $(a\otimes b)\otimes c\leqq\sum_1^n[(a_i\otimes b_i)\otimes c_i]$. Then by Theorem 1, there are non-empty subsets $S_1,\ldots,S_m$ *of* $\boldsymbol{n}$ such that

$$a\otimes b\leqq\sum_{j=1}^{m}\prod_{i\in S_j}(a_i\otimes b_i)=\sum_{j=1}^{m}\left(\prod_{i\in S_j}a_i\right)\otimes\left(\prod_{i\in S_j}b_i\right)$$

and

$$c\leqq\prod_{j=1}^{m}\sum_{i\in S_j}c_i. \tag{1}$$

Again using Theorem 1, we have that there exist non-empty subsets $T_1,\ldots,T_p$ of $\boldsymbol{m}$ such that

$$a\leqq\sum_{k=1}^{p}\prod_{j\in T_k}\left(\prod_{i\in S_j}a_i\right) \tag{2}$$

and

$$b\leqq\prod_{k=1}^{p}\sum_{j\in T_k}\left(\prod_{i\in S_j}b_i\right). \tag{3}$$

Now for $k=1,\ldots,p$, let $U_k=\{i\in S_j:j\in T_k\}$. Then $U_1,\ldots,U_p$ are non-empty subsets of $\boldsymbol{n}$. Then by (2) we have

$$a\leqq\sum_{k=1}^{p}\prod_{i\in U_k}a_i. \tag{4}$$

Using (1) and (3) we have that for $k=1,\ldots,p$, $b\leqq\sum_{j\in T_k}\prod_{i\in S_j}b_i$ and $c\leqq\prod_{j\in T_k}\sum_{i\in S_j}c_i$. Then it follows by Theorem 1 that for $k=1,\ldots,p$, we have $b\otimes c\leqq\sum_{i\in U_k}(b_i\otimes c_i)$. Hence $b\otimes c\leqq\prod_{k=1}^{p}\sum_{i\in U_k}(b_i\otimes c_i)$. Applying Theorem 1 to the preceding result and (4), we obtain $a\otimes(b\otimes c)\leqq\sum_1^n[a_i\otimes(b_i\otimes c_i)]$.

Theorem 5. *Let A, B and C be semilattices. Then $(A\otimes B)\otimes C$ is isomorphic with $A\otimes(B\otimes C)$.*

Proof. The initial remarks made in the proof of Theorem 4 remain valid in this case and we define the map φ in the same way as before. Again it suffices to prove that if $(a\otimes b)\otimes c\leqq\sum_{i=1}^{n}[(a_i\otimes b_i)\otimes c_i]$ then $a\otimes(b\otimes c)\leqq\sum_{i=1}^{n}[a_i\otimes(b_i\otimes c_i)]$. Now it follows by Theorem 2 that this assertion is equivalent to the statement: if there is an n-ary polynomial p such that $a\otimes b\in p((a_1\otimes b_1),\dots,(a_n\otimes b_n))$ and $c\in p^*((c_1),\dots,(c_n))$, then there is an n-ary polynomial t such that $a\in t((a_1),\dots,(a_n))$ and $b\otimes c\in t^*((b_1\otimes c_1),\dots,(b_n\otimes c_n))$.

We will prove this statement by induction on the complexity of the polynomials involved. It is clearly true for polynomials of length 1. Now assume that the statement holds for the n-ary polynomials p and q. Let $a\otimes b\in(p+q)((a_1\otimes b_1),\dots,(a_n\otimes b_n))$ and $c\in p^*q^*((c_1),\dots,(c_n))$. Then $a\otimes b\leqq\sum_{i=1}^{k}(x_i\otimes y_i)+\sum_{i=k+1}^{m}(x_i\otimes y_i)$ where $x_i\in A$ and $y_i\in B$ for $i=1,\dots,m$ and

$$\sum_{1}^{k}(x_i\otimes y_i)\in p((a_1\otimes b_1),\dots,(a_n\otimes b_n))$$

and

$$\sum_{k+1}^{m}(x_i\otimes y_i)\in q((a_1\otimes b_1),\dots,(a_n\otimes b_n)).$$

Thus $a\otimes b\leqq\sum_{1}^{m}(x_i\otimes y_i)$, where for each i either $x_i\otimes y_i\in p((a_1\otimes b_1),\dots,(a_n\otimes b_n))$ or $x_i\otimes y_i\in q((a_1\otimes b_1),\dots,(a_n\otimes b_n))$. Since $c\in p^*((c_1),\dots,(c_n))$ and $c\in q^*((c_1),\dots,(c_n))$ it follows that for all i there is an n-ary polynomial s (where s is either p or q) such that

$$x_i\otimes y_i\in s((a_1\otimes b_1),\dots,(a_n\otimes b_n))$$

and

$$c\in s^*((c_1),\dots,(c_n)).$$

Now by the inductive hypothesis, for each i there is an n-ary polynomial u_i such that

$$x_i\in u_i((a_1),\dots,(a_n))$$

and

$$y_i\otimes c\in u_i^*((b_1\otimes c_1),\dots,(b_n\otimes c_n)).$$

Since $a\otimes b\leqq\sum_{1}^{m}(x_i\otimes y_i)$ it follows by Theorem 2 that there is an n-ary polynomial r such that $a\in r((x_1),\dots,(x_n))$ and $b\in r^*((y_1),\dots,(y_n))$. Let t be the n-ary polynomial $r(u_1,\dots,u_n)$. Since $x_i\in u_i((a_1),\dots,(a_n))$ for all i, it is easy to see that $a\in t((a_1),\dots,(a_n))$. Also since $b\in r^*((y_1),\dots,(y_n))$ it is readily verified that

$$b\otimes c\in r^*((y_1\otimes c),\dots,(y_n\otimes c)).$$

It follows that

$$b\otimes c\in r^*(u_1^*,\dots,u_n^*)((b_1\otimes c_1),\dots,(b_n\otimes c_n))$$

$$=t^*((b_1\otimes c_1),\dots,(b_n\otimes c_n)).$$

Thus the statement holds for the polynomial $p+q$.

Finally, assume the statement holds for p and q and suppose that

$$a \otimes b \in (pq)((a_1 \otimes b_1), \ldots, (a_n \otimes b_n))$$

and

$$c \in (p^* + q^*)((c_1), \ldots, (c_n)).$$

Then

$$a \otimes b \in p((a_1 \otimes b_1), \ldots, (a_n \otimes b_n)),$$

$$a \otimes b \in q((a_1 \otimes b_1), \ldots, (a_n \otimes b_n)),$$

and there exist $x \in A$ and $y \in B$ such that $c \leqq x + y$ where $x \in p^*((c_1), \ldots, (c_n))$ and $y \in q^*((c_1), \ldots, (c_n))$. Now by the inductive hypothesis there exist polynomials r and s such that

$$a \in r((a_1), \ldots, (a_n)), b \otimes x \in r^*((b_1 \otimes c_1), \ldots, (b_n \otimes c_n)),$$

and

$$a \in s((a_1), \ldots, (a_n)), b \otimes y \in s^*((b_1 \otimes c_1), \ldots, (b_n \otimes c_n)).$$

Let t be the polynomial rs. Then $a \in t((a_1), \ldots, (a_n))$ and

$$b \otimes c \leqq b \otimes (x + y) \in (r^* + s^*)((b_1 \otimes c_1), \ldots, (b_n \otimes c_n))$$

so that

$$b \otimes c \in t^*((b_1 \otimes c_1), \ldots, (b_n \otimes c_n)).$$

Thus the statement holds for the polynomial pq.

This completes the induction and the Theorem is now established.

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