BULL. AUSTRAL. MATH. SOC. 54D35, 54F05, 54D40
VOL. 49 (1994) [483–488]

ORDERED COMPACTIFICATIONS WITH COUNTABLE REMAINDERS

D.C. KENT AND T.A. RICHMOND

It is shown that if a partially-ordered topological space X admits a finite-point T_2-ordered compactification, then it admits a countable T_2-ordered compactification if and only if it admits n-point T_2-ordered compactifications for all n beyond some integer m.

1. INTRODUCTION

Countable compactifications of topological spaces have been studied in [**1**, **5**, **7**, **9**]. In [**7**], Magill showed that a locally compact, T_2 topological space X has a countable T_2 compactification if and only if it has n-point T_2 compactifications for every integer $n \geqslant 1$. We generalise this theorem to T_2-ordered compactifications of ordered topological spaces.

Before starting our generalisation of Magill's theorem, we recall two unpleasant facts about ordered compactifications. For the class of ordered topological spaces which allow T_2-ordered compactifications (that is, the $T_{3.5}$-ordered spaces), local compactness does not guarantee the existence of finite-point T_2-ordered compactifications (think of the reals with the usual order and discrete topology); furthermore the existence of an n-point T_2-ordered compactification for some $n > 1$ does not guarantee the existence of a one-point T_2-ordered compactification (think of the reals with the usual order and topology). Here is our main theorem: If a $T_{3.5}$-ordered space X allows a finite-point T_2-ordered compactification, then X allows a countable T_2-ordered compactification if and only if there is a positive integer m such that X allows an n-point T_2-ordered compactification for every $n \geqslant m$. In case the order on X is equality, the result is equivalent to Magill's theorem.

An *ordered topological space*, or simply an *ordered space* is a triple (X, τ, θ) where τ is a topology on the set X and θ is the graph of a partial order on X. An ordered space (X, τ, θ) is T_2-*ordered* if θ is closed in the product space $X \times X$, and is $T_{3.5}$-*ordered* (*completely regular ordered* in [**10**]) if the following conditions are satisfied: (1) If $A \subseteq X$ is closed and $x \in X \backslash A$, then there exist continuous functions $f, g : X \to [0,1]$ with f increasing, g decreasing, $f(x) = g(x) = 1$, and $f(a) \wedge g(a) = 0$ for all $a \in A$;

Received 5th August, 1993.

(2) If x and y are distinct points in X, then there exists a continuous monotone function $f : X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$. Compact T_2-ordered implies $T_{3.5}$-ordered, and $T_{3.5}$-ordered is hereditary.

An *ordered compactification* of (X, τ, θ) is a compact T_2-ordered space (X', τ', θ') such that (X', τ') contains (X, τ) as a dense subset, and $\theta \subseteq \theta'$. We shall usually write (X', τ', θ') simply as X'. An ordered space has an ordered compactification if and only if it is $T_{3.5}$-ordered (see [4] or [10]). An ordered compactification (X', τ', θ') of (X, τ, θ) is *strict* if θ' is the smallest order that makes (X', τ') an ordered compactification, that is, if θ' is the intersection of all closed partial orders on (X', τ') that extend θ.

If X' is an (ordered) compactification of X, the associated *remainder* is the subspace $X' \backslash X$ of X'. An (ordered) compactification whose remainder is finite or countably infinite is called a *finite-point (ordered) compactification* or a *countable (ordered) compactification*, respectively. A relation $\leqslant$ is defined on the set $K(X)$ of all compactifications of a topological space X by $X^* \geqslant X'$ if and only if there exists a continuous function $f : X^* \to X'$ which leaves X pointwise invariant. If $X^* \geqslant X'$ and $X' \geqslant X^*$ then X^* and X' are equivalent compactifications. If we do not distinguish between equivalent compactifications, then $\leqslant$ is a partial order on $K(X)$. The set $K_o(X)$ of all ordered compactifications of ordered space X can be partially ordered in the same manner with the only additional requirement that the *projection function* $f : X^* \to X'$ be increasing.

If θ' is a partial order on X', we shall write $x \leqslant' y$ for $(x, y) \in \theta'$. A set $B \subseteq X'$ is *increasing* if $B = \{x \in X' : b \leqslant' x \text{ for some } b \in B\}$. *Decreasing sets* are defined dually. The discrete order on a set X is $\triangle_X = \{(x, x) : x \in X\}$.

2. Countable Remainders

A locally compact topological space X has a two-point compactification if and only if X has some compactification with disconnected remainder (for example, 6.16 in [2]). We say an ordered space X is *order disconnected* if there exists a continuous increasing surjection $f : X \to \{0,1\}$ where $\{0,1\}$ has the discrete topology and the usual order $0 < 1$. While the existence of an order disconnected remainder does not imply the existence of a two-point ordered compactification (consider $\mathbf{R} \backslash \{0\}$, which has only three-point and four-point ordered compactifications), we do have the following result.

Lemma 2.1. *Suppose X' is an m-point strict ordered compactification of (X, τ, θ) and X^* is a larger ordered compactification of X. Suppose $h : X^* \to X'$ is the projection function and there exists $\alpha \in X' \backslash X$ such that $h^{-1}(\alpha)$ is order disconnected. Then there exists a $(m+1)$-point ordered compactification X'' with $X'' \geqslant X'$, obtained by replacing α in X' by two compactification points.*

Proof: Let X'' be the disjoint union of $X' \backslash \{\alpha\}$ and $\{0,1\}$. Suppose

$g : h^{-1}(\alpha) \to \{0,1\}$ is continuous, increasing, and onto. Define $f : X^* \to X''$ by $f(x) = h(x)$ for $x \in X^* \backslash h^{-1}(\alpha)$ and $f(x) = g(x)$ for $x \in h^{-1}(\alpha)$. If X'' is given the quotient topology τ'' derived from f and X^*, then (X'', τ'') is a topological compactification of X.

Define a relation θ'' on X'' by $a \leqslant'' b$ if and only if there exist points $a = c_0, c_1, \ldots, c_n = b$ in X'' such that for each $i = 1, \ldots, n$, there exists a net (x_λ, y_λ) in θ converging in $X'' \times X''$ to (c_{i-1}, c_i). The points $a = c_0, c_1, \ldots, c_n = b$ are called a *trail* from a to b with length n. In [**12**, Theorem 1.1] it is shown that the analogous relation $\leqslant'$ defined on X' is the strict order on X'. Observe that the nets (x_λ, y_λ) defining a trail are nets in $\theta \subseteq X^2$ and thus (x_λ) and (y_λ) are embedded in X', X'', and X^*. Since X' is a quotient of X'', $x_\lambda \to c_i$ in X'' implies $x_\lambda \to c_i'$ in X', where $c_i' = c_i$ if $c_i \in X'' \backslash \{0,1\} = X' \backslash \{\alpha\}$, and $c_i' = \alpha$ if $c_i \in \{0,1\}$. If $c_0, \ldots, c_n$ is a trail in X'' from c_0 to c_n where $c_0, c_n \in X'' \backslash \{0,1\}$, then $c_0 = c_0', c_1', \ldots, c_n' = c_n$ is a trail in X' from c_0 to c_n, and thus $c_0 \leqslant' c_n$. This shows that θ'' extends $\theta' \cap (X' \backslash \{0,1\})^2$, and therefore extends θ.

We now show that $\leqslant''$ is antisymmetric. Suppose $a \leqslant'' b$ and $b \leqslant'' a$. If $a, b \in X'' \backslash \{0,1\} = X' \backslash \{\alpha\}$, then $a \leqslant' b$ and $b \leqslant' a$, and thus $a = b$. If $a \in X'' \backslash \{0,1\}$ and $b \in \{0,1\}$, then the trails from a to b and from b to a imply $a \leqslant' \alpha$ and $\alpha \leqslant' a$, contrary to the fact that $a \in X'' \backslash \{0,1\} = X' \backslash \{\alpha\}$. Finally, suppose $a, b \in \{0,1\}$, that is, suppose $0 \leqslant'' 1$ and $1 \leqslant'' 0$. Since $0 \leqslant'' 1$, there exists a trail $0 = c_0, \ldots, c_i, \ldots, c_n = 1$ in X'' from 0 to 1. Viewing the nets involved as nets in X' we have $\alpha = 0' \leqslant' c_i' \leqslant' 1' = \alpha$, and thus $c_i \in \{0,1\}$. Thus, the only trail with minimal length from 0 to 1 is $0, 1$. Similarly, $1 \leqslant'' 0$ implies $1, 0$ is the unique minimal trail from 1 to 0. Suppose (x_λ, y_λ) is a net in θ converging to $(0,1)$ and (z_γ, w_γ) is a net in θ converging to $(1,0)$. Now in $X^* \times X^*$, there are convergent subnets $(x_{\sigma(\lambda)}, y_{\sigma(\lambda)}) \to (a^*, b^*)$ and $(z_{\rho(\gamma)}, w_{\rho(\gamma)}) \to (b^\#, a^\#)$ where $a^*, a^\# \in g^{-1}(0)$ and $b^*, b^\# \in g^{-1}(1)$. Since these subnets are in θ and θ^* is closed, it follows that $a^* \leqslant^* b^*$ and $b^\# \leqslant^* a^\#$. But $1 = g(b^\#) \nleqslant g(a^\#) = 0$, contrary to g being increasing. Thus $0 \leqslant'' 1$ and $1 \leqslant'' 0$ is not possible, and $\leqslant''$ is antisymmetric. The relation $\leqslant''$ is easily seen to be reflexive and transitive, and is thus a partial order on X''.

To show that $\leqslant''$ is closed in $X'' \times X''$, it suffices to show that if (A_γ, B_γ) is any net in $\leqslant''$ converging to (A, B), then $A \leqslant'' B$. This can be shown by an induction argument on $\max_\gamma\{\text{length of a minimal trail from } A_\gamma \text{ to } B_\gamma\}$ (which is bounded), as in the proof of Theorem 1.1 of [**12**].

Thus, $(X'', \tau'', \leqslant'')$ is a strict ordered compactification of X with $X'' \geqslant X'$. ∎

The lemma below gives us a supply of order disconnected spaces.

Lemma 2.2. *Every countable $T_{3.5}$-ordered space is order disconnected.*

PROOF: We shall show the stronger result that for any distinct points x and y in a countable $T_{3.5}$-ordered space X, there exists a continuous increasing surjection $g : X \to \{0,1\}$ with $g(x) \neq g(y)$. Let $CI^*(X)$ denote the set of continuous increasing functions from X to $[0,1]$. Since X is $T_{3.5}$-ordered, the evaluation map $e : X \to [0,1]^{CI^*(X)}$ defined by $e(x) = \Pi_{f \in CI^*(X)} f(x)$ is a topological and order embedding (see [4]). Choose $f_o \in CI^*(X)$ such that $f_o(x) \neq f_o(y)$. Since X is countable, there exists an irrational number α strictly between $f_o(x)$ and $f_o(y)$ with $\alpha \notin \pi_{f_o}(e(X))$. Now since the projection π_{f_o} is continuous and increasing, $\pi_{f_o}^{-1}([0,\alpha)) = \pi_{f_o}^{-1}([0,\alpha]) = U$ is a closed, open, decreasing set in $e(X) \approx X$. The function $g : X \to \{0,1\}$ defined by $g(U) = 0$ and $g(X \backslash U) = 1$ has the desired properties. ∎

In [3] Engelking and Sklyarenko show that the supremum of a set $\{X_i\}_{i \in I}$ of compactifications of a topological space X can be constructed by forming the product $P = \Pi_{i \in I} X_i$, identifying X with the subspace $\{z \in P : z = \Pi_{i \in I} x \text{ for some } x \in X\}$, then taking $cl_P X$. This construction also yields the supremum of any set of ordered compactifications. By 1.8 of [8], the remainder of the supremum of a set of (ordered) compactifications is contained in the product of the remainders of these (ordered) compactifications. Thus, we have the following result.

LEMMA 2.3. *If $\{X_i\}_{i \in I}$ is a set of (ordered) compactifications of X with $|X_i \backslash X| < \rho$ for each $i \in I$, then $\sup\{X_i\}_{i \in I}$ is an (ordered) compactification whose remainder has cardinality at most $\rho \times |I|$.*

THEOREM 2.4. *Suppose (X, τ, θ) admits finite-point ordered compactifications. Then X has a countable ordered compactification if and only if X admits n-point ordered compactifications for all integers n greater than some m.*

PROOF: Suppose X has m-point ordered compactification X' and countable ordered compactification X^*. Without loss of generality, we may assume X' is a strict ordered compactification, and $X^* \geqslant X'$ (otherwise, replace θ' by the strict order on (X', τ') and replace X^* by $\sup\{X', X^*\}$). If $h : X^* \to X'$ is the projection function, there must exist $\alpha \in X' \backslash X$ such that $h^{-1}(\alpha)$ is countable. By Lemmas 2.2 and 2.1, X has an $(m+1)$-point ordered compactification X''. Repeating this process shows that X has n-point ordered compactifications for all $n \geqslant m$.

Conversely, if X admits n-point ordered compactifications X_n for all $n > m$, Lemma 2.3 implies that $\sup\{X_n : n > m\}$ is a countable ordered compactification of X. ∎

THEOREM 2.5. *If (X, τ, θ) admits a countable ordered compactification X^* and a finite-point ordered compactification X' with $X' \leqslant X^*$, then X^* is the supremum of all finite-point ordered compactifications below it.*

PROOF: The proof is analogous to that of Theorem 2.3 of [9]. Let $X'' = \sup\{X^\# \leqslant$

https://doi.org/10.1017/S0004972700016580 Published online by Cambridge University Press

$X^* : X^{\#}$ is a finite-point ordered compactification of $X\}$. Clearly $X^* \geqslant X''$. Equality holds if the projection $f : X^* \to X''$ is one-to-one. Suppose $x \neq y$ in X^*. If the projection $h : X^* \to X'$ maps x and y to distinct points, then $f(x) \neq f(y)$. If $h(x) = h(y)$, use the strong statement proved in Lemma 2.2 to find a finite-point ordered compactification $X^{\#} \leqslant X^*$ such that the projection $k : X^* \to X^{\#}$ does separate x and y. □

THEOREM 2.6. *Suppose X admits a finite-point ordered compactification. Then X has a largest finite-point ordered compactification if and only if it has no countable ordered compactification.*

PROOF: If X has no countable ordered compactification, then there is an integer n such that X has an n-point ordered compactification but no m-point ordered compactifications for $m > n$. Now any two n-point ordered compactifications must be topologically equivalent, for otherwise by considering the associated n-stars (see [**6**]) we find that the supremum of the topological compactifications underlying the two n-point ordered compactifications has more than n compactification points. Now by the remarks preceeding Lemma 2.3, the supremum of a set of ordered compactifications is topologically equivalent to the supremum of the set of underlying topological compactifications, which leads to the contradition that X admits an m-point ordered compactification with $m > n$. Thus, all n-point ordered compactifications of X are topologically equivalent; intersecting their orders gives a largest finite-point ordered compactification.

The converse is immediate from Theorem 2.4. □

Although Theorem 2.6 gives necessary and sufficient conditions for the existence of a largest finite-point ordered compactification, no such result is known which guarantees the existence of a smallest ordered compactification, finite-point or otherwise. Indeed, if X is the half-open interval $[0,1)$ with the usual topology and discrete order, there is a unique largest finite-point ordered compactification whose order is also discrete, however there is no smallest ordered compactification of X.

If a $T_{3.5}$-ordered space X admits a finite-point ordered compactification, it obviously admits ordered compactifications whose remainders have minimal finite cardinality; we call any such compactification a *minimal-point ordered compactification.* If X has a smallest finite-point ordered compactification, then all minimal-point ordered compactifications of X have equivalent topologies, but the converse is false as is shown by the example of the preceding paragraph. On the other hand, if all minimal-point ordered compactifications of X have equivalent order, there exists a smallest ordered compactification; again, the converse is false. In general, minimal point ordered compactifications of the same space may have non-equivalent topologies and/or non-equivalent order.

Finally, for the sake of comparing finite-point ordered compactifications with finite-point (non-ordered) compactifications, we mention a few additional facts. A $T_{3.5}$-ordered space may have a largest finite-point (non-ordered) compactification but no largest finite-point ordered compactification (for example, the Euclidean plane); on the other hand, it may have a largest finite-point ordered compactification but no largest finite-point (non-ordered) compactification (for example, the natural numbers). There are also examples of $T_{3.5}$-ordered spaces which have a largest finite-point ordered compactification and a largest finite-point (non-ordered) compactification whose remainders are of different cardinality.

REFERENCES

[1] G.L. Cain, 'Countable compactifications', in *General topology and its relations to modern analysis and algebra VI*, (Z. Frolík, Editor), Proc. Sixth Prague Topological Symposium 1986 (Heldermann Verlag, Berlin, 1988), pp. 69–75.

[2] R.E. Chandler, *Hausdorff compactifications*, Lecture Notes in Pure and Appl. Math. **23** (Marcel Dekker, Inc., New York, 1976).

[3] R. Engelking and E.G. Sklyarenko, 'On compactifications allowing extensions of mappings', *Fund. Math.* **53** (1963), 65–79.

[4] P. Fletcher and W. Lindgren, *Quasi-uniform spaces*, Lecture Notes in Pure and Appl. Math. **77** (Marcel Dekker, Inc., New York, 1982).

[5] T. Kimura, '$\aleph_o$-point compactifications of locally compact spaces and product spaces', *Proc. Amer. Math. Soc.* **93** (1985), 164–168.

[6] K.D. Magill, Jr., 'N-point compactifications', *Amer. Math. Monthly* **72** (1965), 1075–1081.

[7] K.D. Magill, 'Countable compactifications', *Canad. J. Math.* **18** (1966), 616–620.

[8] J.R. McCartney, 'Maximum zero-dimensional compactifications', *Math. Proc. Cambridge Philos. Soc.* **68** (1970), 653–661.

[9] J.R. McCartney, 'Maximum countable compactifications of locally compact spaces', *Proc. London Math. Soc. (3)* **22** (1971), 369–384.

[10] L. Nachbin, *Topology and order*, New York Math. Studies **4** (Van Nostrand, Princeton, N.J., 1965).

[11] T.A. Richmond, 'Finite-point order compactifications', *Math. Proc. Cambridge Philos. Soc.* **102** (1987), 467–473.

[12] T.A. Richmond, 'Posets of ordered compactifications', *Bull. Austral. Math. Soc.* **47** (1993), 59–72.

Department of Pure and Applied Mathematics
Washington State University
Pullman, WA 99164-3113
United States of America

Department of Mathematics
Western Kentucky University
Bowling Green, KY 42101
United States of America