5

Complex analysis of transverse fields

In this chapter, we continue the discussion of transverse fields that are determined by the location of the conductors. In the "central" region, far from the magnet ends, the powerful methods of complex analysis[1] can be applied to the calculation of potentials, magnetic fields, multipoles and forces. Many of the topics in this chapter are based on a series of important papers by Richard Beth and by Klaus Halbach. Beginning with the field from a line current, we consider methods for calculating the fields from current sheets. Then we use the complex form of Green's theorem to express the fields of block conductors in terms of contour integrals.

5.1 Complex representation of potentials and fields

We define the complex potential function as

$$W(z) = u(x,y) + iv(x,y).$$

The real and imaginary parts of W must satisfy the *Cauchy-Riemann equations*, which are expressed in Cartesian coordinates as

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned} \tag{5.1}$$

From Equations 3.2 and 3.26 in free space in two dimensions, we have

$$\begin{aligned} B_x &= \frac{\partial A_z}{\partial y} = -\mu_0 \frac{\partial V_m}{\partial x} \\ B_y &= -\frac{\partial A_z}{\partial x} = -\mu_0 \frac{\partial V_m}{\partial y}. \end{aligned} \tag{5.2}$$

[1] A brief summary of some important results from the theory of complex variables is given in Appendix E.

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

These equations can be put into the form of the Cauchy-Riemann equations by associating

$$u = A_z$$
$$v = \mu_0 V_m.$$

Thus in two dimensions, the vector and scalar potentials are related to each other as the real and imaginary parts of the complex potential function

$$W(z) = A_z + i\mu_0 V_m. \tag{5.3}$$

Now consider the derivative of the complex potential. A complex function with a continuous derivative is known as an *analytic function*. A complex derivative must give the same result independent of the manner that Δz approaches 0. In the case when $\Delta z = \Delta x$, we have

$$\frac{dW}{dz} = \frac{\partial W}{\partial x} = \frac{\partial A_z}{\partial x} + i\mu_0 \frac{\partial V_m}{\partial x} = -B_y - iB_x.$$

If we had chosen $\Delta z = i\Delta y$ instead, we would obtain the same expression for B. So in either case we find

$$i\frac{dW}{dz} = B_x - iB_y.$$

Defining the complex magnetic field as[2]

$$B(z) = B_x + iB_y, \tag{5.4}$$

we find the relation between the magnetic field and the potential is

$$B^*(z) = i\,\frac{dW}{dz}, \tag{5.5}$$

where B^* is the complex conjugate of B.[1]

We can transform the magnetic field between Cartesian and polar coordinates by using the complex rotation variable. Let

$$B_c = B_x + iB_y$$
$$B_p = B_r + iB_\theta$$

[2] Unfortunately, Beth and Halbach use different definitions for the complex magnetic field H and use different systems of units, so some care must be exercised in comparing their results. We follow Halbach's conventions here in defining the components of H the same way as normal complex variables and using the SI system of units.

be the Cartesian and polar representations of a complex variable. Defining

$$R = e^{i\theta} = \cos\theta + i\sin\theta,$$

we can transform between the two representations using

$$\begin{aligned} B_p &= R^* B_c \\ B_c &= R B_p. \end{aligned} \tag{5.6}$$

Consider the analytic function

$$f(z) = u(x,y) + iv(x,y). \tag{5.7}$$

Differentiating the first Cauchy-Riemann Equation 5.1 with respect to x, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}.$$

The fact that the second partial derivative has to exist follows from the analytic nature of $f(z)$.[2] Differentiating the second Cauchy-Riemann equation with respect to y gives

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}.$$

Combining these equations, we find that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Thus $u(x,y)$ satisfies the Laplace equation. Similarly, we can differentiate the first Cauchy-Riemann equation with y and the second with x to show that $v(x,y)$ also satisfies the Laplace equation. It follows that the real and imaginary parts of any analytic function satisfy the Laplace equation.

Returning to Equation 5.7, consider the two curves

$$\begin{aligned} u(x,y) &= \alpha_1 \\ v(x,y) &= \beta_1, \end{aligned}$$

where α_1 and β_1 are fixed values. Differentiating u with respect to x, we find

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0.$$

The slope of the curve is

$$m_\alpha = \frac{dy}{dx} = -\frac{\partial_x u}{\partial_y u}.$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

Differentiating v with respect to x, we find

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$$

and the slope of this curve is

$$m_\beta = -\frac{\partial_x v}{\partial_y v}.$$

The product of the slopes is

$$m_\alpha m_\beta = \frac{\partial_x u}{\partial_y u}\frac{\partial_x v}{\partial_y v}.$$

Rewriting the numerator using the Cauchy-Riemann equations, we find

$$m_\alpha m_\beta = \frac{(\partial_y v)(-\partial_y u)}{\partial_y u \partial_y v} = -1.$$

From analytic geometry, this is the condition that indicates that two lines are perpendicular. Thus the real and imaginary parts of an analytic function describe orthogonal curves. This indicates in particular that the equipotential lines for A_z and V_m cross at right angles.

5.2 Maxwell's equations in complex conjugate coordinates

Instead of defining complex variables as functions of x and y, it is sometimes more convenient to use z and z^* as the independent variables. These are known as *complex conjugate coordinates*.[3] We can write the partial derivatives with respect to x and y in terms of these variables as

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*} \\ \frac{\partial}{\partial y} &= i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right). \end{aligned} \tag{5.8}$$

The corresponding derivatives with respect to z and z^* are

$$\begin{aligned} 2\frac{\partial}{\partial z} &= \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ 2\frac{\partial}{\partial z^*} &= \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \end{aligned} \tag{5.9}$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

We can use Equation 5.8 to write the magnetostatic Maxwell equations in complex coordinates. The divergence equation

$$\partial_x H_x + \partial_y H_y = 0$$

becomes [1]

$$\frac{\partial H}{\partial z} + \frac{\partial H^*}{\partial z^*} = 0, \tag{5.10}$$

where $H = H_x + iH_y$ is the complex magnetic field intensity. The curl equation

$$\partial_x H_y - \partial_y H_x = J_z \equiv \sigma$$

can be transformed using Equation 5.8 into the form

$$-i\partial_z H + i\partial_{z*}\, H^* = \sigma.$$

This can be further simplified using Equation 5.10, resulting in two forms for the curl equation.[1]

$$\begin{aligned} 2i\frac{\partial H^*}{\partial z^*} &= \sigma \\ -2i\frac{\partial H}{\partial z} &= \sigma. \end{aligned} \tag{5.11}$$

Operating on the complex potential in Equation 5.3, we find

$$2\frac{\partial W}{\partial z^*} = (\partial_x A_z - \partial_y \mu_0 V_m) + i(\partial_x \mu_0 V_m + \partial_y A_z).$$

The expressions in parentheses are the Cauchy-Riemann equations, which are thus compactly incorporated into the expression [4]

$$\frac{\partial W}{\partial z^*} = 0. \tag{5.12}$$

The ∇ operator can be written as

$$\begin{aligned} \nabla &= 2\frac{\partial}{\partial z^*} \\ \nabla^* &= 2\frac{\partial}{\partial z} \end{aligned} \tag{5.13}$$

and the Laplacian is

$$\nabla^2 = 4\frac{\partial^2}{\partial z \partial z^*}. \tag{5.14}$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

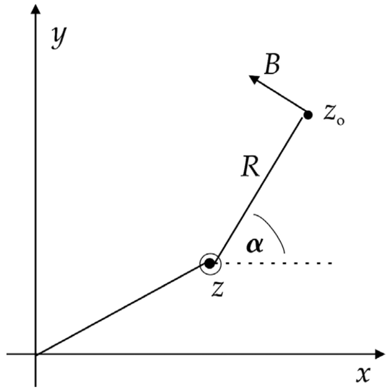

Figure 5.1 Geometry of a line current.

5.3 Field from a line current

Consider a single filament of current that crosses the x-y plane at the location z and an observation point z_o, as shown in Figure 5.1. The displacement between these two points is

$$z_o - z = Re^{i\alpha}. \tag{5.15}$$

We know from Equation 1.26 that the field of the line current is

$$\vec{B} = \frac{\mu_0 I}{2\pi R}(-\hat{x}\sin\alpha + \hat{y}\cos\alpha). \tag{5.16}$$

Every filament in a magnet must have a return filament of the opposite polarity somewhere. It is convenient to assume that all filaments have their currents return through a filament at the coordinate origin. Considering Equation 3.9 for the vector potential of a line current, we assume the complex potential for a filament and its return is given by

$$W = -\frac{\mu_0 I}{2\pi}[\ln(z_o - z) - \ln(z_o - 0)].$$

The second term in this equation is constant for a given field point. In the cross-section of any real magnet, there are equal numbers of filaments with positive and negative currents. Thus summed over all the filaments in a magnet, the second terms cancel out. The first terms do not cancel out in general because the filaments have different positions z. Thus the potential for the line current is

$$W(z_o) = -\frac{\mu_0 I}{2\pi}\ln(z_o - z). \tag{5.17}$$

Using Equations 5.5 and 5.15 with the derivative operating on the coordinates of the field point z_o, we get the magnetic field

$$\begin{aligned} B_x - i\,B_y &= -i\,\frac{\mu_0 I}{2\pi}\,\frac{1}{z_o - z} \\ &= -i\,\frac{\mu_0 I}{2\pi R}\,e^{-i\alpha} \\ &= -i\,\frac{\mu_0 I}{2\pi R}\,(\cos\alpha - i\sin\alpha), \end{aligned} \tag{5.18}$$

which agrees with the result in Equation 5.16.

From Equation 5.15, the complex logarithm is

$$\ln(z_o - z) = \ln R + i\alpha.$$

Using Equations 5.3 and 5.17 we can confirm that the vector potential for a line current is

$$A_z = -\frac{\mu_0 I}{2\pi}\ln R \tag{5.19}$$

and find that the scalar potential is

$$\begin{aligned} V_m &= -\frac{I}{2\pi}\alpha \\ &= -\frac{I}{2\pi}\tan^{-1}\left(\frac{y_o - y}{x_o - x}\right). \end{aligned} \tag{5.20}$$

Although Equations 5.19 and 5.20 appear very different, they both lead to the fields in Equation 5.18.

Assume that a line current is located at position z. Then, according to Equation 5.18, the field intensity at the observation point z_o is

$$H^*(z_o) = -i\frac{I}{2\pi}\,\frac{1}{z_o - z}. \tag{5.21}$$

Integrate H^* over observation points around any closed contour that encloses the point z.

$$\oint H^*\,dz_o = -i\frac{I}{2\pi}\oint\frac{1}{z_o - z}\,dz_o.$$

According to Cauchy's integral formula,[3]

[3] See Appendix E.

$$\oint \frac{1}{z_o - z} dz_o = 2\pi i.$$

Thus we find that the complex form of the Ampère law is

$$\oint H^* dz_o = I, \tag{5.22}$$

where I is the total current enclosed by the contour.

Now consider a line current in the vicinity of a plane surface of infinite permeability iron. As we saw in Chapter 2, the effect of the iron on the field of a conductor filament is equivalent to the presence of an image filament on the other side of the iron surface. The direction of the image current is the same as the conductor current. In the case of a circular boundary of radius R, the positions of the conductor and image filaments are

$$\begin{aligned} z &= \rho e^{i\phi} \\ z_I &= \frac{R^2}{\rho} e^{i\phi} = \frac{R^2}{z^*}. \end{aligned} \tag{5.23}$$

5.4 Field from a current sheet

We can consider a current sheet as a collection of parallel line currents. The sheet is assumed to have a finite width, but to have infinitesimal thickness. Then using Equation 5.17, the potential for the current sheet is

$$W(z_o) = -\frac{\mu_0}{2\pi} \int K(s) \ln[z_o - z(s)] ds, \tag{5.24}$$

where s is the arc length along the sheet and $K = dI/ds$ is the sheet current density.

Example 5.1: potential for a straight sheet with constant K
Consider the straight sheet with width b shown in Figure 5.2. The current density is $K = \frac{I}{b}$ and the filaments making up the sheet are located at

$$z(s) = z_1 + s e^{i\theta}.$$

It follows that $dz = e^{i\theta} ds$ and

$$z_2 = z_1 + b e^{i\theta}. \tag{5.25}$$

From Equation 5.24, the potential is

$$W(z_o) = -\frac{\mu_0 I}{2\pi b} \int \ln[z_o - z] e^{-i\theta} dz.$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

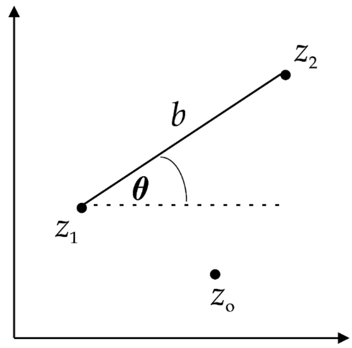

Figure 5.2 A straight current sheet.

We can write

$$\begin{aligned}\ln(z_o - z) &= \ln(z - z_o) + \ln(-1)\\ &= \ln(z - z_o) + i\pi.\end{aligned}$$

The second term is independent of z_o and gets absorbed into the constant term for the potential. Defining $u = z - z_o$, we get

$$W(z_o) = -\frac{\mu_0 I}{2\pi b} e^{-i\theta} \int_{u_1}^{u_2} \ln u \, du.$$

Substituting for $e^{-i\theta}$ from Equation 5.25 and evaluating the integral[4] gives

$$W(z_o) = -\frac{\mu_0 I}{2\pi(z_2 - z_1)} [u_2 \ln u_2 - u_1 \ln u_1 - u_2 + u_1].$$

The last two terms in the square bracket give

$$\begin{aligned}-u_2 + u_1 &= -(z_2 - z_o) + (z_1 - z_o)\\ &= z_1 - z_2.\end{aligned}$$

This term is also independent of z_o and gets absorbed into the constant term for the potential. Thus the potential for the straight sheet is given by [5]

$$W(z_o) = -\frac{\mu_0 I}{2\pi(z_2 - z_1)} [u_2 \ln u_2 - u_1 \ln u_1]. \tag{5.26}$$

Summing up the contributions to the magnetic field from the field of individual line currents given in Equation 5.21, we find the field of a current sheet is given by

[4] CRC 377.

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

$$H^*(z_o) = \frac{i}{2\pi}\int \frac{K(s)}{z(s) - z_o} ds, \tag{5.27}$$

where s is the distance along the sheet. If the position along the sheet is specified by the polar angle ϕ, this can be written as

$$H^*(z_o) = \frac{i}{2\pi}\int \frac{dI/d\phi}{z(\phi) - z_o} d\phi. \tag{5.28}$$

It is possible to determine a unique current distribution $dI/d\phi$ for circular or elliptic current sheets that can produce any desired two-dimensional field compatible with Maxwell's equations in the magnet aperture.[6]

Example 5.2: field due to a circular arc sheet with constant current density
Let us consider a current sheet in the form of a circular arc, as shown in Figure 5.3. The magnetic field from the sheet is given by Equation 5.28.

$$H^*(z_o) = -\frac{i}{2\pi}\frac{dI}{d\phi}\int_{\phi_1}^{\phi_2} \frac{d\phi}{z_o - z(\phi)}.$$

Since $z = ae^{i\phi}$, we can write this as

$$H^*(z_o) = -\frac{1}{2\pi}\frac{dI}{d\phi}\mathbb{I},$$

where[5]

$$\begin{aligned} \mathbb{I} &= \int_{z_1}^{z_2} \frac{dz}{z\,(z_o - z)} \\ &= \frac{2i}{z_o}\tan^{-1}\left[i\left(\frac{z_o - 2z}{z_o}\right)\right]. \end{aligned} \tag{5.29}$$

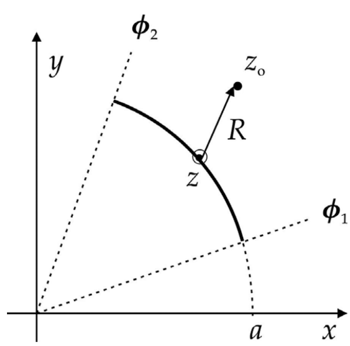

Figure 5.3 Circular arc current sheet.

[5] GR 2.172.

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

Therefore the field is

$$H^*(z_o) = -\frac{i}{\pi z_o}\frac{dI}{d\phi}\left\{\tan^{-1}\left[i\left(\frac{z_o - 2z}{z_o}\right)\right]\right\}_{z_1}^{z_2}.$$

Using the relation[6]

$$\tan^{-1}z = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right),$$

we can write the field of the circular arc as

$$H^*(z_o) = -\frac{i}{2\pi z_o}\frac{dI}{d\phi}\left[\ln\left(\frac{z_2}{z_o - z_2}\right) - \ln\left(\frac{z_1}{z_o - z_1}\right)\right]. \qquad (5.30)$$

From Equation 5.28, the field of the arc conductor at the origin is

$$\begin{aligned} H^*(0) &= \frac{i}{2\pi}\ \frac{dI}{d\phi}\int_{\phi_1}^{\phi_2}\frac{d\phi}{a\,e^{i\phi}} \\ &= -\frac{1}{2\pi a}\ \frac{dI}{d\phi}\left[e^{-i\phi_2} - e^{-i\phi_1}\right]. \end{aligned}$$

If the angular arc completes a *full* circle, we have a current shell and the integral in Equation 5.29 becomes

$$\mathbb{I} = -\oint\frac{dz}{z(z - z_o)}.$$

A point where the denominator of the integrand becomes zero is called a *pole*. If z_o is inside the circle, then the contour integral has simple poles at $z = 0$ and $z = z_o$. The *residue*[7] for the pole at $z = 0$ is

$$\lim_{z\to 0} z\frac{1}{z(z - z_o)} = -\frac{1}{z_o}$$

and the residue for the pole at $z = z_o$ is

$$\lim_{z\to z_o}(z - z_o)\frac{1}{z(z - z_o)} = \frac{1}{z_o}.$$

Therefore, by the residue theorem, the value of the integral is zero and the field inside the shell vanishes. When z_o is outside the shell, the integral only has the pole at $z = 0$ and the residue theorem gives

[6] GR 1.622.3. [7] See Appendix E.

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

$$\mathbb{I} = -2\pi i \left(-\frac{1}{z_o} \right) = \frac{2\pi i}{z_o}.$$

Therefore, the field outside the shell is

$$H^*(z_o) = -\frac{i}{z_o} \frac{dI}{d\phi}.$$

Since the total current is

$$I = 2\pi \frac{dI}{d\phi},$$

the field can be written as

$$H^*(z_o) = -i \frac{I}{2\pi z_o}.$$

This is the same as Equation 5.21 for the field of a line current located at the center of the shell.

Let us apply the Ampère law, Equation 5.22, for an infinitesimal rectangular contour across a current sheet, as shown in Figure 5.4. Then we have,

$$H_1^*(z_o)dz - H_2^*(z_o)dz = dI,$$

where dI is the current enclosed in the contour. In the limit where the distance perpendicular to the sheet approaches 0, the path of the observation points approach the path along the sheet and this results in the "current sheet theorem."[7]

$$H_1^*(z) - H_2^*(z) = \frac{dI}{dz}. \tag{5.31}$$

In addition to determining fields, this result has been used for calculations of magnetic stored energy and Lorentz forces.[8]

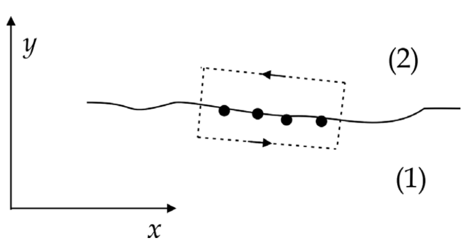

Figure 5.4 The sheet theorem.

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

5.5 cos ϕ current sheets

We consider here two examples of using complex methods to study the properties of current sheets that have a cos ϕ azimuthal current distribution.

Example 5.3: field from cos ϕ current distribution using contour integration
Consider a closed circular sheet of radius a. We have

$$\frac{dI}{d\phi} = I_0 \cos\phi$$
$$z = ae^{i\phi}$$
$$\cos\phi = \frac{e^{i\phi}+e^{-i\phi}}{2} = \frac{z+z^*}{2a}.$$

Substituting into Equation 5.28, we get

$$\begin{aligned} H^*(z_o) &= \frac{i\,I_0}{4\pi a}\oint \frac{z+z^*}{z-z_o}\,\frac{dz}{i\,z} \\ &= \frac{I_0}{4\pi a}\oint \left[\frac{1}{z-z_o} + \frac{z^*}{z(z-z_o)}\right] dz \\ &= \frac{I_0}{4\pi a}[\mathbb{I}_1 + \mathbb{I}_2]. \end{aligned} \tag{5.32}$$

It follows from Cauchy's theorem that the first integral

$$\mathbb{I}_1 = \oint \frac{dz}{z-z_o} = \begin{cases} 2\pi i & \text{if } z_o < a \\ 0 & \text{if } z_o > a \end{cases}.$$

Using the method of partial fractions,[9] the denominator of the second integral can be written

$$\frac{1}{z(z-z_o)} = \frac{A}{z} + \frac{B}{z-z_o}$$
$$1 = (z-z_o)A + zB.$$

Equating powers of z, we find that

$$A = -\frac{1}{z_o}$$
$$B = \frac{1}{z_o}.$$

Then we can write

$$\mathbb{I}_2 = -\mathbb{I}_3 + \mathbb{I}_4,$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

where

$$\mathbb{I}_3 = \frac{1}{z_o} \oint \frac{z^*}{z}\, dz = \frac{ia}{z_o} \int_0^{2\pi} e^{-i\phi}\, d\phi = 0$$

and

$$\mathbb{I}_4 = \frac{1}{z_o} \oint \frac{z^*}{z - z_o}\, dz. \tag{5.33}$$

Using $z^* = a^2/z$, we can write this as

$$\mathbb{I}_4 = \frac{a^2}{z_o} \oint \frac{dz}{z(z - z_o)} = \frac{a^2}{z_o} \left[-\frac{1}{z_o} \oint \frac{dz}{z} + \frac{1}{z_o} \oint \frac{dz}{z - z_o} \right].$$

For z_o inside the contour, the factor in square brackets vanishes because of the residue theorem and $\mathbb{I}_4 = 0$. Then from Equation 5.32,

$$H^*(z_o) = H_x - iH_y = \frac{I_0}{4\pi a} [2\pi i + 0] = i \frac{I_0}{2a}.$$

From this, we see that the field inside the current sheet is

$$H_x = 0 \qquad H_y = -\frac{I_0}{2a}. \tag{5.34}$$

The field is only in the vertical direction and has constant magnitude everywhere inside the sheet in agreement with Equation 4.36.

For z_o outside the contour, we have[8]

$$\mathbb{I}_4 = \oint \frac{z^*}{z_o(z - z_o)}\, dz = \frac{ia^2}{z_o} \int_0^{2\pi} \frac{1}{a\, e^{i\phi} - z_o}\, d\phi = -\frac{a^2}{z_o^2}\, 2\pi i.$$

[8] GR 2.313.1.

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

Substituting these results back into Equation 5.32, the field outside the contour is

$$H^*(z_o) = \frac{I_0}{4\pi a}\left[0 - \frac{a^2}{z_o^2}2\pi i\right] = -i\frac{I_0 a}{2z_o^2}.$$

If we write $z_o = x + iy$ and multiply the numerator and denominator by $(z_o^*)^2$, we find that

$$H^*(z_o) = -\frac{I_0 a}{r^4}xy - i\frac{I_0 a}{2r^4}(x^2 - y^2).$$

Equating real and imaginary parts, we find the Cartesian field components outside the sheet are

$$\begin{aligned} H_x &= -\frac{I_0 a}{r^4}xy = -\frac{I_0 a}{2r^2}\sin 2\phi \\ H_y &= \frac{I_0 a}{2r^4}(x^2 - y^2) = \frac{I_0 a}{2r^2}\cos 2\phi. \end{aligned} \tag{5.35}$$

On the midplane ($y = 0$), H is positive, along the y direction, and falls off with distance like $1/x^2$.

Example 5.4: field from cos ϕ current distribution using the sheet theorem
Assume again that we have a circular sheet with radius a. The current elements are located at

$$\begin{aligned} z &= ae^{i\phi} \\ dz &= izd\phi, \end{aligned}$$

so we have

$$\begin{aligned} \frac{dI}{dz} = \frac{dI}{d\phi}\frac{d\phi}{dz} &= -i\frac{I_0}{z}\cos\phi \\ &= -i\frac{I_0}{z}\left(\frac{e^{i\phi} + e^{-i\phi}}{2}\right) \\ &= -i\frac{I_0}{z}\left(\frac{z}{a} + \frac{a}{z}\right). \end{aligned}$$

Using the current sheet theorem, Equation 5.31,

$$H_1^*(z) - H_2^*(z) = -i\left(\frac{I_0}{2a} + \frac{I_0 a}{2z^2}\right).$$

The field inside the sheet H_{in} must be finite at $z = 0$ and for current in the positive z direction in the first quadrant of the circle, the field must go in the negative y direction. Therefore we identify H_2 with H_{in} and get

$$-H_{in}^{*}(z) = -H_{in,x} + iH_{in,y} = -i\frac{I_0}{2a}.$$

Equating real and imaginary parts, we find that

$$\begin{aligned} H_{in,x} &= 0 \\ H_{in,y} &= -\frac{I_0}{2a} \end{aligned} \tag{5.36}$$

in agreement with Equation 5.34. We identify the field exterior to the current sheet H_{ext} with H_1 in the sheet theorem.

$$H_{ext,x} - i\,H_{ext,y} = -i\frac{I_0 a}{2z^2}$$

5.6 Green's theorems in the complex plane

So far we have examined the fields due to current filaments and current sheets. We next want to proceed to the case of conductors with finite cross-sectional areas. However, before doing that, we need to review some important theorems that allows us to replace two-dimensional integrations over the conductor surface with contour integrals around the boundary of the surface. Besides the practical importance of reducing computation times in numerical calculations, this allows us to make use of some powerful results from the theory of complex contour integration.

Recall from Equation 3.79 that Green's theorem in the plane is

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint (P\,dx + Q\,dy),$$

where $P(x,y)$ and $Q(x,y)$ are continuous functions with continuous partial derivatives in a region R that is bounded by a curve C. Define the complex function

$$F(z,z^*) = P(x,y) + iQ(x,y).$$

Using Equation 5.9, the derivative of F can be written as

$$2\frac{\partial F}{\partial z^*} = \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right). \tag{5.37}$$

The closed integral of F around C is

$$\oint F\,dz = \oint (P\,dx - Q\,dy) + i\oint (Q\,dx + P\,dy).$$

Applying Green's theorem in the plane, we have

$$\oint F\,dz = i\iint \left[\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)\right] dx\,dy.$$

Replacing the integrand on the right-hand side using Equation 5.37, we find the first complex Green's theorem.[1]

$$\int \frac{\partial F}{\partial z^*} dS = \frac{1}{2i}\oint F\,dz \tag{5.38}$$

Following similar arguments, we have

$$2\frac{\partial F}{\partial z} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) + i\left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)$$

and

$$\oint F\,dz^* = -i\iint \left[\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) + i\left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)\right] dx\,dy.$$

After substitution, we obtain the second complex Green's theorem.[1]

$$\int \frac{\partial F}{\partial z} dS = -\frac{1}{2i}\oint F\,dz^*. \tag{5.39}$$

5.7 Field from a block conductor

We next want to consider the case of a block conductor, which we define as one with finite cross-sectional area. If we consider the conductor block as made up from an array of current filaments, we can use Equation 5.21 and express the field as

$$H^* = -\frac{i}{2\pi}\int \frac{\sigma}{z_o - z} dS, \tag{5.40}$$

where σ is the current density in the block. If we assume the current density is constant, we can rewrite this as

$$H^* = \frac{i\sigma}{2\pi}\int \frac{dS}{z - z_o}. \tag{5.41}$$

Powerful methods have been developed that allow the fields from block conductors to be evaluated using contour integration.[1] Consider the Green's theorem, Equation 5.38. For our application, the integrand of the surface integral is associated with the expression for the magnetic field in Equation 5.41. The integral has

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

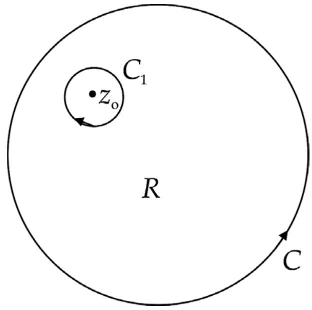

Figure 5.5 Contour for the Green's theorem calculation.

a singularity for the case when z_o is inside C, as shown in Figure 5.5. We can isolate the singularity by constructing a small circular contour C_1 around it. Then we can use Green's theorem in the region between the two contours to transform the surface integral into a contour integration. However, this requires evaluation on both contours C and C_1. Halbach proposed adding a constant term to the function F in Green's theorem that is (a) analytic in R and (b) makes the contour integration around C_1 vanish.[1] He assumed that F could be written as the product of two functions F_1 and F_2 with the property

$$\frac{\partial F}{\partial z^*} = F_1(z)\frac{\partial F_2(z^*)}{\partial z^*}, \tag{5.42}$$

where F_1 contains the singularity. Then F must have the form

$$F(z,z^*) = F_1(z)[F_2(z^*) - F_2(z_o^*)]. \tag{5.43}$$

Following this procedure, we define

$$F_1(z) = \frac{i\sigma}{2\pi}\frac{1}{z - z_o}$$
$$F_2(z^*) = z^*.$$

Then for use in Green's theorem, we have

$$F = \frac{i\sigma}{2\pi}\frac{1}{z - z_o}(z^* - z_o^*)$$

and

$$\frac{\partial F}{\partial z^*} = \frac{i\sigma}{2\pi}\frac{1}{z - z_o},$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

which is the integrand from Equation 5.41. Applying Green's theorem, we find that

$$H^* = \frac{1}{2i}\oint \frac{i\sigma}{2\pi}\,\frac{1}{z - z_o}(z^* - z_o^*)dz,$$

which simplifies to [1, 10]

$$H^* = \frac{\sigma}{4\pi}\oint \frac{z^* - z_o^*}{z - z_o}dz. \tag{5.44}$$

Let us confirm that Equation 5.44 does indeed vanish for the circular contour C_1. Let

$$z - z_o = re^{i\theta}.$$

Then for the contour C_1,

$$\oint \frac{z^* - z_o^*}{z - z_o}\, dz = ir\int_0^{2\pi} e^{-i\theta}\, d\theta = 0.$$

Thus we can ignore the contours around isolated singularities inside the conductor region and only evaluate Equation 5.44 on the outer boundary of the conductor.

Other quantities of interest can also be conveniently expressed in terms of contour integrals. For example, the area A of a current block is given by [10, 11]

$$A = \frac{1}{2i}\oint z^* dz. \tag{5.45}$$

Expressions have also been derived for the stored energy.[1, 12]

5.8 Block conductor examples

We consider three examples of using Equation 5.44 to find the field of a block conductor. The first example, the cylindrical conductor, was treated already in Chapter 1 using the Ampère law. Even though the calculation presented here is considerably more complicated, we carry it out to demonstrate some of the techniques involved and to compare with a result where we know the answer. The other two examples cannot be computed straightforwardly using the Ampère law.

Example 5.5: field of a solid cylindrical conductor
Assume we have a solid cylindrical conductor with radius a, as shown in Figure 5.6. Let

$$z = ae^{i\phi}$$
$$z_o = re^{i\theta}.$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

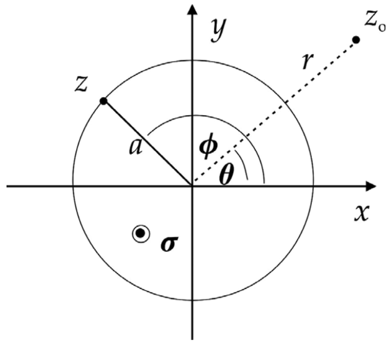

Figure 5.6 Solid cylindrical conductor.

Then Equation 5.44 gives

$$\begin{aligned} H^* &= \frac{\sigma}{4\pi}\int_0^{2\pi} \frac{ae^{-i\phi} - re^{-i\theta}}{ae^{i\phi} - re^{i\theta}}\, i\, ae^{i\phi}\, d\phi \\ &= i\,\frac{\sigma a}{4\pi}\left[a\,\mathbb{I}_1 - re^{-i\theta}\,\mathbb{I}_2\right], \end{aligned} \tag{5.46}$$

where

$$\begin{aligned} \mathbb{I}_1 &= \int_0^{2\pi} \frac{d\phi}{ae^{i\phi} - re^{i\theta}} \\ \mathbb{I}_2 &= \int_0^{2\pi} \frac{e^{i\phi}}{ae^{i\phi} - re^{i\theta}}\, d\phi. \end{aligned} \tag{5.47}$$

Case 1: z_o inside the conductor

When z_o is inside the conductor, we can use theorems from complex analysis to evaluate the integrals. Define $u = e^{i\phi}$ and $\beta = z_o/a$. Then

$$\mathbb{I}_1 = \frac{1}{ia}\oint \frac{du}{(u-\beta)u}. \tag{5.48}$$

We would like to convert the denominator into a simple pole so that we can use the residue theorem to evaluate the integral. To do this, expand the denominator using the method of partial fractions. Then we can write Equation 5.48 as

$$\mathbb{I}_1 = \frac{1}{ia}\left\{-\frac{1}{\beta}\oint \frac{du}{u-0} + \frac{1}{\beta}\oint \frac{du}{u-\beta}\right\}.$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

Applying the residue theorem, we obtain

$$\mathbb{I}_1 = \frac{1}{ia} 2\pi i \left(-\frac{1}{\beta} + \frac{1}{\beta} \right) = 0.$$

Turning next to $\mathbb{I}_2$,

$$\mathbb{I}_2 = \frac{1}{ia} \oint \frac{du}{(u-\beta)}$$

we apply the residue theorem and find

$$\mathbb{I}_2 = \frac{1}{ia} 2\pi i \ (1) = \frac{2\pi}{a}.$$

Returning now to Equation 5.46,

$$\begin{aligned} H^* &= i\frac{\sigma a}{4\pi}(-r\, e^{-i\theta})\frac{2\pi}{a} \\ &= -i\frac{\sigma r}{2}[\cos\theta - i \sin\theta]. \end{aligned}$$

Thus the field inside the conductor is

$$\begin{aligned} H_x &= -\frac{\sigma r}{2}\sin\theta \\ H_y &= \frac{\sigma r}{2}\cos\theta. \end{aligned} \tag{5.49}$$

Case 2: z_o outside the conductor

In this case, there are no singularities inside the contour, so we can treat $\mathbb{I}_1$ and $\mathbb{I}_2$ as ordinary integrals. Performing the first integration gives[9]

$$\mathbb{I}_1 = \frac{1}{-ire^{i\theta}} \left[i\phi - \ln(-re^{i\theta} + ae^{i\phi}) \right]_0^{2\pi}.$$

The logarithm term cancels because it has the same value at 0 and 2π. Thus we find that

$$\mathbb{I}_1 = -\frac{2\pi}{re^{i\theta}}.$$

[9] GR 2.313.1.

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

For the integral $\mathbb{I}_2$, let $\beta = r/a$ and $u = e^{i\phi}$. Then

$$\mathbb{I}_2 = \frac{1}{ia}\oint \frac{du}{u - \beta e^{i\theta}}$$
$$= \frac{1}{ia}\ln\left[e^{i\phi} - \beta e^{i\theta}\right]_0^{2\pi}.$$

The second term in the logarithm is constant and the first term has the same value at the two limits. Therefore, $\mathbb{I}_2 = 0$, and Equation 5.46 gives

$$H^* = -i\frac{\sigma a^2}{2re^{i\theta}}$$
$$= -i\frac{\sigma a^2}{2r}[\cos\theta - i\sin\theta].$$

Equating real and imaginary parts, we find the field outside the conductor is

$$H_x = -\frac{\sigma a^2}{2r}\sin\theta$$
$$H_y = \frac{\sigma a^2}{2r}\cos\theta. \tag{5.50}$$

The field of an elliptical block conductor has also been found using similar methods.[12, 13]

Example 5.6: field outside a rectangular conductor
Assume we have a rectangular conductor oriented at an angle θ with respect to the x axis, as shown in Figure 5.7. We look for the field at the observation point z_o. In terms of the variables (z, z*), a straight line from the vertex n to vertex $n + 1$ has the equation [1, 10]

$$z^* = z_n^* + \Delta z^*\left(\frac{z - z_n}{\Delta z}\right), \tag{5.51}$$

where

$$\Delta z = z_{n+1} - z_n. \tag{5.52}$$

Since the rectangle has four sides and has to close, we identify $z_5 = z_1$. For the side beginning with vertex 1, we define

$$\beta_1 = \frac{\Delta z^*}{\Delta z} = \frac{ae^{-i\theta}}{ae^{i\theta}} = e^{-2i\theta},$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

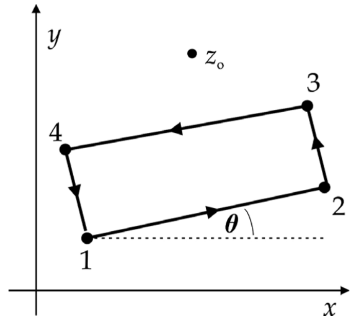

Figure 5.7 Rectangular conductor.

where a is the length of side 1. The second equation comes from considering z_1 as the origin of a line to z_2 in polar coordinates. Note that β is a constant because it is defined in terms of fixed z locations. In a rectangle, the angles of the other sides with respect to the x axis increase by 90° at each of the vertices. Thus for the side from vertex 2 to vertex 3,

$$\beta_2 = e^{-2i(\theta+\pi/2)} = e^{-i\pi}e^{-2i\theta} = -\beta_1.$$

Similarly we find,

$$\beta_3 = \beta_1$$
$$\beta_4 = -\beta_1.$$

Then Equation 5.44 gives

$$H^* = \frac{\sigma}{4\pi}\int_{z_1}^{z_2} \frac{[z_1^* + \beta_1(z - z_1) - z_o^*]}{z - z_o}\, dz + \cdots$$

with similar expressions for the remaining three sides. Defining the constant

$$\alpha_n = z_n^* - \beta_n z_n - z_o^*,$$

we can write

$$H^* = \frac{\sigma}{4\pi}[\alpha_1 \mathbb{I}_1 + \beta_1 \mathbb{I}_2] + \cdots. \tag{5.53}$$

For points z_o outside the contour,

$$\mathbb{I}_1 = \int_{z_1}^{z_2} \frac{dz}{z - z_o}$$
$$= [\ln(z - z_o)]_{z_1}^{z_2}$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

and[10]

$$\mathbb{I}_2 = \int_{z_1}^{z_2} \frac{z}{z - z_o}\, dz$$
$$= \left[z + z_o \ln(z - z_o)\right]_{z_1}^{z_2}.$$

Substituting into Equation 5.53,

$$H^* = \frac{\sigma}{4\pi}\left\{\alpha_1 \ln\left(\frac{z_2 - z_o}{z_1 - z_o}\right) + \beta_1[z_2 + z_o\ln(z_2 - z_o) - z_1 - z_o\ln(z_1 - z_o)]\right\} + \cdots$$
$$= \frac{\sigma}{4\pi}\left\{\alpha_1 \ln\left(\frac{z_2 - z_o}{z_1 - z_o}\right) + \beta_1 z_o \ln\left(\frac{z_2 - z_o}{z_1 - z_o}\right) + \beta_1(z_2 - z_1)\right\} + \cdots$$
$$= \frac{\sigma}{4\pi}\left\{[\alpha_1 + \beta_1 z_o]\ln\left(\frac{z_2 - z_o}{z_1 - z_o}\right) + \beta_1(z_2 - z_1)\right\} + \cdots.$$

Writing out the third term for all four sides gives

$$\beta_1[(z_2 - z_1) - (z_3 - z_2) + (z_4 - z_3) - (z_1 - z_4)] = 2\beta_1[-z_1 + z_2 - z_3 + z_4].$$

For a rectangle, the directed line segments

$$z_4 - z_3 = -(z_2 - z_1),$$

so this term cancels. Thus we find the field at z_o due to the rectangular conductor block is [5, 10]

$$H^* = \frac{\sigma}{4\pi}\sum_{n=1}^{4} h_n$$
$$h_n = [(z_n - z_o)^* - \beta_n(z_n - z_o)]\ \ln\left(\frac{z_{n+1} - z_o}{z_n - z_o}\right). \tag{5.54}$$

Example 5.7: on-axis field for annular sector conductor

For our last example, consider a conductor with the shape of an annular sector, as shown in Figure 4.11. We look for the field at the center of the circular arcs. Thus we have

$$z = re^{i\phi}$$
$$z_o = 0$$

and the contour in Equation 5.44 can be broken into the four parts.

[10] GR 2.112.1.

$$H^*(0) = \frac{\sigma}{4\pi}\left\{\int_{r_1}^{r_2}\frac{re^{-i\phi_1}}{re^{i\phi_1}}e^{i\phi_1}dr + \int_{\phi_1}^{\phi_2}\frac{r_2e^{-i\phi}}{r_2e^{i\phi}}r_2 i\, e^{i\phi}d\phi\right.$$
$$\left. + \int_{r_2}^{r_1}\frac{re^{-i\phi_2}}{re^{i\phi_2}}e^{i\phi_2}dr + \int_{\phi_2}^{\phi_1}\frac{r_1e^{-i\phi}}{r_1e^{i\phi}}r_1 i\, e^{i\phi}d\phi\right\}.$$

Simplifying and performing the integrals, we get

$$H^*(0) = \frac{\sigma}{4\pi}\left\{\int_{r_1}^{r_2} e^{-i\phi_1}dr + i\int_{\phi_1}^{\phi_2} r_2e^{-i\phi}d\phi + \int_{r_2}^{r_1} e^{-i\phi_2}dr + i\int_{\phi_2}^{\phi_1} r_1e^{-i\phi}d\phi\right\}$$
$$= -\frac{\sigma}{2\pi}(r_2 - r_1)(e^{-i\phi_2} - e^{-i\phi_1}).$$

Expanding the exponentials, we find that the field of the annular sector conductor is

$$H^*(0) = -\frac{\sigma}{2\pi}(r_2 - r_1)[(\cos\phi_2 - \cos\phi_1) - i(\sin\phi_2 - \sin\phi_1)], \quad (5.55)$$

which agrees with Equation 4.50. Note that the field strength is proportional to the radial thickness.

5.9 Field from image currents

We now consider the magnetic field produced by a current distribution in the presence of infinite permeability iron. The case of a filament near a planar iron surface is shown in Figure 5.8. The current in the filament induces image currents on the surface of the iron, which can be represented by an equivalent image filament inside the iron. We have seen in Chapter 2 that the image current is in the same direction as the conductor filament and is located the same distance from the iron surface as the conductor filament. The field of the image filament is given by

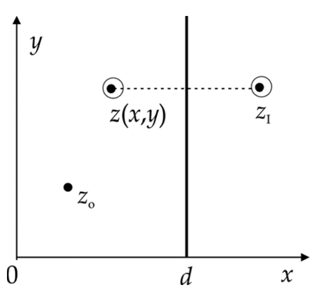

Figure 5.8 Line current near an iron slab.

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

$$H_I^*(z_o) = i\frac{I}{2\pi}\frac{1}{z_I - z_o}. \tag{5.56}$$

The location of the image filament is

$$\begin{aligned} z_I &= d + (d - x) + i\,y \\ &= 2d - z^*. \end{aligned}$$

For a uniform distribution of current, we have

$$H_I^*(z_o) = i\frac{\sigma}{2\pi}\int \frac{dS}{2d - z^* - z_o}.$$

To convert this surface integral to a contour integral, we use the complex Green's theorem, Equation 5.39. Choosing

$$F = \frac{i\sigma}{2\pi}\frac{z}{2d - z^* - z_o},$$

we find that the contribution to the field from the image current in a planar iron surface is

$$H_I^*(z_o) = -\frac{\sigma}{4\pi}\oint \frac{z}{2d - z^* - z_o}dz^*. \tag{5.57}$$

We are also interested in the image currents near a circular iron surface at radius R, as shown in Figure 5.9. From Chapter 2, we know the image current is in the same direction as the conductor current. If ρ is the distance of the conductor filament from the center of the circle, then the image filament is a distance R^2/ρ from the center. Thus we have

$$\begin{aligned} z &= \rho e^{i\phi} \\ z_I &= \frac{R^2}{\rho}e^{i\phi} = \frac{R^2}{z^*}. \end{aligned}$$

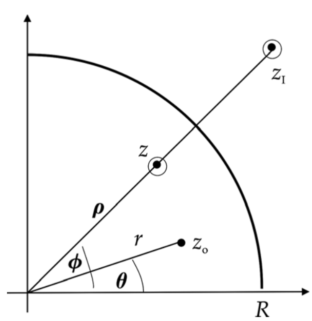

Figure 5.9 Line current near a circular iron cavity.

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

For a sheet conductor, the contribution of the images in a circular iron cavity to the field is a sum over the corresponding image currents. Using Equation 5.27, we obtain

$$\begin{aligned} H_I^*(z_o) &= \frac{i}{2\pi}\int \frac{K(z)}{z_I - z_o}dz \\ &= \frac{i}{2\pi}\int \frac{Kz^*}{R^2 - z_o z^*}dz, \end{aligned} \tag{5.58}$$

where $K(z) = dI/dz$.

Example 5.8: image field for cos ϕ current sheet in an iron cavity
Let us examine the image field at the origin for a closed circular sheet with radius a and a cos ϕ angular current distribution. When $z_o = 0$, the integrand does not have a singularity. Applying Equation 5.58,

$$\begin{aligned} H_I^*(0) &= \frac{i}{2\pi}\int_0^{2\pi} \frac{I_0 \cos\phi}{iz}\frac{ae^{-i\phi}}{R^2} izd\phi \\ &= \frac{iI_0 a}{2\pi R^2}\int_0^{2\pi}\frac{\cos\phi}{e^{i\phi}}d\phi \\ &= \frac{iI_0 a}{4\pi R^2}\int_0^{2\pi}\frac{e^{i\phi}+e^{-i\phi}}{e^{i\phi}}d\phi. \end{aligned}$$

After the integration, we find the contribution of the image field at the origin is

$$H_I^*(0) = i\frac{I_0 a}{2R^2} = H_{Ix} - iH_{Iy},$$

which gives the field components

$$\begin{aligned} H_{Ix} &= 0 \\ H_{Iy} &= -\frac{I_0 a}{2R^2}. \end{aligned} \tag{5.59}$$

The contribution of the image field is in the same direction that we saw in Equation 5.34 for the field of the conductor. The enhancement of the field due to the presence of the iron is

$$\begin{aligned} E(0) &= \frac{H^*(0) + H_I^*(0)}{H^*(0)} \\ &= 1 + \frac{a^2}{R^2}. \end{aligned} \tag{5.60}$$

This shows that the iron cavity can contribute up to a factor of 2 to the field at the origin.

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

For a block conductor with constant current density, the image current in circular iron is

$$H_I^*(z_o) = \frac{i\sigma}{2\pi}\int \frac{z^*}{R^2 - z_o z^*}\, dS.$$

Using Equation 5.39 for Green's theorem and choosing

$$F = \frac{i\sigma}{2\pi}\ \frac{zz^*}{R^2 - z_o z^*},$$

we find that the field due to the image current in the circular iron is [1]

$$H_I^*(z_o) = -\frac{\sigma}{4\pi}\oint \frac{zz^*}{R^2 - z_o z^*}\, dz^*. \tag{5.61}$$

Example 5.9: on-axis image field in circular iron for annular sector conductor Consider an annular sector conductor extending from radius r_1 to r_2 inside an iron cavity of radius R. The image field at the origin is given by Equation 5.61 with $z_o = 0$.

$$H_I^*(0) = -\frac{\sigma}{4\pi R^2}\oint zz^*\, dz^*$$

$$= -\frac{\sigma}{4\pi R^2}\left\{ e^{-i\phi_1}\int_{r_1}^{r_2} r^2 dr - i r_2^3 \int_{\phi_1}^{\phi_2} e^{-i\phi} d\phi + e^{-i\phi_2}\int_{r_2}^{r_1} r^2 dr - i r_1^3 \int_{\phi_2}^{\phi_1} e^{-i\phi} d\phi \right\}.$$

Evaluating the integrals and simplifying gives the field contribution due to the iron.

$$H_I^*(0) = -\frac{\sigma}{6\pi R^2}(r_2^3 - r_1^3)[(\cos\phi_2 - \cos\phi_1) - i(\sin\phi_2 - \sin\phi_1)]. \tag{5.62}$$

Note that this expression has the same sign and angular dependence as the field from the conductor given in Equation 5.55. The presence of the iron gives the enhancement factor at the origin [14]

$$E(0) = 1 + \frac{r_2^2 + r_1 r_2 + r_1^2}{3R^2}. \tag{5.63}$$

5.10 Multipole expansion

Since the magnetic potential, Equation 5.3, is an analytic function, it can be expanded in a power series

$$W(z_o) = \sum_{n=0}^{\infty} w_n\, z_o^n.$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

The magnetic field can then be expressed as

$$\begin{aligned} H^*(z_o) &= \frac{i}{\mu_0}\,\frac{dW}{dz_o} \\ &= \sum_{n=1}^{\infty} \frac{i\,n}{\mu_0} w_n z_o^{n-1}. \end{aligned}$$

Redefining the coefficients, we write the field as the power series

$$H^*(z_o) = \sum_{n=1}^{\infty} c_n z_o^{n-1}. \tag{5.64}$$

The field can also be expressed in terms of the integral in Equation 5.41.

$$\begin{aligned} H^*(z_o) &= \frac{i}{2\pi}\int \frac{\sigma}{z - z_o}\,dS \\ &= \frac{i}{2\pi}\int \frac{\sigma}{z\left(1 - \frac{z_o}{z}\right)}\,dS. \end{aligned}$$

Expand the factor in the denominator in a geometric series.

$$H^*(z_o) = \frac{i}{2\pi}\int \frac{\sigma}{z}\left[1 + \frac{z_o}{z} + \left(\frac{z_o}{z}\right)^2 + \cdots\right] dS$$

This series converges for observation points inside the magnet aperture up to the closest conductor. Equating this expression with Equation 5.64 gives

$$\begin{aligned} \sum_{n=1}^{\infty} c_n z_o^{n-1} &= \frac{i}{2\pi}\int \frac{\sigma}{z}\sum_{n=1}^{\infty}\left(\frac{z_o}{z}\right)^{n-1} dS \\ &= \frac{i}{2\pi}\sum_{n=1}^{\infty}\int \frac{\sigma}{z}\left(\frac{z_o}{z}\right)^{n-1} dS. \end{aligned}$$

The z_o factor cancels from both sides of the equation. Then matching term by term, we find

$$c_n = \frac{i\sigma}{2\pi}\int z^{-n} dS. \tag{5.65}$$

We can convert this surface integral into a contour integral by using the Green's theorem, Equation 5.39, with

$$F = \frac{i\sigma}{2\pi}\,\frac{z^{1-n}}{1-n}.$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

Thus Equation 5.65 becomes [1]

$$\begin{aligned} c_n &= -\frac{1}{2i}\oint \frac{i\sigma}{2\pi}\ \frac{z^{1-n}}{1-n} dz^* \\ &= \frac{\sigma}{4\pi(n-1)}\oint z^{1-n} dz^* \end{aligned} \tag{5.66}$$

for $n > 1$. For the case $n = 1$, we return to Equation 5.65 and find

$$c_1 = \frac{i\sigma}{2\pi}\int \frac{1}{z} dS.$$

This time we use the Green's theorem Equation 5.38 with

$$F = \frac{i\sigma}{2\pi z} z^*$$

to find that [1]

$$c_1 = \frac{\sigma}{4\pi}\oint \frac{z^*}{z} dz. \tag{5.67}$$

Example 5.10: multipoles for an annular sector conductor
We consider an annular sector conductor with radius between r_1 and r_2 that has constant current density σ. Let $z = re^{i\phi}$. For multipoles with $n > 1$, we have using Equation 5.66

$$c_n = \frac{\sigma}{4\pi(n-1)}\left\{\int_{r_1}^{r_2} (re^{i\phi_1})^{1-n} e^{-i\phi_1} dr - i\int_{\phi_1}^{\phi_2} (r_2 e^{i\phi})^{1-n} r_2 e^{-i\phi} d\phi \right.$$
$$\left. + \int_{r_2}^{r_1} (re^{i\phi_2})^{1-n} e^{-i\phi_2} dr - i\int_{\phi_2}^{\phi_1} (r_1 e^{i\phi})^{1-n} r_1 e^{-i\phi} d\phi \right\}.$$

After performing the integrations and simplifying the algebraic results, we find that

$$c_n = -\frac{\sigma}{2\pi n(2-n)}(r_2^{2-n} - r_1^{2-n})(e^{-in\phi_2} - e^{-in\phi_1}). \tag{5.68}$$

Because of the factor in the denominator, this relation cannot be used when $n = 2$. For that case, we return to Equation 5.66 and find

$$c_2 = \frac{\sigma}{4\pi}\oint z^{-1} dz^*.$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

For the annular sector, performing the integrals and summing terms, we find the quadrupole multipole is

$$c_2 = -\frac{\sigma}{4\pi} \ln\left(\frac{r_2}{r_1}\right)\left(e^{-2i\phi_2} - e^{-2i\phi_1}\right). \tag{5.69}$$

We can get the $n = 1$ term from Equation 5.67. The dipole multipole is

$$c_1 = -\frac{\sigma}{2\pi}(r_2 - r_1)\left(e^{-i\phi_2} - e^{-i\phi_1}\right). \tag{5.70}$$

Errors in the construction of magnet coils can lead to the introduction of additional unwanted multipole contributions to the field.[1, 15] These errors can include left-right and up-down asymmetries in the shape of the coils, displacements, rotations, and errors in the excitation currents.

5.11 Field due to a magnetized body

We next look at the magnetic field produced by a magnetized body. This is the case, for example, for a permanent magnet with net magnetization in the x-y plane. Consider a pair of parallel filaments with currents flowing in opposite directions located a distance d apart, as shown in Figure 5.10. There is a net field component in the x-y plane, oriented perpendicular to the axis connecting the two filaments. Such an arrangement is known as a current *doublet*.[16, 17] The field for the two filaments is

$$H^*(z_o) = \frac{iI}{2\pi}\left[\frac{1}{z_2 - z_o} - \frac{1}{z_1 - z_o}\right].$$

Let

$$d = z_2 - z_1 = |d|e^{i\alpha}$$
$$z_d = ½(z_1 + z_2),$$

where α is the angle between d and the x axis. Substituting, we find

$$H^*(z_o) = -\frac{iI}{2\pi}\left[\frac{d}{(z_d - z_o)^2 - \dfrac{d^2}{4}}\right].$$

Recall that the magnetic dipole moment is

$$m = IA = I\,l\,d,$$

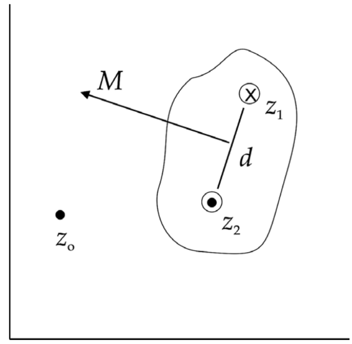

Figure 5.10 Model for a magnetized body.

where l is a unit distance along the z direction. Define m' as the magnetic moment per unit length. Then in the limit as $d \to 0$,

$$I\,d \to m'$$

$$\frac{d^2}{4} \to 0$$

and the field at z_o due to the doublet at z is

$$H^*(z_o) = -\frac{i}{2\pi}\frac{m'\,e^{i\alpha}}{(z - z_o)^2}.$$

We now want to express the field in terms of the magnetization M. The direction of m' is rotated by $\pi/2$ with respect to the direction of d. Let us define β to be the direction of M with respect to the x axis.

$$\beta = \alpha - \frac{\pi}{2}$$

$$e^{i\alpha} = e^{i\beta}\,e^{i\pi/2} = ie^{i\beta}.$$

Then summing up all the magnetic moments in the magnetized body, we have [18]

$$H^*(z_o) = \frac{1}{2\pi}\int \frac{M}{(z - z_o)^2}\,dS. \tag{5.71}$$

If the magnetization is constant in the body, we can convert this to a contour integral by using the Green's theorem, Equation 5.39. Defining

$$F = -\frac{M}{2\pi}\,\frac{1}{z - z_o},$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

we find that the field due to the magnetized body is [18]

$$H^*(z_o) = \frac{M}{4\pi i}\oint \frac{dz^*}{z - z_o}. \tag{5.72}$$

Example 5.11: triangular block with constant magnetization
Consider a triangular block of magnetic material with vertices $\{z_1, z_2, z_3\}$. Assume the magnetization has the constant value M. Define

$$\Delta z_n = z_{n+1} - z_n.$$

Since the triangle is closed, $z_4 = z_1$. The slopes of the sides are

$$\beta_n = \frac{\Delta z_n^*}{\Delta z_n},$$

so we can change the integration variable in Equation 5.72 from z^* to z for side n through the relation

$$dz^* = \beta_n dz.$$

The field produced by the block is

$$\begin{aligned} H^*(z_o) &= \frac{M}{4\pi\, i}\left[\int_{z_1}^{z_2} \frac{\beta_1}{z - z_o}\, dz + \cdots\right] \\ &= \frac{M}{4\pi\, i}\left[\beta_1 \ln(z_2 - z_o) - \beta_1 \ln(z_1 - z_o) + \cdots\right]. \end{aligned}$$

Collecting terms, the field of the triangular magnetized block is

$$\begin{aligned} H^*(z_o) = \frac{M}{4\pi i}\{&(\beta_3 - \beta_1)\ln(z_1 - z_o) + (\beta_1 - \beta_2)\ln(z_2 - z_o) \\ &+(\beta_2 - \beta_3)\ln(z_3 - z_o)\}. \end{aligned} \tag{5.73}$$

5.12 Force

The vector force dF on a current filament is

$$\overrightarrow{\mathrm{dF}} = I\overrightarrow{dl} \times \overrightarrow{B}.$$

If the filament is directed along the z direction, B is in the x-y plane, and so is F. The force can then be written as the complex variable $F = F_x + iF_y$, where

$$\begin{aligned} dF_x &= -\mu_0 I H_y dz \\ dF_y &= \mu_0 I H_x dz. \end{aligned}$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

Define f to be the force per unit length in the z direction.

$$f = \frac{dF}{dz} = i\mu_0 \, I \, H$$

For a distributed current distribution, we can generalize this as

$$f = i\mu_0 \int \sigma \, H dS. \tag{5.74}$$

Using Equation 5.11 for σ, we have

$$f = 2\mu_0 \int H \frac{\partial H}{\partial z} dS.$$

To express this as a contour integral, use the complex Green's theorem, Equation 5.39, with

$$F = \mu_0 H^2,$$

which gives [1]

$$f = \frac{i\mu_0}{2} \oint H^2 dz^*. \tag{5.75}$$

This shows that the transverse force per unit length is proportional to the square of the magnetic field intensity. Examples of complex force calculations can be found in references.[1, 19]

5.13 Conformal mapping

Operating on a complex variable z with some function f

$$w = f(z)$$

produces another complex variable w. This can be interpreted as a mapping from the z plane onto another w plane. Suppose that two curves in the z plane intersect at a point with the angle θ between them. A mapping is called *conformal* if the two corresponding curves in the w plane also intersect with the same angle θ between them. If $f(z)$ is an analytic function with $df/dz \neq 0$ inside a region R, then the mapping is conformal. Conformal mappings have the property that the function in the w plane is also analytic, so the real and imaginary parts of the mapped function are solutions of the Laplace equation.

Conformal mapping can frequently be used to transform a problem with complicated boundaries in the z plane, for example, into a simpler problem in the upper

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

half-plane or the interior of the unit circle in the w plane. Once the solution is found for the problem in the w plane, an inverse mapping $z = g(w)$ can be used to obtain the solution to the original problem. The theory of conformal mapping is a major subject in its own right. We only have space here to briefly introduce the subject and present a few examples. Fortunately, approximately half the book by Binns and Lawrenson is devoted to using conformal mapping in the solution of electric and magnetic field problems.[20] The interested reader can find many useful examples there.

The bilinear transformation combines the operations of translation, rotation, stretching, and inversion.[21]

$$w = \frac{\alpha z + \beta}{\gamma z + \delta},$$

where α, β, γ, and δ are complex numbers with the property that

$$\alpha\delta - \beta\gamma \neq 0.$$

This transformation can map circles and lines in the z plane into circles and lines in the w plane. It can be used, for example, to map a pair of separated circles to concentric circles. The bilinear transformation has the property that a quantity known as the cross-ratio is conserved.

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \tag{5.76}$$

This expression can be used to create a transformation that maps three given points in the z plane to three corresponding points in the w plane. An important bilinear transformation that maps any point z_o in the upper half of the z plane into the interior of the unit circle in the w plane is given by [22]

$$w = e^{i\theta_0}\left(\frac{z - z_o}{z - z_o^*}\right). \tag{5.77}$$

The points on the x axis are mapped to the boundary of the circle.

Example 5.12: line current in an iron cavity

Suppose we have a line current at the point w_1 inside a circular cavity with unit radius that is made from infinitely permeable iron, as shown in Figure 5.11. We use Equation 5.77 to map between the physical situation in the w plane and the upper half of the z plane. To determine the two unknown constants θ_0 and z_o, we associate the points

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

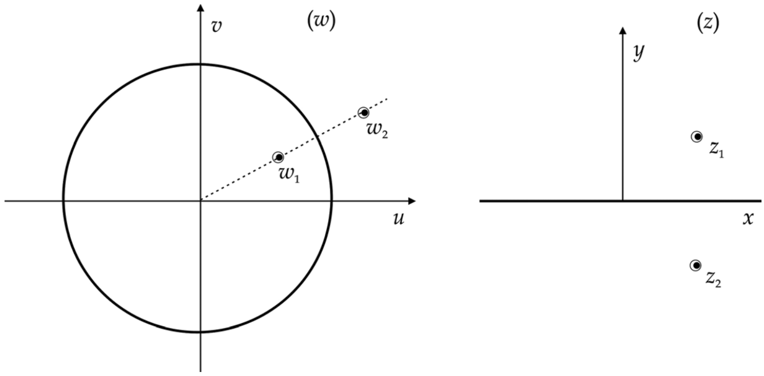

Figure 5.11 Line current in an iron cavity.

$$z = i \leftrightarrow w = 0$$
$$z = \infty \leftrightarrow w = -1,$$

which requires that

$$z_o = i$$
$$e^{i\theta_0} = -1.$$

This gives the specific mapping function between the planes

$$w = \frac{i - z}{i + z}.$$

The known line current at w_1 maps to a line current at z_1 in the z plane and the circular iron boundary maps to an iron plane along the real axis in the z plane. The mapping between the two line currents is

$$w_1 = u_1 + iv_1 = \frac{i - x_1 - iy_1}{i + x_1 + iy_1}.$$

Normalizing the denominator, we find

$$u_1 + iv_1 = \frac{[1 - x_1^2 - y_1^2] + i\,[2x_1]}{x_1^2 + (1 + y_1)^2}.$$

The real and imaginary parts of this equation can be solved for x_1 and y_1 as

$$x_1 + iy_1 = \frac{[2v_1] + i\,[1 - u_1^2 - v_1^2]}{u_1^2 + v_1^2 + 2u_1 + 1}.$$

In the z plane, we know there is an image current below the iron plane at the location $z_2 = z_1^*$. We can then use the mapping function to find the location w_2 of the image current in the w plane. After some algebraic simplifications, we find that

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

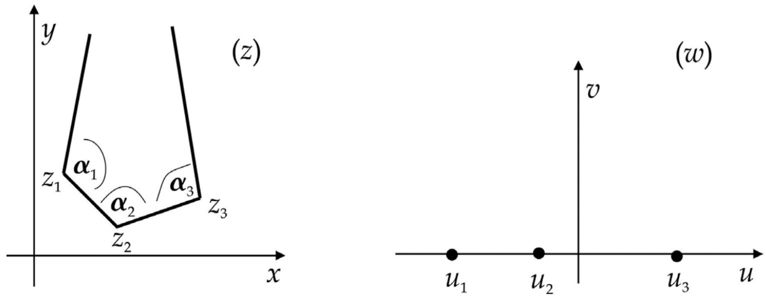

Figure 5.12 Schwarz-Christoffel transformation.

$$w_2 = u_2 + iv_2 = \frac{-v_1 + i(u_1 + 1)}{v_1 + i(u_1^2 + v_1^2 + u_1)}$$
$$= \frac{u_1 + iv_1}{u_1^2 + v_1^2}.$$

Representing w_1 and w_2 in polar coordinates, we find that

$$r_2 = \frac{1}{r_1}$$
$$\theta_2 = \theta_1,$$

which agrees with the result from the method of images.

Suppose that the boundary of some region in the z plane is made up of a series of straight line segments, as shown in Figure 5.12. The line segments meet at the vertices $z_1, z_2, \ldots$ It is possible to map this boundary to the real axis in the w plane by using the Schwarz-Christoffel transformation,[23] which takes the form of the differential equation

$$\frac{dz}{dw} = G(w - u_1)^{\alpha_1/\pi - 1}(w - u_2)^{\alpha_2/\pi - 1} \cdots (w - u_n)^{\alpha_n/\pi - 1}, \tag{5.78}$$

where G is a complex constant and the α_i are the interior angles. The points $u_1, u_2, \ldots$ on the real axis of the w plane correspond to the vertices in the z plane. The interior of the figure in the z plane maps to the upper half of the w plane.

Example 5.13: potential of a line current near the corner of two perpendicular planes

Consider a line current near the perpendicular intersection of two infinitely permeable plane surfaces, as shown in Figure 5.13. We solve the problem by using the Schwarz-Christoffel transformation, which in this case takes the form

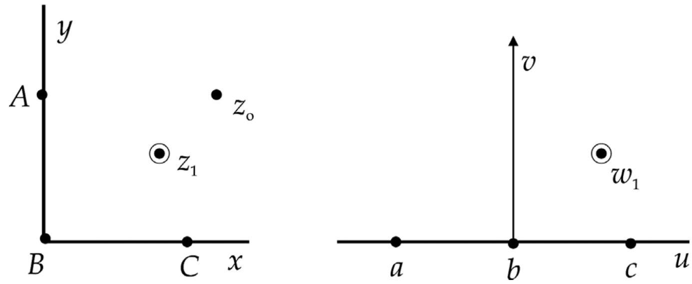

Figure 5.13 Line current near a corner. *ABC* and *abc* lie on infinitely permeable boundary surfaces. The line current is at z_1.

$$\frac{dz}{dw} = G(w - u_1)^{-1/2},$$

since the vertex angle $\alpha_1 = \pi/2$. Integrating this equation, we find

$$z = 2G\sqrt{w - u_1} + H,$$

where H is another complex constant. Solving for w, we get

$$w - u_1 = \frac{z^2 - 2H\,z + H^2}{4G^2}.$$

Breaking this equation into real and imaginary parts, leads to

$$u - u_1 + iv = \frac{x^2 - y^2 - 2H\,x + H^2 + 2i(x\,y - H\,y)}{4G^2}. \tag{5.79}$$

We choose three points A, B, C on the boundary in the z plane and demand that they correspond to three points a, b, c along the real axis in the w plane according to the following prescription:

$$\begin{aligned} A &: x = 0, y = 1 \leftrightarrow a : u = -1, v = 0 \\ B &: x = 0, y = 0 \leftrightarrow b : u = 0, v = 0 \\ C &: x = 1, y = 0 \leftrightarrow c : u = 1, v = 0 \end{aligned}$$

Applying these constraints to Equation 5.79, we find that

$$\begin{aligned} u_1 &= 0 \\ H &= 0 \\ 4G^2 &= 1 \end{aligned}$$

and the resulting transformation equation is

$$w = z^2.$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

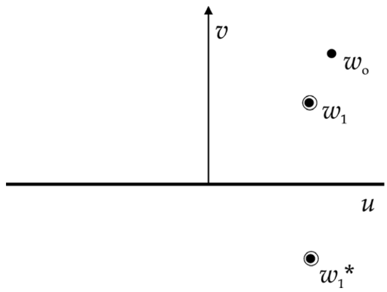

Figure 5.14 Image current in the *w* plane.

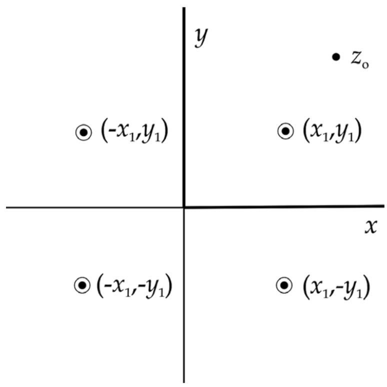

Figure 5.15 Line current and three images in the *z* plane.

In the *w* plane, the potential is due to the line current w_1 and the image current due to the plane boundary of the infinitely permeable material, as shown in Figure 5.14. The potential is given by

$$W(w_o) = \frac{\mu_0 I}{2\pi}[\ln(w_o - w_1) + \ln(w_o - w_1^*)].$$

Transforming back to the *z* plane, we have

$$\begin{aligned} W(z_o) &= \frac{\mu_0 I}{2\pi}[\ln(z_o^2 - z_1^2) + \ln(z_o^2 - z_1^{*2})] \\ &= \frac{\mu_0 I}{2\pi}\{\ln[(z_o - z_1)(z_o + z_1)] + \ln[(z_o - z_1^*)(z_o + z_1^*)]\} \\ &= \frac{\mu_0 I}{2\pi}\left[\ln(z_o - z_1) + \ln(z_o + z_1) + \ln(z_o - z_1^*) + \ln(z_o + z_1^*)\right]. \end{aligned}$$

This shows that the potential in the *z* plane is due to the physical line current at z_1 together with three image currents,[24] as shown in Figure 5.15. The four currents lie on a circle centered at the corner of the iron surfaces.

5.14 Integrated potentials

Suppose that $\Phi(r,\ \theta,\ s)$ is a scalar potential describing some three-dimensional magnetic configuration. Assume the conductors have a finite extent in the s direction, so that Φ vanishes as $s \to \pm\infty$. Define

$$E(r,\theta) = \int_{-\infty}^{\infty} \Phi(r,\theta,s)\, ds.$$

Taking the derivative with respect to r gives

$$\frac{\partial E}{\partial r} = \int_{-\infty}^{\infty} \frac{\partial \Phi}{\partial r}\, ds = -\int_{-\infty}^{\infty} B_r\, ds, \tag{5.80}$$

while the derivative with respect to θ yields

$$\frac{1}{r}\,\frac{\partial E}{\partial \theta} = \int_{-\infty}^{\infty} \frac{1}{r}\,\frac{\partial \Phi}{\partial \theta}\, ds = -\int_{-\infty}^{\infty} B_\theta\, ds. \tag{5.81}$$

We can likewise define $A(r,\theta,s)$ as the vector potential describing the same three-dimensional magnetic configuration. Since the conductors have a finite extent in the s direction, A_s also vanishes as $s \to \pm\infty$. Define

$$F(r,\theta) = \int_{-\infty}^{\infty} A_s(r,\theta,s)\, ds.$$

Considering the integral of B_r, we find that

$$\begin{aligned}\int_{-\infty}^{\infty} B_r ds &= \int (\nabla \times \vec{A})_r ds \\ &= \int \left(\frac{1}{r}\frac{\partial A_s}{\partial \theta} - \frac{\partial A_\theta}{\partial s}\right) ds \\ &= \frac{1}{r}\frac{\partial}{\partial \theta}\int A_s ds - A_\theta\Big|_{-\infty}^{\infty}.\end{aligned}$$

Assuming that A_θ has the same value at $\pm\infty$, we find that

$$\int_{-\infty}^{\infty} B_r\, ds = \frac{1}{r}\,\frac{\partial F}{\partial \theta}. \tag{5.82}$$

Similarly, the integral of B_θ

$$\begin{aligned}\int_{-\infty}^{\infty} B_\theta\, ds &= \int \left(\frac{\partial A_r}{\partial s} - \frac{\partial A_s}{\partial r}\right) ds \\ &= A_r\Big|_{-\infty}^{\infty} - \frac{\partial}{\partial r}\int A_s\, ds,\end{aligned}$$

https://doi.org/10.1017/9781009291156.006 Published online by Cambridge University Press

so that

$$\int_{-\infty}^{\infty} B_\theta \, ds = -\frac{\partial F}{\partial r}. \tag{5.83}$$

Equating the expressions for the integrated values of B_r and B_θ, we find that

$$\frac{1}{r}\frac{\partial F}{\partial \theta} = -\frac{\partial E}{\partial r}$$

$$\frac{\partial F}{\partial r} = \frac{1}{r}\frac{\partial E}{\partial \theta}.$$

These two equations have the same form as the Cauchy-Riemann equations in polar coordinates. Thus F and E represent the real and imaginary parts of the analytic potential function

$$W(z) = F(r,\theta) + iE(r,\theta).$$

This potential can be used to describe the influence of a magnet end on the field quality of a long magnet.[25]

References

[1] K. Halbach, Fields and first order perturbation effects in two dimensional conductor dominated magnets, *Nuc. Instr. Meth.* 78:185, 1970.

[2] K. Miller, *Introduction to Advanced Complex Calculus*, Dover, 1970, p. 66.

[3] M. Spiegel, *Complex Variables, Schaum's Outline Series*, McGraw-Hill, 1964, p. 7, 69–70, 83.

[4] R. Beth, Complex methods for three-dimensional magnetic fields, *IEEE Trans. Nuc. Sci*. 18:901, 1971.

[5] R. Beth, Complex representation and computation of two-dimensional magnetic fields, *J. Appl. Phys.* 37:2568, 1966.

[6] R. Beth, Elliptical and circular current sheets to produce a prescribed internal field, *IEEE Trans. Nuc. Sci*. 14:386, 1967.

[7] R. Beth, Some extensions of complex methods for two-dimensional fields, *Proc. 6th Int. Conf. on High Energy Accelerators*, Cambridge Electron Accelerator Lab, Cambridge, MA, 1967, p. 387.

[8] F. Toral, et al., Further developments on Beth's current sheet theorem: computation of magnetic field, energy and mechanical stresses in the cross section of particle accelerator magnets, *IEEE Trans. Appl. Superconductivity* 14:1886, 2004.

[9] M. Protter & C. Morrey, *College Calculus with Analytic Geometry*, Addison-Wesley, 1964, p. 465–468.

[10] R. Beth, Evaluation of current produced two-dimensional magnetic fields, *J. Appl. Phys.* 40:4782, 1969.

[11] M. Spiegel, *op. cit*., p. 114.

[12] R. Beth, Analytic design of superconducting multipolar magnets, Proc. 1968 *Summer Study on Superconducting Devices and Accelerators*, Brookhaven National Laboratory, p. 843–859.

[13] R. Beth, An integral formula for two-dimensional fields, *J. Appl. Phys.* 38:4689, 1967.
[14] J. Blewett, Iron shielding for air core magnets, Proc. 1968 *Summer Study on Superconducting Devices and Accelerators*, Brookhaven National Laboratory, p. 1042–1051.
[15] A. Jain, Basic theory of magnets, in S. Turner (ed.), *CERN Accelerator School on Measurement and Alignment of Accelerator and Detector Magnets*, CERN 98–05, 1998, p. 1.
[16] K. Binns & P. Lawrenson, *Analysis and Computation of Electric and Magnetic Field Problems*, 2nd ed., Pergamon Press, 1973, p. 48–49.
[17] M. Green, Modeling the behavior of oriented permanent magnet material using current doublet theory, *IEEE Trans. Mag.* 24:1528, 1988.
[18] K. Halbach, Design of permanent multipole magnets with oriented rare earth cobalt material, *Nuc. Instr. Meth.* 169:1, 1980.
[19] R. Beth, Currents and coil forces as contour integrals in two dimensional magnetic fields, *J. Appl. Phys.* 40:2445, 1969.
[20] K. Binns & P. Lawrenson, *op. cit*., chapters 6–10 and appendix III.
[21] K. Miller, *op. cit*., p. 202–204.
[22] M. Spiegel, *op. cit*., p. 203–204, 216.
[23] J. Dettman, Applied Complex Variables, Dover, 1984, p. 260–265.
[24] K. Binns & P. Lawrenson, *op. cit*., p. 43.
[25] F. Mills & G. Morgan, A flux theorem for the design of magnet coil ends, *Part. Acc.* 5:227, 1973.