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ON THE NUMBER OF NEARLY SELF-CONJUGATE PARTITIONS

BERNARD L. S. LIN and SITONG SUN

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Abstract

A partition λ of n is said to be nearly self-conjugate if the Ferrers graph of λ and its transpose have exactly $n-1$ cells in common. The generating function of the number of such partitions was first conjectured by Campbell and recently confirmed by Campbell and Chern ('Nearly self-conjugate integer partitions', submitted for publication). We present a simple and direct analytic proof and a combinatorial proof of an equivalent statement.

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1. Introduction

A partition λ of a positive integer n is a finite weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$. The terms λ_i are called the parts of λ and the number of parts of λ is called the length of λ, denoted $\ell(\lambda)$. The weight of λ is the sum of its parts, denoted by $|\lambda|$. We use $m(\lambda)$ to denote the largest part of λ. The Ferrers graph of λ is an array of left-justified cells with λ_i cells in the ith row. The Durfee square of a partition is the largest possible square contained within the Ferrers graph and anchored in the upper left-hand corner of the Ferrers graph. The conjugate of λ, denoted λ^T, is the partition whose Ferrers graph is obtained from that of λ by interchanging rows and columns.

A partition λ is said to be self-conjugate if $\lambda = \lambda^T$. It is well known that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts. This result is due to Sylvester [5]. Andrews and Ballantine [2] recently proved that the number of parts in all self-conjugate partitions of n is almost always equal to the number of partitions of n in which no odd part is repeated and there is exactly one even part (possibly repeated).

There is an interesting variation of self-conjugate partitions. Assuming that λ is a partition of n, then λ is said to be nearly self-conjugate if the Ferrers graphs of λ and

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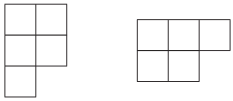

FIGURE 1. The two nearly self-conjugate partitions of 5.

its conjugate have exactly $n-1$ cells in common. For example, there are two nearly self-conjugate partitions of 5, which are depicted in Figure 1.

Let $\mathrm{nsc}(n)$ count the number of nearly self-conjugate partitions of n. The sequence $\{\mathrm{nsc}(n)\}_{n\geq 0}$ seems to be first considered by Campbell, who conjectured the generating function of $\mathrm{nsc}(n)$ (see [4]).

CONJECTURE 1.1. We have

$$\sum_{n\geq 0}\mathrm{nsc}(n)q^n = \frac{2q^2}{1-q^2}(-q^3;q^2)_\infty. \tag{1.1}$$

Throughout the paper, we adopt the following q-series notation:

$$\begin{aligned}(a;q)_0 &= 1,\\ (a;q)_n &= \prod_{k=1}^{n}(1-aq^{k-1}) \quad \text{for } n\geq 1,\\ (a;q)_\infty &= \prod_{k=1}^{\infty}(1-aq^{k-1}).\end{aligned}$$

To interpret the right-hand side of (1.1), Campbell and Chern (‘Nearly self-conjugate integer partitions’, submitted for publication) introduced symplectic partitions, in which the smallest part equals 2 and all others are distinct odd parts. Clearly, the generating function of twice the number of symplectic partitions equals the expression on the right-hand side of (1.1). Campbell and Chern, established a correspondence showing that the number of nearly self-conjugate partitions of n is equal to twice the number of symplectic partitions of n, which confirms Conjecture 1.1.

In this note, we aim to prove Conjecture 1.1 by analysing the structure of nearly self-conjugate partitions and using elementary techniques (see Section 2). In Section 3, we give an equivalent statement of Conjecture 1.1 and provide a combinatorial proof.

2. Analytic proof

Let $\mathscr{N}$ denote the set of nearly self-conjugate partitions and $\mathscr{D}$ denote the set of partitions into distinct parts. Let $\mathscr{G}$ denote the set of gap-free partitions (partitions in which every integer less than the largest part must appear at least once) and $\mathscr{G}_k$ denote the subset of $\mathscr{G}$ consisting of the partitions with the largest part being k. Given a partition $\lambda\in\mathscr{G}$, we define

$$\mathrm{rep}(\lambda) = |\{i : i \text{ appears more than once in } \lambda \text{ and } i \text{ is not the largest part in } \lambda\}|.$$

That is, $\mathrm{rep}(\lambda)$ counts the number of repeated parts less than the largest part of λ. For example, if $\lambda = (4, 4, 3, 2, 2, 1, 1)$, then $\mathrm{rep}(\lambda) = 2$. For a partition $\lambda \in \mathscr{D}$, we use $c(\lambda)$ to denote the number of separate sequences of consecutive integers of λ. For example, if $\mu = (8, 7, 5, 4, 3, 1)$, then $c(\mu) = 3$, counting the sequences

$$(8, 7), (5, 4, 3), (1).$$

REMARK 2.1. $c(\lambda)$ serves as an important statistic in Sylvester's refinement of Euler's partition identity [3, page 88], namely, the number of odd partitions of n with exactly k different parts equals the number of distinct partitions of n into k separate sequences of consecutive integers.

We first give an auxiliary result.

LEMMA 2.2. *We have*

$$\sum_{\lambda \in \mathscr{N}} q^{|\lambda|} = 2 \sum_{\lambda \in \mathscr{G}} (\mathrm{rep}(\lambda) + 1) q^{2|\lambda|+1-m(\lambda)}.$$

PROOF. Recall the Frobenius symbol [3] of λ, which is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix}$$

with $a_1 > a_2 > \cdots > a_k \geq 0$ and $b_1 > b_2 > \cdots > b_k \geq 0$, where a_i (respectively, b_i) counts the number of cells to the right of (respectively, below) the ith diagonal entry of λ in its Ferrers graph and k is the size of the Durfee square of λ. Then,

$$|\lambda| = \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i + k.$$

Now we add one to each term of the Frobenius symbol of λ to get a modified Frobenius symbol,

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \beta_1 & \beta_2 & \cdots & \beta_k \end{pmatrix}$$

with $\alpha_1 > \alpha_2 > \cdots > \alpha_k \geq 1$ and $\beta_1 > \beta_2 > \cdots > \beta_k \geq 1$.

If $\lambda \in \mathscr{N}$, there exists exactly one j such that $|\alpha_j - \beta_j| = 1$ and $\alpha_i = \beta_i$ for all $i \neq j$. Assuming that $\alpha_j - \beta_j = 1$,

$$|\lambda| = 2 \sum_{i=1}^{k} \beta_i + 1 - k.$$

Conversely, for a partition $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathscr{D}$, there exists exactly $c(\beta)$ separate sequences of consecutive integers. For each i with $1 \leq i \leq c(\beta)$, let β_{j_i} be the first integer in the ith sequence of consecutive integers in β. We have

$$\beta_1 = \beta_{j_1} > \beta_{j_2} > \cdots > \beta_{j_{c(\beta)}}.$$

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We now construct a partition $\alpha^i = (\alpha^i_1, \alpha^i_2, \dots, \alpha^i_k)$ for each i with $1 \le i \le c(\beta)$ by

$$\alpha^i_r = \begin{cases} \beta_r + 1 & \text{if } r = j_i, \\ \beta_r & \text{otherwise.} \end{cases}$$

Then, we have a modified Frobenius symbol with the first row being α^i and the second row being β, which corresponds to a nearly self-conjugate partition. Thus, in total, there are $c(\beta)$ nearly self-conjugate partitions with such modified Frobenius symbols.

If $\beta_j - \alpha_j = 1$, we can exchange α and β in the Frobenius symbol and obtain the same conclusion. Consequently, we can conclude that

$$\sum_{\lambda\in\mathcal{N}} q^{|\lambda|} = 2\sum_{\beta\in\mathscr{D}} c(\beta) q^{2|\beta|+1-\ell(\beta)}.$$

For a partition $\lambda \in \mathscr{D}$, its conjugate λ^T must be a gap-free partition and it follows that $\ell(\lambda) = m(\lambda^T)$ and $c(\lambda) = \mathrm{rep}(\lambda^T) + 1$. Thus,

$$\sum_{\lambda\in\mathscr{D}} c(\lambda) q^{2|\lambda|+1-\ell(\lambda)} = \sum_{\lambda\in\mathscr{G}} (\mathrm{rep}(\lambda)+1) q^{2|\lambda|+1-m(\lambda)}.$$

This completes the proof. □

We are now in a position to prove (1.1). By standard combinatorial arguments,

$$\begin{aligned}\sum_{\lambda\in\mathscr{G}_k} z^{\mathrm{rep}(\lambda)+1} q^{|\lambda|} &= \sum_{j\ge 0} q^{jk} z q^{k(k+1)/2} \prod_{i=1}^{k-1} (1 + zq^i + zq^{2i} + \cdots) \\ &= zq^{k(k+1)/2} \frac{\prod_{i=1}^{k-1}(1 - q^i + zq^i)}{(q;q)_k}.\end{aligned}$$

Differentiating the above equation with respect to z, putting $z = 1$ and replacing q by q^2,

$$\begin{aligned}\sum_{\lambda\in\mathscr{G}_k} (\mathrm{rep}(\lambda)+1) q^{2|\lambda|} &= \frac{1}{(q^2;q^2)_k}\Big(q^{k(k+1)} + q^{k(k+1)} \sum_{i=1}^{k-1} q^{2i}\Big) \\ &= \frac{q^{k(k+1)}}{(q^2;q^2)_k} \sum_{i=0}^{k-1} q^{2i} = \frac{q^{k(k+1)}(1-q^{2k})}{(q^2;q^2)_k(1-q^2)}.\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}\sum_{\lambda\in\mathcal{N}} q^{|\lambda|} &= 2\sum_{\lambda\in\mathscr{G}} (\mathrm{rep}(\lambda)+1) q^{2|\lambda|+1-m(\lambda)} \\ &= 2\sum_{k\ge 1} q^{1-k} \sum_{\lambda\in\mathscr{G}_k} (\mathrm{rep}(\lambda)+1) q^{2|\lambda|} \\ &= 2\sum_{k\ge 1} \frac{q^{1-k} q^{k(k+1)}(1-q^{2k})}{(q^2;q^2)_k(1-q^2)}\end{aligned}$$

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$$= 2\sum_{k\geq 1}\frac{q^{k^2+1}(1-q^{2k})}{(q^2;q^2)_k(1-q^2)}$$
$$= \frac{2q^2}{1-q^2}\sum_{k\geq 1}\frac{q^{k^2-1}}{(q^2;q^2)_{k-1}}. \tag{2.1}$$

Recall Euler's identity [1, page 19],

$$(-zq;q)_\infty = \sum_{k\geq 0}\frac{z^k q^{k(k+1)/2}}{(q;q)_k}.$$

Replacing q by q^2 and z by q,

$$(-q^3;q^2)_\infty = \sum_{k\geq 0}\frac{q^{k(k+2)}}{(q^2;q^2)_k}.$$

From (2.1) and the above identity,

$$\sum_{\lambda\in\mathcal{N}} q^{|\lambda|} = \frac{2q^2}{1-q^2}\sum_{k\geq 0}\frac{q^{k^2+2k}}{(q^2;q^2)_k} = \frac{2q^2}{1-q^2}(-q^3;q^2)_\infty.$$

This completes the proof.

3. Equivalent statement

Multiplying both sides of (1.1) by $1-q^2$ and comparing the coefficients of q^n,

$$\mathrm{nsc}(n) - \mathrm{nsc}(n-2) = 2\mathrm{do}_{\geq 3}(n-2),$$

where $\mathrm{do}_{\geq 3}(n)$ denotes the number of partitions of n into distinct odd parts with the smallest part at least 3.

Based on the classical bijection [5] between the partitions of n into distinct odd parts and the self-conjugate partitions of n, Campbell and Chern ('Nearly self-conjugate integer partitions', submitted for publication) established the following result.

LEMMA 3.1 (Campbell and Chern). *The number of nearly self-conjugate partitions of n equals twice the number of partitions of n with exactly one even part such that the differences between parts are at least* 2.

Let $\mathscr{O}(n)$ denote the set of partitions of n with exactly one even part and the differences between parts being at least 2. Thus, Conjecture 1.1 could be restated as follows.

PROPOSITION 3.2. *For $n \geq 2$,*

$$|\mathscr{O}(n)| - |\mathscr{O}(n-2)| = \mathrm{do}_{\geq 3}(n-2).$$

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PROOF. We first introduce an injection $\varphi : \mathscr{O}(n-2) \to \mathscr{O}(n)$ for every $n \geq 2$. Assuming that $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathscr{O}(n-2)$, we define $\varphi(\lambda) = \mu \in \mathscr{O}(n)$ as follows.

Case 1: If λ has only one part, then this unique part must be even. Hence, we can assume that $\lambda = (2m)$ and define $\mu = (2m+2)$.

Case 2: If $\ell(\lambda) \geq 2$ and the smallest part is odd, we assume that

$$\lambda = (\lambda_1, \ldots, \lambda_{i-1}, 2m, \lambda_{i+1}, \lambda_{i+2}, \ldots).$$

We define $\mu = (\lambda_1, \ldots, \lambda_{i-1}, 2m+1, \lambda_{i+1}+1, \lambda_{i+2}, \ldots)$. It is easy to see that $\ell(\mu) = \ell(\lambda) \geq 2$ and the largest part of μ is odd.

Case 3: If $\ell(\lambda) \geq 2$ and the smallest part is even, we suppose that

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, 2m)$$

and define $\mu = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_{i-1}, 2m+1)$. It is clear that $\ell(\mu) = \ell(\lambda) \geq 2$ and the largest part of μ is even and the smallest part of μ is greater than or equal to 3.

We observe that each partition in $\mathscr{O}(n) \backslash \varphi(\mathscr{O}(n-2))$ is a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ with $r \geq 2$, μ_1 being even and $\mu_r = 1$. Obviously, $\mu_1 - \mu_2 \geq 3$. Subtracting 1 from the largest part and removing the smallest part, we get a partition into distinct odd parts with each part at least 3, which is counted by $\mathrm{do}_{\geq 3}(n-2)$. This completes the proof. □

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BERNARD L. S. LIN, School of Science,
Jimei University, Xiamen 361021, PR China
e-mail: linlsjmu@163.com

SITONG SUN, School of Science,
Jimei University, Xiamen 361021, PR China
e-mail: sunstong@163.com