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Abstract

We classify fibrations of abstract 3-regular GKM graphs over 2-regular ones, and show that all fibrations satisfying the known necessary conditions for realizability are, in fact, realized as the projectivization of equivariant complex rank-2 vector bundles over quasitoric 4-manifolds or S^4. We investigate the existence of invariant (stable) almost complex, symplectic, and Kähler structures on the total space. In this way, we obtain infinitely many Kähler manifolds with Hamiltonian non-Kähler actions in dimension 6 with prescribed one-skeleton, in particular with a prescribed number of isolated fixed points.

Contents

O. Goertsches, P. Konstantis and L. Zoller

1. Introduction

While, as shown by Karshon [Kar99], any effective Hamiltonian circle action on a compact symplectic 4-manifold with finite fixed point set extends to a toric action, an analogous statement in higher dimensions is no longer true. In fact, Tolman [Tol98] gave the first example of a Hamiltonian T^2-action on a compact 6-dimensional symplectic manifold with finite fixed point set that does not admit an invariant Kähler structure. Her proof relied solely on the shape of the image of the moment map, or rather the x-ray which also contains the subpolytopes given by the images of the lower-dimensional orbit type strata. In the toric setting, the moment image contains the entire information of the x-ray and by Delzant's theorem [Del88] there is a one-to-one correspondence between Delzant polytopes and toric manifolds. If one had variants of this correspondence outside of the toric case, this would in theory enable the construction of Hamiltonian non-Kähler actions just by drawing specific x-rays. This is the core idea of the present article.

Our language of choice is not the x-ray but the GKM graph [GKM98] which encodes the one-skeleton of the space in a labelled graph. In our setting, which is T^2-actions in dimension 6 with finite fixed point set (in particular, these are actions of complexity one [KT03] which are not tall), this contains, up to lengths of edges, the same information as the x-ray. GKM graphs have the advantage of not being bound to Hamiltonian actions but rather being able to model arbitrary GKM actions while further geometric structures (almost complex, symplectic, or Kähler) are reflected in properties of the graph (see § 2). Regarding beginnings of a Delzant-type correspondence, we proved in [GKZ19] that in dimension 6, for GKM actions with connected stabilizers on simply connected manifolds (these conditions are automatic in the toric setting), the GKM graph does encode the non-equivariant diffeomorphism type. From this we deduced that Tolman's original example is diffeomorphic to Eschenburg's twisted flag manifold $\mathrm{SU}(3)//T^2$ (see [Esc84, Esc92]), which is the projectivization of a complex T^2-equivariant rank-2 vector bundle over $\mathbb{CP}^2$. This implied, in particular, that Tolman's example is Kähler, although of course not in an equivariant fashion.

From the point of view of GKM theory, the fact that Tolman's example is a projectivized equivariant bundle is reflected in the fact that its GKM graph fibers over the GKM graph of $\mathbb{CP}^2$,

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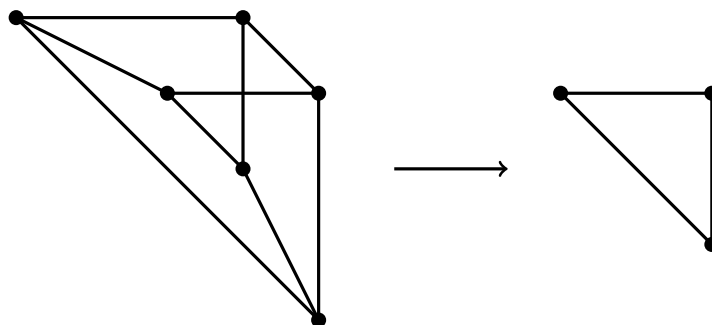
see also Example 4.8. Formally, we make use of the notion of a fibration of abstract GKM graphs, introduced by Guillemin, Sabatini, and Zara [GSZ12], which we review in § 3.

In this paper, we extend this viewpoint on Tolman's example to the more general setting of 6-dimensional GKM T^2-manifolds with an arbitrary (finite) number of fixed points. As stated initially, the purpose here is twofold: on the one hand we contribute to the realization problem of abstract GKM graphs by showing that many GKM fibrations in dimension 6 are, in fact, realizable. On the other hand, we closely investigate how Tolman's original example embeds in this context which uncovers a rich variety of new examples of Hamiltonian non-Kähler actions for any possible number of fixed points. In a little more detail, our results can be summarized as follows.

Realization. We prove that every fiberwise signed GKM fibration with values in $\mathbb{Z}^2$ (see Definition 3.4) of a 3-regular abstract GKM graph over an effective 2-regular abstract GKM graph can be realized geometrically by a fibration of GKM T^2-manifolds (see Theorem 5.1). The realization is given as the (6-dimensional) projectivization of a rank-2 complex T^2-vector bundle over a 4-dimensional T^2-manifold. The first step to construct the bundle is to do so separately over each invariant two-sphere in the base such that its projectivization is a specific Hirzebruch surface. We proceed to glue those to obtain a bundle over the entire one-skeleton and finally extend the bundle to the whole base with the use of equivariant obstruction theory. Conversely, we argue in Proposition 3.8 that the aforementioned conditions on the abstract graph fibrations are, in fact, also necessary in order to be realized by an equivariant fiber bundle of effective GKM T^2-manifolds in these dimensions.

Geometric structures. We go on to show that certain properties of the graphs lead to corresponding geometric structures on the realizations: a signed GKM structure on the base graph lets us choose almost complex realizations and if, furthermore, the base graph is the boundary of a Delzant polytope, then our realizations are Hamiltonian actions. In the latter case, the T^2-invariant symplectic form on the total space also admits a compatible complex structure (in fact, the manifold is even projective). However, the complex structure is not necessarily T^2-invariant.

(Non-)Existence of invariant Kähler structures. Regarding the T^2-invariance of the complex structures which are present in the Hamiltonian case, we prove the following: if the GKM fibration is graph theoretically a Möbius band and $n - 1$ of the $2n$ fixed points ($n \neq 4$) map to the interior of the moment image, then there cannot exist a T^2-invariant Kähler structure (Theorem 7.1). This is done by classifying all possible signed GKM structures that a T^2-invariant almost complex structure could induce on the underlying GKM graph and proving individually that they do not come from a Kähler action. In combination with the previous results this yields that every fibration of abstract GKM graphs of the above type gives rise to a T^2-action in the spirit of Tolman's original example, i.e. a Hamiltonian action with isolated fixed points on a simply connected compact 6-dimensional manifold such that no T^2-invariant Kähler structure exists, while the symplectic form itself is non-equivariantly Kähler. Whether the stabilizers of the realizations are connected, as they are in Tolman's example, depends on the respective GKM graph (see Theorem 5.1). In contrast to the Tolman-type scenario, in case the GKM fibration is graph theoretically of product type there always exists a T^2-invariant complex structure (§ 6.3).

Classification of GKM fibrations. Finally, we quantify the new examples by classifying fiberwise signed GKM fibrations over a fixed base graph. Up to isomorphism they correspond bijectively to $((\mathbb{Z} - 0)^n / \pm) \times \{0, 1\}$ (Proposition 4.5). In case B is the boundary of a Delzant polytope

the Hamiltonian non-Kähler actions described above correspond to the elements of the form $([k_1, \ldots, k_n], 1)$ such that k_i and k_{i-1} have the same sign for all but one $i \in \{1, \ldots, n\}$, where we set $k_0 = -k_n$ (Proposition 4.10). In particular, every such tuple in combination with a 2-dimensional Delzant polytope gives rise to an exotic Hamiltonian action in the previous sense. We prove that different tuples (up to signs and permutation) produce examples of different equivariant homotopy type (Corollary 8.3). In addition, we compute the (non-equivariant) cohomology rings as well as the Chern classes and verify that even in the case of manifolds with 6 fixed points, which fiber over $\mathbb{CP}^2$, our method produces infinitely many pairwise not homotopy equivalent examples of manifolds carrying such an exotic Hamiltonian action (§ 9).

2. GKM theory and geometric structures

The purpose of this section is to review the basics of GKM theory and see how different geometric structures on manifolds leave their mark on the GKM graph.

2.1 GKM manifolds

For an action of a compact torus T on a connected, compact manifold M, we consider its fixed point set $M^T = \{p \in M \mid T \cdot p = \{p\}\}$ as well as its *one-skeleton* $M_1 = \{p \in M \mid \dim T \cdot p \leq 1\}$. In *GKM theory*, named after Goresky, Kottwitz, and MacPherson [GKM98], one puts certain assumptions on the action that allow us to encode the structure of the one-skeleton in a labelled graph. More precisely, we say that the action *satisfies the GKM conditions*, or that it is a *GKM action*, if M is orientable, M^T is a finite set of points, and M_1 a finite union of T-invariant 2-spheres. A manifold together with a GKM (T-)action will also be referred to as a *GKM (T-)manifold*. Note that a finite non-empty fixed point set forces the manifold to be even-dimensional.

In this setting, the orbit space of the one-skeleton M_1/T has the structure of a graph Γ, with one vertex for each fixed point, and one edge for each invariant 2-sphere. The vertex set of a graph Γ will be denoted by $V(\Gamma)$, and the set of edges by $E(\Gamma)$. Formally, we include each edge twice, once for each orientation, in the edge set $E(\Gamma)$. For an oriented edge $e \in E(\Gamma)$ we denote its initial vertex by $i(e)$ and its terminal vertex by $t(e)$; the edge e with the opposite orientation will be denoted $\bar{e}$. We write $E(\Gamma)_v$ for the set of edges $e \in E(\Gamma)$ emanating from v. At each fixed point p of the action the isotropy representation decomposes into n two-dimensional summands, where $2n$ is the dimension of M. The n weights of these irreducible submodules are elements of $\mathbb{Z}_{\mathfrak{t}}^*/ \pm 1$, where $\mathbb{Z}_{\mathfrak{t}}^* \subset \mathfrak{t}^*$ is the weight lattice of T. Any such weight corresponds uniquely to an invariant 2-sphere containing p, and we put it as a label of the corresponding edge of Γ. In total, we obtain a map $\alpha : E(\Gamma) \to \mathbb{Z}_{\mathfrak{t}}^*/ \pm 1$ which we call an *axial function*, following [GZ99]. The graph Γ, together with the axial function α, will be called the *GKM graph* of the T-action.

Remark 2.1. Often, one includes the vanishing of the odd-dimensional cohomology groups of M as part of the GKM conditions, in order to make the connection between the GKM graph and (equivariant) cohomology. The focus of this paper is the realization of certain GKM graphs and their geometrical properties. All our examples will automatically satisfy this condition.

Independent of this geometric setting, one can define abstract GKM graphs [GZ01]. The graphs one considers have finite vertex and edge sets; we allow multiple edges between vertices, but no loops, i.e. edges that connect a vertex to itself.

Definition 2.2. A *connection* on a graph Γ consists of a bijective map $\nabla_e : E(\Gamma)_{i(e)} \to E(\Gamma)_{t(e)}$ for each $e \in E(\Gamma)$, such that:

(1) $\nabla_e e = \bar{e}$; and
(2) $(\nabla_e)^{-1} = \nabla_{\bar{e}}$ for all $e \in E(\Gamma)$.

Definition 2.3. An *(abstract) GKM graph* (Γ, α) consists of an n-valent connected graph Γ and a map $\alpha : E(\Gamma) \to \mathbb{Z}^m / \pm 1$, called an *axial function*, such that there exists a connection ∇ on Γ for which the following hold.

(1) For every $v \in V(\Gamma)$ and distinct edges $e, f \in E(\Gamma)_v$ we have that $\alpha(e)$ and $\alpha(f)$ are linearly independent.
(2) The connection ∇ is *compatible with* α, i.e. for every $v \in V(\Gamma)$ and edges $e, f \in E(\Gamma)_v$ and any sign choices for the elements in $\mathbb{Z}^m / \pm$ we have

$$\alpha(\nabla_e f) = \varepsilon \alpha(f) + c \alpha(e)$$

for some $\varepsilon \in \{\pm 1\}$ and $c \in \mathbb{Z}$.
(3) For every $e \in E(\Gamma)$ we have $\alpha(\bar{e}) = \alpha(e)$.

Note that the linear independence of $\alpha(e)$ and $\alpha(f)$ in the first condition, which is defined via preimages under $\mathbb{Z}^m \to \mathbb{Z}^m / \pm 1$, is well defined.

Remark 2.4. In the definition of an abstract GKM graph, no acting torus is fixed. Whenever a GKM graph is associated to an action of a torus T^m or realization of abstract GKM graphs is considered in the context of a fixed torus, we identify its Lie algebra $\mathfrak{t}$ with $\mathbb{R}^m$ in such a way that $\mathbb{Z}^*_{\mathfrak{t}}$ corresponds to $\mathbb{Z}^m \subset \mathbb{R}^m \cong (\mathbb{R}^m)^*$. In analogy with the geometric situation we also refer to the values of the axial function as weights or labels of edges.

Remark 2.5. Given an action of a torus T on a connected, compact manifold M satisfying the GKM conditions, the GKM graph of the action admits a compatible connection, see [GW22, Proposition 2.3] or [GZ01]. Note that there exist different conventions in the literature of whether the connection is part of the structure of an abstract GKM graph or not. In general, the connection is not unique and there is no canonical choice, which is why we chose to not fix a connection for the abstract object in order to keep the passage from geometry to graphs canonical. In addition, an equivariant diffeomorphism induces a map on the graph level (see the notion of isomorphism below) which is not necessarily compatible with specific choices of connections. This is why it is not handy to have connections as part of the abstract data in particular for classification purposes.

Definition 2.6. Let (Γ, α) be a GKM graph and Γ' a connected regular subgraph of Γ. If $(\Gamma', \alpha|_{E(\Gamma')})$ is a GKM graph (i.e. it admits a compatible connection), then we call it a *GKM subgraph* of (Γ, α).

Following [FY19] we make the following definition.

Definition 2.7. An *isomorphism* $(\Gamma, \alpha) \to (\Gamma', \alpha')$ between GKM graphs consists of bijections f and g between the vertex and edge sets and an automorphism φ of $\mathbb{Z}^m$ such that for any $e \in E(\Gamma)$ we have:

(1) $f(i(e)) = i(g(e))$;
(2) $f(t(e)) = t(g(e))$;
(3) $\varphi(\alpha(e)) = \alpha'(g(e))$.

We remark that any equivariant homeomorphism between two GKM manifolds induces such an isomorphism of GKM graphs, with the automorphism φ equal to the identity.

Definition 2.8. We call a GKM graph with labels in $\mathbb{Z}^m$ *effective* if at one (and, hence, every) vertex, the labels of the outgoing edges lift to a generating set of $\mathbb{Z}^m$.

The terminology is justified by the fact that a GKM T^m-action on a manifold is effective if and only if the corresponding GKM graph is effective: for any vertex v of the GKM graph, the kernel of the action is given as the intersection of the kernels of the $\alpha(e)$, where e varies over all edges emanating from v. Here, we interpret the $\alpha(e)$ as homomorphisms $T^m \to S^1$.

2.2 Almost complex structures

Recall that an almost complex structure on a smooth manifold M is a smooth endomorphism J of TM satisfying $J^2 = -\mathrm{Id}$. If such an endomorphism is present on the bundle $TM \oplus \mathbb{R}^k$ for some k, then we speak of a stable almost complex structure. If an almost complex structure is integrable, then we refer to it as a complex structure.

Given a T-invariant almost complex structure on M, the weights of the isotropy representation have a well-defined sign, i.e. are elements of $\mathbb{Z}_{\mathfrak{t}}^*$. We will speak of the *signed GKM graph* of the action when we consider the graph Γ with these weights as labels. Formally, the axial function becomes a map $\alpha : E(\Gamma) \to \mathbb{Z}_{\mathfrak{t}}^*$, by associating to e the weight of the corresponding summand of the isotropy representation at $i(e)$. Abstractly, we make the following definition.

Definition 2.9. An *abstract signed GKM graph* (Γ, α) consists of an n-valent connected graph Γ and a map $\alpha : E(\Gamma) \to \mathbb{Z}^m$, called *axial function*, such that there exists a compatible connection ∇ on Γ for which the following hold.

(1) For every $v \in V(\Gamma)$ and distinct edges $e, f \in E(\Gamma)_v$ we have that $\alpha(e)$ and $\alpha(f)$ are linearly independent.
(2) The connection ∇ is *compatible with* α, i.e. for every $v \in V(\Gamma)$ and edges $e, f \in E(\Gamma)_v$ we have

$$\alpha(\nabla_e f) = \alpha(f) + c\alpha(e)$$

for some $c \in \mathbb{Z}$.
(3) For every $e \in E(\Gamma)$ we have $\alpha(\bar{e}) = -\alpha(e)$.

Remark 2.10. By composing the axial function of a signed GKM graph with the projection $\mathbb{Z}^m \to \mathbb{Z}^m/\pm$ one obtains an (unsigned) GKM graph. Note that a connection which is compatible with the signed GKM graph is, in particular, compatible with the (unsigned) GKM graph. We also call the signed graph a *compatible signed structure* of the underlying GKM graph. There might be different signed structures compatible with a single GKM graph. If these were induced by an almost complex GKM action as explained previously, then different signed structures would correspond to different homotopy classes of compatible almost complex structures. The existence of a compatible signed structure is clearly an obstruction to the existence of an invariant almost complex structure. For example, it is easy to check that the GKM graph

$$(1,0)$$

$$(0,1)$$
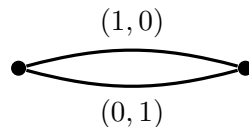
of the standard T^2-action on $S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}$ does not carry a compatible signed structure.

The definitions of GKM subgraphs and isomorphisms of GKM graphs carry over to the signed setting in an obvious fashion. Moreover, any equivariant diffeomorphism between two

GKM manifolds that respects T-invariant almost complex structures induces an isomorphism of the corresponding signed GKM graphs.

2.3 Symplectic structures

Just as the existence of invariant almost complex structures impacts the GKM graph of a GKM manifold, the existence of a compatible symplectic structure forces certain properties onto the GKM graph. Recall that an almost complex structure J is called compatible with a symplectic form ω if the two-form $v \otimes w \mapsto \omega(v, Jw)$ defines a Riemannian metric. For any T-invariant symplectic form there exists a T-invariant compatible almost complex structure, hence combinatorial consequences of symplectic structures contain those from the almost complex case.

More precisely, we consider for a Hamiltonian GKM action of a torus T on a symplectic manifold M a moment map $\mu\colon M \to \mathfrak{t}^*$ into the dual of the Lie algebra of T. An invariant two sphere $S^2 \subset M$ gets mapped under μ to an affine linear interval in $\mathfrak{t}^*$ whose boundary points are the images of the fixed points. By piecing those together we obtain a linear realization of the underlying graph of the GKM graph, i.e. an edge-wise affine linear map from the topological realization of Γ to $\mathfrak{t}^*$ (not an embedding), whose image is $\mu(M_1)$. If (Γ, α) is the signed GKM graph associated to an almost complex structure which is compatible with the symplectic form, then the labels are encoded in the linear realization as follows: if e is an oriented edge, then the slope of the corresponding affine linear segment in $\mathfrak{t}^*$ is given by $\alpha(e) \in \mathbb{Z}_{\mathfrak{t}}^* \subset \mathfrak{t}^*$. Thus, the linear realization determines (Γ, α) up to multiples of the weights. The labels are uniquely determined if we add the assumption that they are primitive.

Remark 2.11. Having such a linear realization is a non-trivial obstruction for a signed GKM graph to come from a Hamiltonian action. For example, the signed structures of type III in Theorem 7.1 do not since the cone spanned by the weights of the outgoing edges at every point is all of $\mathfrak{t}^*$.

The convexity theorem, which implies that the image $\mu(M)$ of the moment map is the convex hull of $\mu(M_1)$, gives additional obstructions. The preimage of an outer edge of the resulting polytope is contained in M_1. Thus, every outer edge of the convex hull $\mu(M)$ must be an edge in $\mu(M_1)$. As a counterexample, the linear realization
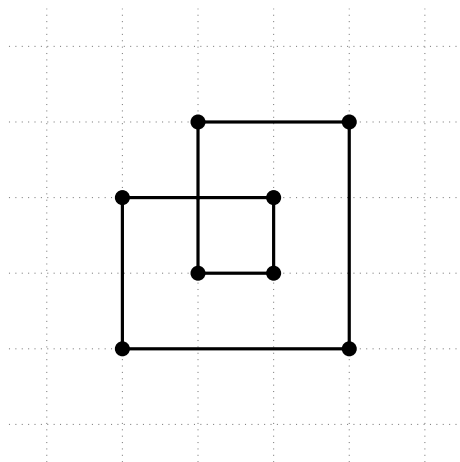
does not have this property (the grid represents the standard basis of $\mathbb{Z}^2$) and it follows from the results in §7.3 that, more generally, the signed GKM graph which is uniquely defined by having this realization and primitive weights does not admit any linear realization with the above convexity property. Hence, it cannot come from a Hamiltonian action.

Note that the convexity obstruction from the above remark does not concern the inner edges of the linear realization. However, in order for a GKM graph to come from a Hamiltonian T-action additional obstructions for inner edges do exist: for any subtorus $T' \subset T$, the action of T on the submanifold $M^{T'}$ is again Hamiltonian, so certain subgraphs of the original GKM graph do again satisfy the convexity criterion explained in Remark 2.11. These subtleties will not play a role in our low-dimensional considerations, so we refrain from introducing a precise definition outside of the case below. Note that GKM graphs of 4-dimensional Hamiltonian manifolds do indeed satisfy the following.

DEFINITION 2.12. We say that a signed 2-regular GKM graph with labels in $\mathbb{Z}^m$ is of *polytope type* if, graph theoretically, it is given by the edges of a simple convex 2-polytope in $\mathbb{R}^m$ and the labels of the oriented edges are given by integral representatives of the slopes of the edges.

2.4 Kähler structures

As a last step in the hierarchy we state a certain property of signed GKM graphs coming from invariant Kähler structures which goes beyond the previous obstructions for Hamiltonian actions. Recall that a symplectic form is called Kähler if it admits a compatible complex structure. Note that unlike for T-invariant symplectic forms and T-invariant almost complex structures, the T-invariance of a Kähler form does not immediately imply the existence of a compatible T-invariant complex structure. In fact this is false in general as follows from the result of Tolman [Tol98] mentioned in the introduction as well as from the main results of this article.

In [Tol98, Lemma 3.5] the following is shown. Consider a Hamiltonian T-action on (M, ω) with ω a Kähler form admitting a T-invariant compatible complex structure. We also refer to this as a T-invariant Kähler structure. Let $V \subset T_pM$ be a sum of irreducible summands of the isotropy representation at a fixed point p such that the weights of those summands form a minimal generating set of a convex cone in $\mathfrak{t}^*$. Then there is a Hamiltonian submanifold of M containing p whose tangent space at p is V. As a special case, this implies the following.

COROLLARY 2.13. *For any pair of adjacent edges in the signed GKM graph associated to an action of GKM type with an invariant Kähler structure, there is a 2-regular GKM subgraph of polytope type containing those edges.*

An example where this fails while the obstructions for general Hamiltonian actions hold is given by the GKM graph with primitive labels and the linear realization
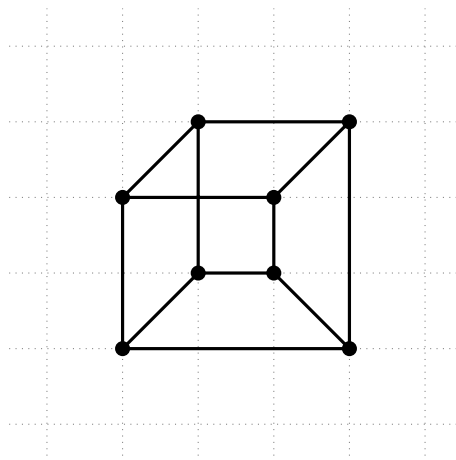
since the only signed GKM subgraph containing the edges between the three inner vertices is the graph from Remark 2.11. See also § 7.2 where we will consider this obstruction to the Kähler property for a large class of graphs.

3. GKM fibrations

Let us review the definition of a GKM fibration, introduced by Guillemin, Sabatini, and Zara in [GSZ12].

A *morphism* of graphs $\pi : \Gamma \to B$ consists of a map sending vertices of Γ to vertices of B, as well as a map, also denoted by π, sending an edge between vertices $p, q \in V(\Gamma)$ with $\pi(p) \neq \pi(q)$ to an edge between $\pi(p)$ and $\pi(q)$. On oriented edges we require that $i(\pi(e)) = \pi(i(e))$. Edges in Γ between $p, q \in V(\Gamma)$ with $\pi(p) = \pi(q)$ are called *vertical*; the other edges are called *horizontal*. For $p \in V(\Gamma)$ the set of horizontal edges emanating from p is denoted by H_p.

Remark 3.1. In [GSZ12] morphisms of graphs are defined only on vertices, not on edges. As we allow multiple edges between vertices, we need to specify images of edges as well.

The morphism π is a *graph fibration* if for all $p \in V(\Gamma)$ the map $\pi : H_p \longrightarrow E(B)_{\pi(p)}$ is a bijection. In other words, graph fibrations have a unique path-lifting property: given a vertex $p \in \Gamma$ and an edge $e \in E(B)$ with $i(e) = \pi(p)$, there exists a unique horizontal edge which lies over e and starts at p.

Of course a fibration of GKM graphs should be compatible with the additional structure. There are analogous versions of this notion for the signed and the unsigned case.

DEFINITION 3.2. Let (Γ, α) and (B, α_B) be (signed) GKM graphs. A graph fibration $\pi : \Gamma \to B$ is a *(signed) GKM fibration* if there exist connections ∇ and ∇^B which are compatible with the (signed) GKM structures on Γ and B such that additionally the following hold.

(1) For any edge e of B and any lift $\tilde{e}$ of e we have $\alpha_B(e) = \alpha(\tilde{e})$.
(2) For every edge e of Γ, the connection ∇_e sends vertical edges to vertical edges (and thus horizontal edges to horizontal edges).
(3) For two edges e, e' of B with $i(e) = i(e')$ and lifts $\tilde{e}, \tilde{e}'$ of e and e' with $i(\tilde{e}) = i(\tilde{e}')$, the edge $\nabla_{\tilde{e}} \tilde{e}'$ is the lift of $(\nabla^B)_e e'$ at $t(\tilde{e})$.

Remark 3.3. Note that our definition deviates from that in [GSZ12] in that we do not fix connections as part of the data of GKM fibrations. For the sake of completeness, we also note that there is the stronger notion of a GKM fiber bundle, which was introduced in [GSZ12] in the signed case. However, our main interest in this article lies in dimension 6 and the GKM fibrations we consider will automatically fulfil the stronger requirements of GKM fiber bundles. Thus, there is no need for us to introduce this more restrictive notion.

The GKM fibrations we can realize geometrically through our main result will have almost complex fibers. However the base will not need to have an almost complex structure. The natural setting for this is given by the following definition which is an intermediate notion between unsigned and signed GKM fibrations.

DEFINITION 3.4. Let $\pi : (\Gamma, \alpha) \to (B, \alpha_B)$ be a GKM fibration (of unsigned graphs). Let $F \subset E(\Gamma)$ be the set of vertical edges and $\tilde{\alpha} : F \to \mathbb{Z}^m$ a lift of $\alpha : E(\Gamma) \to \mathbb{Z}^m/\pm$ satisfying $\tilde{\alpha}(e) = -\tilde{\alpha}(\overline{e})$. Then we call π together with $\tilde{\alpha}$ a *fiberwise signed fibration* if the connections ∇ and ∇^B

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as in Definition 3.2 can be chosen in a way such that $\tilde{\alpha}(\nabla_e e') \equiv \tilde{\alpha}(e') \bmod \alpha(e)$ for any $e' \in F$ and $e \in E(\Gamma)$.

LEMMA 3.5. *Every signed fibration of signed GKM graphs gives rise to a fiberwise signed fibration of the underlying GKM graphs. Conversely, if $(\pi, \tilde{\alpha})$ is a fiberwise signed fibration as above, then any signed structure $(B, \tilde{\alpha}_B)$ compatible with the base graph gives rise to a unique signed structure on Γ which extends $\tilde{\alpha}$ such that π becomes a signed fibration.*

Proof. The first statement is clear. For the second statement, note that an extension of $\tilde{\alpha}$ to $E(\Gamma)$ such that $\pi \colon (\Gamma, \tilde{\alpha}) \to (B, \tilde{\alpha}_B)$ is a signed fibration is unique: on every horizontal edge $e \in E(\Gamma)$ we need to define $\tilde{\alpha}(e) = \tilde{\alpha}_B(\pi(e))$. So it remains to check the existence of a compatible connection. Let ∇ be a connection on Γ as in Definition 3.4 and ∇^B be a connection compatible with the signed graph $(B, \tilde{\alpha}_B)$. We define a new connection ∇' as follows: for any edge $e \in E(\Gamma)$ we define ∇'_e as $\nabla'_e(e') = \nabla_e(e')$ if e' is vertical. On horizontal edges, we define ∇'_e as

$$H_{i(e)} \xrightarrow{\pi} E(B)_{i(\pi(e))} \xrightarrow{\nabla^B_{\pi(e)}} E(B)_{t(\pi(e))} \xrightarrow{\pi^{-1}} H_{t(e)}$$

if e is horizontal and as

$$H_{i(e)} \xrightarrow{\pi} E(B)_{\pi(i(e))} = E(B)_{\pi(t(e))} \xrightarrow{\pi^{-1}} H_{t(e)}$$

if e is vertical. Then ∇' and ∇^B satisfy the requirements for connections compatible with signed fibrations. $\square$

DEFINITION 3.6. We call two (signed) GKM fibrations $\pi \colon (\Gamma, \alpha) \to (B, \alpha_B)$ and $\pi' \colon (\Gamma', \alpha') \to (B, \alpha_B)$ *equivalent* if there is an isomorphism $(f, g, \varphi) \colon (\Gamma, \alpha) \to (\Gamma', \alpha')$ of (signed) GKM graphs as in Definition 2.7 with $\varphi = \mathrm{id}_{\mathbb{Z}^m}$, which respects the decomposition into vertical and horizontal edges and commutes with the fibrations on vertices and horizontal edges. Two fiberwise signed fibrations $(\pi, \tilde{\alpha})$ and $(\pi', \tilde{\alpha}')$ are called *equivalent* if there is an equivalence $(f, g, \mathrm{id}_{\mathbb{Z}}^m)$ of the underlying GKM fibrations such that additionally $\tilde{\alpha}'(g(e)) = \tilde{\alpha}(e)$ for every vertical edge $e \in E(\Gamma)$.

We conclude this section with the observation that equivariant fiber bundles of GKM manifolds indeed give rise to the combinatoric structure of GKM fibrations. This has been observed in the case of flag manifolds in [GSZ12]. However, we are not aware of a general account of this fact, hence we want to give it here. Furthermore, we take this opportunity to geometrically interpret the newly introduced notion of a fiberwise signed fibration.

Let $f \colon M \to N$ be a T-equivariant map of GKM T-manifolds. Then, in particular, it restricts to a map $M_1 \to N_1$ between one-skeleta. This maps fixed points to fixed points. Furthermore, if S is an invariant 2-sphere in M_1 containing the fixed points $p, q \in S^T$, then either $f(p) = f(q)$ or $f(S)$ intersects a two-sphere $S' \subset N_1$ outside of a fixed point. In this case the kernel of the action on S has to be contained in the kernel of the action on S'. This implies that the weight associated to S' in $\mathbb{Z}_{\mathfrak{t}}^* / \pm 1$ is a multiple of the weight of S. As by assumption the weights of adjacent 2-spheres are linearly independent, there is exactly one 2-sphere S' with this property. Hence, on the level of GKM graphs (Γ_M, α_M) and (Γ_N, α_N) of M and N we obtain a map $\pi \colon V(\Gamma_M) \to V(\Gamma_N)$ as well as for any $e \in E(\Gamma_M)$ with $\pi(i(e)) \neq \pi(t(e))$ a unique corresponding edge $\pi(e) \in E(\Gamma_N)$, where we map oriented edges to oriented edges in a way compatible with the map on vertices.

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PROPOSITION 3.7. *Let $F \to M \xrightarrow{f} N$ be a smooth fiber bundle in which M, N, and all fibers are GKM T-manifolds and f is T-equivariant.*

(i) *Then the induced map π on the GKM graphs defined above is a GKM fibration.*
(ii) *If M and N carry T-invariant almost complex structures commuting with $df \colon TM \to TN$, then π is a signed GKM fibration with respect to the induced signed structures.*
(iii) *If $\ker df \to M$ admits the structure of a T-invariant complex vector bundle, then π admits the structure of a fiberwise signed fibration.*

Proof. If $q \in N$, then we denote the fiber over q with F_q, and similarly if $p \in M$, we denote by F_p the fiber containing p. Let $p \in M^T$, then F_p is preserved by T and the tangent representation splits as $T_pM = T_pF_p \oplus H_p$. All 2-spheres corresponding to the irreducible summands in T_pF_p belong to vertical edges. We want to argue that all irreducible summands in H_p do indeed correspond to horizontal edges, i.e. we need $\pi(i(e)) \neq \pi(t(e))$ for those edges. Let $U \subset T$ be the codimension 1 subgroup given by the kernel of an irreducible subrepresentation of H_p. The map df induces a T-equivariant isomorphism $H_p \cong T_{f(p)}N$ hence we find a corresponding summand in $T_{f(p)}N$ and a corresponding 2-sphere $S \subset N$ connecting $f(p)$ to another fixed point $q \in N^T$. Now let $\eta \colon I \to S$ be an embedded path from $f(p)$ to q. Note that giving I the trivial U-action makes η U-equivariant. Hence, restricting the F-bundle $M \to N$ to η we obtain a U-equivariant subbundle. The U-equivariant isomorphism class of the resulting bundle $F \to E \to I$ depends only on the equivariant homotopy class of the map η, see [Las82, Corollary 2.11]. This is trivial and, hence, $E \cong F_p \times I$ is equivariantly a product. Hence, $E^U = F_p^U \times I$ and it follows that every T-fixed point in F_p lies in a common connected component of M^U with some point in the fiber F_q. We claim that every such component of M^U does, in fact, contain a point in F_q^T. Indeed, by assumption, F_q is again GKM as a T-manifold and, hence, every component of F_q^U contains a T-fixed point. Hence, we have shown that the 2-sphere leaving our initially chosen fixed point p and belonging to a subrepresentation of H_p, does indeed contain a fixed point in another fiber F_q. In particular, the corresponding edge is horizontal. Since $H_p \cong T_{f(p)}N$ the horizontal edges correspond bijectively to the edges in the base graph such that labels are preserved.

To finish the proof it remains to show that appropriate connections can be chosen. Using that compatible connections exist on the total space and the base, statements (i) and (ii) could be deduced by repeating the argument from Lemma 3.5. However, we recall the general geometric argument for the existence of connections (cf. [GW22, Proposition 2.3] or [GZ01]) as we need to apply it to the fiberwise signed case. Consider a real or complex T-equivariant vector bundle $V \to S^2$, where T acts on S^2 via a weight $\alpha \in \mathbb{Z}_{\mathfrak{t}}^*$. Let $U \subset T$ denote the kernel of the action on S^2, let $p, q \in S^2$ be the two fixed points and consider the T-representations V_p and V_q over the fixed points. Then $V_p \cong V_q$ as U-representations. One way to see this is to connect p and q by a path $\eta \colon I \to S^2$, which we consider as a U-equivariant map and note that it is U-equivariantly homotopic to the constant map at p. Hence, from [Las82, Corollary 2.11] it follows that the pullback of V to I along η is isomorphic (as a real or complex U-vector bundle) to the product $V_p \times I$ and, in particular, $V_p \cong V_q$ as U-representations. In that case, the irreducible T-representations of V_p can be associated bijectively to those at V_q with the property that they become isomorphic after restricting to U. For the weights β, γ of two such irreducible representations the last condition is equivalent to $\beta \equiv \gamma \mod \alpha$ in the case of complex vector bundles or to $\beta \equiv \pm\gamma \mod \alpha$ in the case of real bundles (after choosing a sign for the weights). Applying this to the case of TM concludes the classical proof of existence of a connection by

taking any bijection with the above property (not canonical) of the irreducible tangent representations at adjacent fixed points and using it to define a bijection of the corresponding edges. In case TM has a T-invariant almost complex structure, this will produce a connection for the signed graph.

Returning to the case of fibrations, let $p, q \in M^T$ be two fixed points connected by an invariant 2-sphere S_e corresponding to a horizontal edge e. In order to define transport along e suppose a connection ∇^N has been fixed for N. We have a decomposition $T_p M \cong T_p F_p \oplus H_p$ and $H_p \cong T_{f(p)} N$ as T-representations. Then the bijection of irreducible subrepresentations of $T_{f(p)} N$ and $T_{f(q)} N$ provided by ∇^N lifts to a map between subrepresentations of H_p and H_q. This tells us how to define the transport ∇^M_e of horizontal edges along horizontal edges. In case (ii) we choose ∇^N to be compatible with the signed structure. Then since $H_p \cong T_{f(p)} N$ is by assumption an isomorphism of complex representations it follows that ∇^M will also be compatible with the signed structure on the horizontal edges. When transporting vertical edges along any horizontal or vertical edge, we just apply the previously described general procedure of defining a connection to the subbundle $\ker df \to M$. Recall that in cases (ii) and (iii) the almost complex structure on $\ker df$ defines unique signs for the vertical weights by associating to an oriented edge the weight of the corresponding irreducible complex subrepresentation at its starting point. By the definition of the connection, it respects these signs. Finally, when transporting a horizontal edge along a vertical edge e from p to q, we just use the isomorphism $H_p \cong T_{f(p)} N \cong H_q$ provided by df. In case (ii) this is, in fact, an isomorphism of complex representations, hence the connection respects the signed structures. $\square$

In view of the requirements for our main realization result we observe that the fiberwise signed condition is indeed necessary when realizing GKM fibrations with 2-dimensional fibers.

Corollary 3.8. *Let $F \to M \xrightarrow{f} N$ be as above and assume that F is of dimension 2. Then the associated GKM fibration admits a fiberwise signed structure.*

Proof. By assumption, M and N are orientable. This implies that $\ker df$ is an orientable T-equivariant real rank-2 vector bundle. We choose a T-invariant metric on $\ker df$. Identifying the fibers with $\mathbb{C}$ via a positively oriented orthonormal frame induces an almost complex structure on $\ker df$. By T-invariance of the metric, this structure is T-invariant as well. Now apply part (iii) of Proposition 3.7. $\square$

4. GKM fibrations in dimension 6

In this section we consider GKM fibrations $\Gamma \to B$ where Γ is 3-regular, B is 2-regular, and weights in $\mathbb{Z}^2 / \pm 1$ (on the geometric side this would, in the case of realizability, correspond to an equivariant fibration of a 6-dimensional T^2-manifold over a 4-dimensional T^2-manifold). All fibrations will be assumed to be of this form even if not explicitly stated. Note that graph theoretically there is not much variety to what can happen: B is necessarily an n-gon (since it is 2-regular and connected).

Definition 4.1. If the lifts of a path around the n-gon B are closed in Γ, then we say Γ is of *product type*. If not, then we say Γ is of *twisted type*.

Example 4.2. The following are examples of linear realizations of total spaces of GKM fibrations of twisted type over a 5- respectively 6-gon.
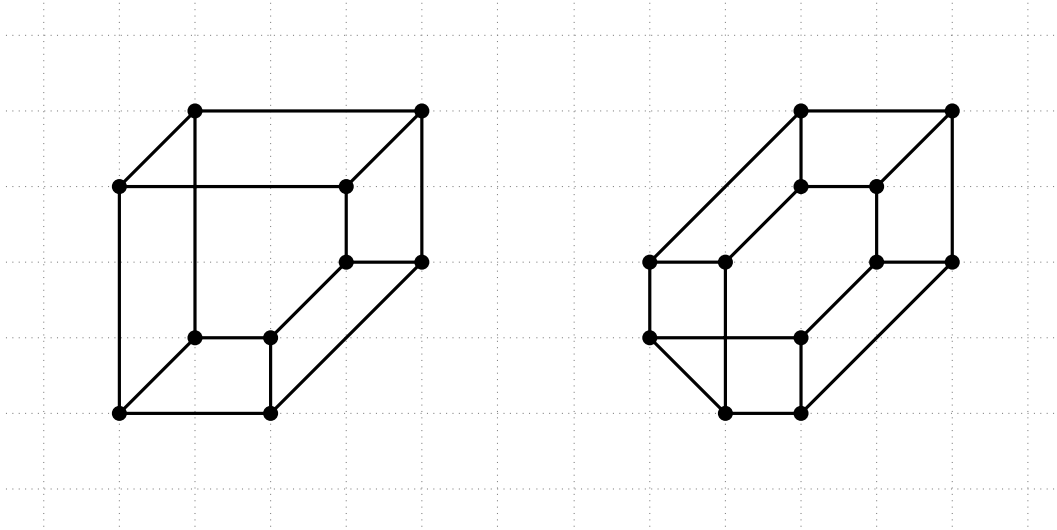
It is not hard to see that the underlying graph of Γ is determined up to isomorphism by whether it is of product or of twisted type: it either looks like a Möbius band or the product of a circle with an interval. However, if we add the additional structure of the labels to the picture, the situation becomes more interesting.

LEMMA 4.3. *If $\pi\colon (\Gamma, \alpha) \to (B, \alpha_B)$ admits a compatible structure of a fiberwise signed fibration, then there are precisely two possible choices for the lift $\tilde{\alpha}\colon F \to \mathbb{Z}^2$ of α on vertical edges. Both choices are equivalent as fiberwise signed fibrations.*

Proof. Suppose we have two lifts $\tilde{\alpha}, \tilde{\alpha}'\colon F \to \mathbb{Z}^2$ of α where $F \subset E(\Gamma)$ are the vertical edges. Let $e \in E(\Gamma)$ be a horizontal edge. If e' and e'' are the unique vertical edges emanating from $i(e)$ and $t(e)$, then a compatible connection ∇ necessarily satisfies $\nabla_e(e') = e''$. Thus, $\tilde{\alpha}(e') \equiv \tilde{\alpha}(e'')$ mod $\alpha(e)$ and $\tilde{\alpha}'(e') \equiv \tilde{\alpha}'(e'') \mod \alpha(e)$. It follows that $\tilde{\alpha}$ and $\tilde{\alpha}'$ either agree or disagree on both, e' and e''. Inductively, this extends to all vertical edges. Conversely if $\tilde{\alpha}$ defines a fiberwise signed structure, then $-\tilde{\alpha}$ clearly does as well. An equivalence of $(\pi, \tilde{\alpha})$ and $(\pi, -\tilde{\alpha})$ is given by the isomorphism that interchanges the vertices of each vertical edge. $\qquad\square$

It follows from the lemma above that two fiberwise signed fibrations are equivalent as such if and only if they are equivalent as unsigned GKM fibrations. Thus, equivalence classes of fiberwise signed fibrations naturally form a subset of equivalence classes of (unsigned) GKM fibrations.

Remark 4.4. There is an involution on the set of equivalence classes of GKM fibrations: given $\Gamma \to B$, choose two horizontal edges covering the same edge in the base graph. In one of the fibers we detach the edges from their vertices and reglue them but with the fiber vertices interchanged. The labels of the edges stay the same. This construction is, of course, self-inverse. It maps GKM fibrations which admit the structure of a fiberwise signed fibration to GKM fibrations which do not carry such a structure and vice versa. A classification of fiberwise signed fibrations extends to a classification of all GKM fibrations.

PROPOSITION 4.5. *Let B be an effective 2-valent GKM graph with n vertices. Then there is a bijective correspondence*

$$\textit{fiberwise signed 3-valent GKM fibrations over } B/\sim \quad \longleftrightarrow \quad ((\mathbb{Z} - 0)^n/\pm) \times \{0, 1\},$$

where $\sim$ denotes equivalence of fibrations. For a fixed signed structure on B this induces a bijection

$$\text{signed 3-valent GKM fibrations over } B/\sim \quad \longleftrightarrow \quad ((\mathbb{Z}-0)^n/\pm) \times \{0,1\}.$$

Remark 4.6. The above correspondence is not canonical and depends on a fixed choice of data in the GKM graph B which we state here separately for later reference. Let $v_1, \ldots, v_n$ be the vertices of B and $e_1, \ldots, e_n$ its edges, where e_i goes from v_i to v_{i+1}. We extend the notation for all $i \in \mathbb{Z}$ by setting $v_{i+n} := v_i$ and $e_{i+n} := e_i$.

If B comes with a signed structure, then we have unique signs for the weights γ_i associated to the e_i and we use these to define the correspondence. Without the signed structure there are choices to make: let $\gamma_1, \gamma_2 \in \mathbb{Z}^2$ be the weights associated to e_1 and e_2 in B, where we choose the signs arbitrarily. Now we choose representatives for the weights $\gamma_i \in \mathbb{Z}^2$ of all e_i with the unique sign such that

$$\gamma_i \equiv -\gamma_{i+2} \mod \gamma_{i+1}.$$

This is possible thanks to the existence of a compatible connection. Again, it turns out handy to extend the notation for all $i \in \mathbb{Z}$ such that γ_i and γ_{i+n} correspond to the same edge and thus agree up to sign. These will occasionally play a role and will be denoted through the equation $\gamma_i = (-1)^{\varepsilon_i}\gamma_{i+n}$. Note that the value of ε_i only depends on whether i is even or odd. The ε_i vanish if and only if the γ_i come from a signed compatible structure.

We will define the correspondence explicitly in the course of the proof; briefly, we map

$$(\pi : \Gamma \to B) \longmapsto ([k_1, \ldots, k_n], \eta),$$

where $\eta \in \{0,1\}$ describes if π is of product or of twisted type, and the numbers k_i are determined by the equation

$$\alpha_i = k_i\gamma_{i-1} - k_{i-1}\gamma_i$$

(see (1)), where the α_i are the weights in the fiber over v_i chosen with consistent orientations.

Proof of Proposition 4.5. Note first that the statement on signed fibrations follows directly from the statement on fiberwise signed fibrations with the help of Lemma 3.5.

We begin by associating an element on the right-hand side to a fiberwise signed GKM fibration $\Gamma \to B$. The $\{0,1\}$ component is determined by the graph structure of Γ: we set it to be 0 if Γ is of product type and 1 if it is of twisted type. In order to define the $(\mathbb{Z}-0)^n/\pm$ component we associate weights γ_i with fixed signs to the edges of B as described in Remark 4.6 and consider the induced assignment of the γ_i to the horizontal edges of Γ. Now choose an orientation of the edge in the fiber over v_1 and let $\alpha_1 \in \mathbb{Z}^2$ be the associated weight (with unique sign). A compatible connection allows us to inductively choose orientations for the vertical edges over v_i in a compatible way such that the associated weights satisfy

$$\alpha_i \equiv \alpha_{i+1} \mod \gamma_i,$$

for $i \in \mathbb{Z}$. Note, however, that if Γ is of twisted type, then transporting a vertical edge around Γ once reverses its orientation and thus the orientation used for the definition of α_i might differ from that of α_{i+n}. We have $\alpha_i = (-1)^\eta \alpha_{i+n}$.

By assumption, the weights of two adjacent edges in B form a basis of $\mathbb{Z}^2$. Thus, for $i \in \mathbb{Z}$, there are unique integers k_i, l_i such that $\alpha_i = k_i\gamma_{i-1} + l_i\gamma_i$. We claim that $k_i = -l_{i+1}$. Transporting the vertical edges along the horizontal ones we find integers d_i, d'_i, $i = 1, \ldots, n$, such that $\alpha_{i+1} = \alpha_i + d_i\gamma_i$ and $\gamma_{i+1} = -\gamma_{i-1} + d'_i\gamma_i$. We obtain $\alpha_i = (k_{i+1} - d_i + l_{i+1}d'_i)\gamma_i - l_{i+1}\gamma_{i-1}$.

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Uniqueness of the k_i and l_i yields the claim. In particular, we have

$$\alpha_i = k_i \gamma_{i-1} - k_{i-1} \gamma_i, \tag{1}$$

for $i \in \mathbb{Z}$. The $(\mathbb{Z} - 0)^n / \pm$ component on the right-hand side of the correspondence is now defined by the equivalence class of $(k_1, \ldots, k_n)$. Recall that in the construction of the k_i we made a choice for the orientation of the edge over v_1 giving rise to α_1. A different choice would lead to a global sign change for the k_i so we obtain a well-defined element of $(\mathbb{Z} - 0)^n / \pm$. This association is easily seen to be invariant under equivalences of GKM fibrations.

Conversely, we check that the construction can be reversed. Given an element on the right-hand side of the correspondence, choose a representative $(k_1, \ldots, k_n, \eta) \in (\mathbb{Z} - 0)^n \times \{0, 1\}$. If $\eta = 1$ let Γ be the unique 3-regular abstract graph of twisted type that fibers over B. Otherwise, let Γ be the unique such graph of product type. We need to construct labels for the edges of Γ that turn the graph fibration $\pi : \Gamma \to B$ into a GKM fibration. To do this set $k_0 = (-1)^{\varepsilon_1 + \eta} k_n$, where ε_1 is defined as in Remark 4.6. Then we assign labels to Γ as follows: the horizontal edges over e_i are labelled by γ_i. For the vertical edges, consider a lift of a path that goes around the n-gon B once and let p_i be the vertex in that path which lies over v_i. To the directed vertical edge emanating from p_i we associate the weight

$$\alpha_i = k_i \gamma_{i-1} - k_{i-1} \gamma_i,$$

for $i = 1, \ldots, n$. Let ∇^B be the unique connection on B and define a compatible connection ∇ on Γ as follows: if $e \in E(\Gamma)$ is vertical, then ∇_e is defined as

$$H_{i(e)} \xrightarrow{\pi} E(B)_{\pi(i(e))} = E(B)_{\pi(t(e))} \xrightarrow{\pi^{-1}} H_{t(e)}.$$

Transport along horizontal edges is uniquely defined by the condition that it respects vertical and horizontal edges. The connections ∇ and ∇^B are easily seen to be compatible with the labels, the most interesting step being to verify that for a lift $\tilde{e}_n$ of e_n the connection $\nabla_{\tilde{e}_n}$ satisfies the congruence relations for the labels. To do this recall that the orientation of an edge, when transported along the lift of a path around B, gets reversed if and only if Γ is of twisted type. Thus, we need to have $\alpha_n \equiv (-1)^\eta \alpha_1 \mod \gamma_n$. The left-hand side, however, is given by

$$\alpha_n = k_n \gamma_{n-1} - k_{n-1} \gamma_n \equiv -k_n \gamma_{n+1} \equiv (-1)^{\eta+1} k_0 \gamma_1 \equiv (-1)^\eta \alpha_1 \mod \gamma_n.$$

The connections are also clearly compatible with the fibration.

The equivalence class does not depend on the chosen lift of the path around B as the other lift will result in the same labels but with a global sign change for the α_i. By Lemma 4.3, these two fibrations are equivalent. The same is accomplished by a global sign change of the k_i so the construction factors through $((\mathbb{Z} - 0)^n / \pm) \times \{0, 1\}$. $\qquad\square$

Example 4.7. Consider the T^2-equivariant $\mathbb{CP}^1$-fibration $\mathrm{SU}(3)/T^2 \to \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)) = \mathbb{CP}^2$. Its GKM fibration is as follows, see [GSZ12, § 2.1]:
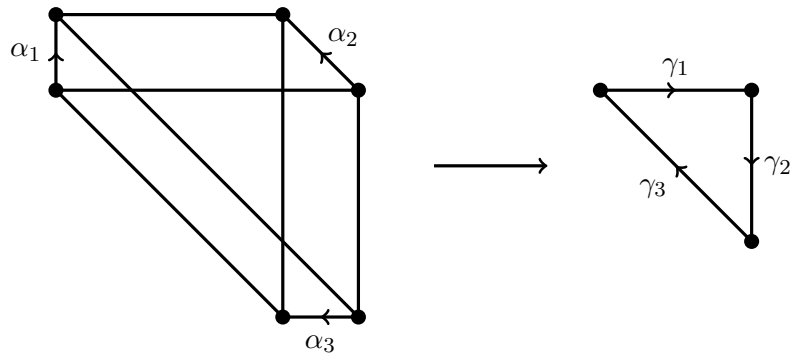
With the indicated fiber orientation, we have $\alpha_1 = \gamma_3 + \gamma_1$, $\alpha_2 = -\gamma_1 - \gamma_2$, and $\alpha_3 = \gamma_2 + \gamma_3$. Thus, this fibration corresponds to $([k_1, k_2, k_3], \eta) = ([1, -1, 1], 1)$.

Example 4.8. Tolman's example [Tol98], Woodward's variant [Woo98], and Eschenburg's twisted flag manifold $\mathrm{SU}(3)//T^2$ (see [Esc84, Esc92, GKZ20]), which are (non-equivariantly) diffeomorphic by [GKZ19], fiber equivariantly over $\mathbb{CP}^2$. Their associated GKM fibrations are as follows.
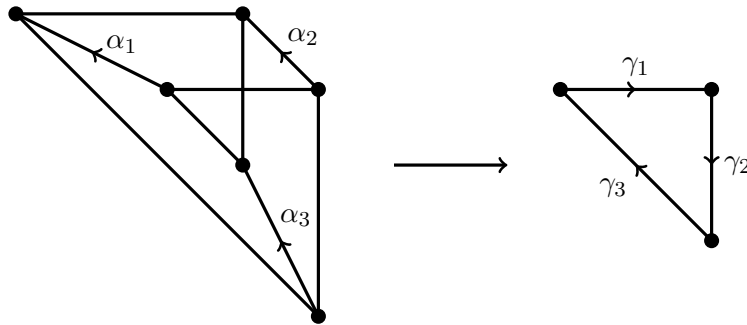
Here we have $\alpha_1 = \gamma_3 - \gamma_1$, $\alpha_2 = -\gamma_1 - \gamma_2$, and $\alpha_3 = -\gamma_2 + \gamma_3$. Thus, this fibration corresponds to $([k_1, k_2, k_3], \eta) = ([1, -1, -1], 1)$.

DEFINITION 4.9. We call a vertex of a 3-valent signed GKM graph an *interior vertex* if the cone spanned by the labels of the three edges emanating from it is equal to $\mathbb{R}^2$, otherwise it is an *exterior vertex*.

   This notation is motivated by the fact that if we are given a Hamiltonian T^2-action with this GKM graph, the momentum image of a fixed point is in the interior of the momentum image if and only if the corresponding vertex of the GKM graph is interior.

PROPOSITION 4.10. *Let B be a signed 2-valent GKM graph and $\Gamma \to B$ the signed GKM fibration associated to $([k_1, \ldots, k_n], \eta) \in ((\mathbb{Z} - 0)^n / \pm) \times \{0, 1\}$ as in Proposition 4.5. Then for $i = 2, \ldots, n$ the fiber over v_i contains exactly one interior vertex of Γ if and only if k_{i-1} and k_i have the same sign. Otherwise, both vertices in the fiber are exterior. The fiber over v_1 contains exactly one interior vertex if and only if k_n and $(-1)^\eta k_1$ have the same sign. Otherwise, both vertices are exterior.*

Proof. Let p_i and q_i be the vertices in the fiber over v_i for $i \in \{1, \ldots, n\}$. Assume that γ_i has been chosen as the weight of the directed edge from v_i to v_{i+1}. Then without loss of generality

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the set of weights of the edges emanating from p_i and q_i are

$$\{-\gamma_{i-1}, \ \gamma_i, \ k_i\gamma_{i-1} - k_{i-1}\gamma_i\} \quad \text{and} \quad \{-\gamma_{i-1}, \ \gamma_i, \ -k_i\gamma_{i-1} + k_{i-1}\gamma_i\}.$$

In general, if $e_1, e_2 \in \mathbb{Z}^2$ is a basis, then the cone spanned by e_1, e_2, and $ae_1 + be_2$ is $\mathbb{R}^2$ if and only if $a, b < 0$. Thus, if k_{i-1} and k_i have the same sign, then exactly one of the set of weights of p_i and q_i has this property. For the statement on v_1 recall that in the construction of the fibration we had $k_0 = (-1)^{\eta+\varepsilon_1}k_n$. Since B is signed, it follows that $\varepsilon_1 = 0$, hence k_0 and k_1 have the same sign if and only if k_n and $(-1)^\eta k_1$ do. $\qquad\square$

COROLLARY 4.11. *Let $\Gamma \to B$ be a fibration of signed GKM graphs of twisted type as above, where the 2-valent GKM graph B is effective and has n vertices. If n is odd, then the number of interior vertices of Γ is an even number between 0 and $n-1$. If n is even, then it is an odd number between 1 and $n-1$.*

5. Realization of GKM fibrations

The following is the main theorem of this paper. In what follows, $T = T^2$ is a 2-dimensional compact torus with Lie algebra $\mathfrak{t}$. Prior results on abstract GKM graphs are translated to graphs with labels in $\mathbb{Z}_{\mathfrak{t}}^*$ as explained in Remark 2.4. In the following theorem, a fiber bundle of almost complex manifolds $f\colon (M, J_M) \to (N, J_N)$ is a fiber bundle such that $df\colon TM \to TN$ commutes with the almost complex structures. We say a fiber bundle $f\colon (M, \omega_M) \to (N, \omega_N)$ is compatible with symplectic structures if ω_M restricts to a symplectic form on every fiber and furthermore the restriction of ω_M to the fiberwise symplectic complement C is homotopic to $f^*\omega_N|_C$ through symplectic (i.e. non-degenerate) forms on C.

THEOREM 5.1.

(i) *Let $\pi\colon \Gamma \to B$ be a fiberwise signed GKM fibration, where Γ is a 3-valent abstract GKM graph and B an effective 2-valent abstract GKM graph with n vertices (both with respect to $\mathbb{Z}_{\mathfrak{t}}^*$). Then π is geometrically realized as the projectivization $\mathbb{P}(E)$ of a T-equivariant complex vector bundle $E \to X$ over a 4-dimensional T-manifold X which can be taken to be S^4 if $n = 2$ and quasitoric if $n \geq 3$. Furthermore, $\mathbb{P}(E)$ and X have T-invariant stable almost complex structures compatible with the fibration. The realization $\mathbb{P}(E)$ has the property that all its isotropy groups are connected if and only if, in the notation of Proposition 4.5 and Remark 4.6, the fiberwise signed GKM fibration π corresponds to $([k_1, \ldots, k_n], \eta)$, with all $k_i = \pm 1$.*

(ii) *If $\pi\colon \Gamma \to B$ is a fibration of signed GKM graphs, then its geometrical realization as in part (i) can be chosen to be a fiber bundle of almost complex manifolds such that the induced fibration of signed GKM graphs is precisely π.*

(iii) *If, in addition, B is the boundary of a two-dimensional Delzant polytope (i.e. X can be chosen as a 4-dimensional toric manifold), then any realization $\mathbb{P}(E)$ as in part (i) admits a T-invariant symplectic structure such that the action is Hamiltonian and $\mathbb{P}(E) \to X$ is compatible with the symplectic structure on $\mathbb{P}(E)$ and a T-invariant Kähler structure on X. Moreover the symplectic form on $\mathbb{P}(E)$ is symplectomorphic to a Kähler form with a compatible complex structure for which $\mathbb{P}(E) \to X$ is holomorphic.*

Remark 5.2. In part (iii) of the above theorem note that the T-invariant symplectic form on $\mathbb{P}(E)$ does admit a compatible complex structure since it is symplectomorphic to a Kähler form. However, this complex structure will not necessarily be compatible with the T-action since the Kähler form and the symplectomorphism are not. In fact, we will show in §7 that in the case

of a twisted-type fibration with the maximal number of interior fixed points, the compatible complex structure can never be T-invariant. In contrast, we will show in §6.3 that for product-type fibrations we always obtain a T-invariant Kähler structure on $\mathbb{P}(E)$. The example of the standard flag manifold $U(3)/T^3$ shows that such structures can also exist in the twisted type case. However, it is not clear whether they always exist outside of the case with maximal number of interior fixed points.

We will prove the theorem in this section and the following. In §5 we construct the vector bundle E, see Theorem 5.7. The statements on the geometric structures on $\mathbb{P}(E)$ are proved in §6.

5.1 Realization in dimension 4

As a starting point, we need to geometrically realize the base graph of the fibration which corresponds to a 2-dimensional torus action on a 4-manifold. Actions of tori of dimension half the dimension of the manifold are quite well studied so we can draw on the existing theory of quasitoric manifolds. Recall that a *quasitoric manifold* is a smooth compact $2n$-dimensional manifold X with a locally standard smooth action of the n-dimensional torus T^n which is such that its orbit space X/T^n is homeomorphic, as a manifold with corners, to a simple convex polytope P. Via the homeomorphism $X/T^n \cong P$, the facets of P correspond to the so-called *characteristic submanifolds* of X which are the preimages of the facets under the projection. The *characteristic function* of a quasitoric manifold X is the assignment sending a facet of $P \cong X/T^n$ to the isotropy circle of the corresponding characteristic submanifold.

Let us describe the inverse construction to this [DJ91], see also [BP15, §7.3]: we start with a *characteristic pair* (P, λ) consisting of a simple convex polytope P with facets $F_1, \ldots, F_m$, together with a map λ sending a facet F_j to a subcircle $\lambda(F_j) \subset T^n$, with the property that the multiplication map $\lambda(F_{j_1}) \times \cdots \times \lambda(F_{j_k}) \to T^n$ is injective whenever the intersection $F_{j_1} \cap \cdots \cap F_{j_k}$ is non-empty. We extend λ to all faces of P, i.e. we denote by $\lambda(F)$ the subtorus of T^n generated by all $\lambda(F_i)$, where F_i is a facet containing F. Moreover, we denote by $F(x)$, for $x \in P$, the smallest face of P containing x. In this situation, we define $X := P \times T^n / \sim$, where the equivalence relation $\sim$ is generated by $(x, s) \sim (x, t) \Leftrightarrow s^{-1}t \in \lambda(F(x))$. Then X, equipped with the T^n-action defined as the multiplication on the second factor of $P \times T^n$, is a quasitoric manifold with orbit space P and characteristic function λ; it is called the *canonical model of* (P, λ).

Furthermore, we recall that an *omniorientation* of a quasitoric manifold is an orientation of X and a choice of orientations of its characteristic submanifolds. Note that given an orientation of X, an orientation of a characteristic submanifold is equivalent to an orientation of the normal bundle. For a fixed point $p \in X^{T^n}$ we have a decomposition $T_pX = V_1 \oplus \cdots \oplus V_n$ where the V_i are the irreducible 2-dimensional subrepresentations and, in fact, normal spaces to the characteristic submanifolds containing p. An omniorientation thus induces an orientation on all the V_i and, hence, on T_pX. The *sign* of the fixed point is said to be 1 if this orientation agrees with that coming from the orientation on X and -1 if it does not. A *positive omniorientation* is an omniorientation such that the signs of all fixed points are positive. Now any omniorientation induces a complex structure on the tangent spaces of the fixed points as explained below and Kustarev proved in [Kus09] that this extends to a T^n-invariant almost complex structure on X if and only if the omniorientation is positive (see also [BP15, Theorem 7.3.24]).

To define the complex structure on $T_pX = V_1 \oplus \cdots \oplus V_n$ induced by an omniorientation recall that V_i is normal to a unique characteristic submanifold containing p which corresponds to some

facet F_i. Let $\lambda(F_i) \subset T^n$ denote the kernel of the action on the characteristic submanifold belonging to F_i. Then $\lambda(F_i)$ acts effectively on V_i. There is now a unique way to identify $\lambda(F_i) \cong S^1$ such that there is an orientation-preserving S^1-equivariant isomorphism $V_i \cong \mathbb{C}$ to the standard S^1-representation on $\mathbb{C}$ with the orientation given by the standard complex structure. This complex structure on $\mathbb{C}$ then induces a T^n-invariant complex structure on all the V_i and, hence, on $T_p X$.

PROPOSITION 5.3. *Let B be an effective 2-regular GKM graph with n vertices. Then B is the GKM graph of a T-GKM action on a 4-dimensional manifold X.*

  (i) *If $n = 2$, then X can be taken to be S^4. For $n \geq 3$, we can choose X as a quasitoric manifold.*
 (ii) *If B has a compatible structure of a signed GKM graph, then X as in part (i) can be chosen such that it carries a T-invariant almost complex structure which induces this signed structure.*
(iii) *If B is of polytope type, then X can be chosen to be a toric manifold.*

Proof. Part (iii) is Delzant's theorem. In part (i), the statement holds for $n = 2$ since any such graph can be realized by a T-action on $S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}$ which acts on the $\mathbb{C}^2$ factor as a pullback of the standard representation along some automorphism of T; for $n \geq 3$, we translate the canonical model of 4-dimensional quasitoric manifolds, which we recalled above in arbitrary dimension, to GKM graphs. As B is an n-gon with $n \geq 3$, the underlying graph is realized as the boundary of a convex polytope $P \subset \mathbb{R}^2$. For an edge e of P we set $\lambda(e) := \ker \alpha \subset T$, where α is the weight (up to sign) associated to e in B, interpreted as a homomorphism $T \to S^1$. Then (P, λ) is a characteristic pair, and we consider the corresponding canonical model, which is a 4-dimensional quasitoric manifold X. By construction, the GKM graph of X is B.

Regarding part (ii), the existence of invariant almost complex structures on quasitoric manifolds is already well understood, as explained above the statement of the proposition, and it only remains to draw the connection to the signed GKM graph. Assume B admits a signed structure. Then necessarily $n \geq 3$, and by part (i) we may choose X as a quasitoric 4-manifold. In view of the discussion above, it suffices to find a positive omniorientation of X such that the induced complex structures on the tangent spaces of all fixed points yield the weights prescribed by the signed structure.

As a first step we show that the signed structure gives rise to canonical parametrizations of the isotropy circles of all characteristic submanifolds. Consider $p \in X^T$, let e, e' be the edges emanating from p, and let $\alpha(e), \alpha(e') \in \mathbb{Z}_{\mathfrak{t}}^*$ be the corresponding weights. Let $\alpha(e)^*, \alpha(e')^* \in \mathbb{Z}_{\mathfrak{t}}$ be the associated dual basis. Note that a facet of P is just an edge and these correspond to the edges of the GKM graph. Let S_e denote the two-sphere corresponding to e, i.e. S_e is the characteristic submanifold belonging to e. Now the kernel $T_e \subset T$ of the action on S_e is a circle and its Lie algebra is generated by $\alpha(e')^*$. The element $\alpha(e')^*$, hence, fixes an identification $T_e \cong S^1$. We claim that this is well defined for the characteristic submanifold S_e and does not depend on the initial choice of fixed point p in the characteristic submanifold. Indeed, let q be the terminal vertex of e and let $\bar{e}, e''$ be the emanating edges at q. As before, we have weights $\alpha(\bar{e}), \alpha(e'') \in \mathbb{Z}_{\mathfrak{t}}^*$ and a dual basis $\alpha(\bar{e})^*, \alpha(e'')^* \in \mathbb{Z}_{\mathfrak{t}}$. The compatible connection implies $\alpha(e'') \equiv \alpha(e') \mod \alpha(e)$ and, thus, $\alpha(e) = -\alpha(\bar{e})$ implies $\alpha(e')^* = \alpha(e'')^*$. Hence, the latter is indeed a canonical generator of the Lie algebra of T_e arising from the signed structure and it will be denoted by $\lambda(e) \in \mathbb{Z}_{\mathfrak{t}}$.

We now fix an orientation of the polytope P. Together with this orientation the datum of the $\lambda(e)$ for all characteristic submanifolds e is called a combinatorial quasitoric pair in [BP15]. It carries the information of the characteristic pair as well as orientations for the polytope and

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all isotropy circles. It is shown in [BP15, Proposition 7.3.11] that there is an associated omniorientation on the canonical model for which, by [BP15, Proposition 7.3.18], the weights at a fixed point p, with signs given by the complex structure on T_pM associated to the omniorientation, is precisely the dual basis of $\lambda(e), \lambda(e')$ where e, e' are the edges adjacent to p. By construction, this dual basis is precisely the weights $\alpha(e), \alpha(e')$ provided by the signed structure. Hence, there is an omniorientation inducing the weights prescribed by the signed graph.

It remains to show that the omniorientation is positive. Let p, q, e, e', e'' be as above. By [BP15, Lemma 7.3.21] as well as the subsequent remark, we may compute the sign at the fixed point p as follows: let e, e' be the order of the edges coming out of p such that the orientation provided by vectors along e, e' agrees with the fixed orientation of P. Then the sign at p is $\det(\alpha(e), \alpha(e'))$ where the determinant is understood via the identification $\mathbb{Z}_{\mathfrak{t}}^* \cong \mathbb{Z}^2$ using the dual of the standard basis. If the signs of all fixed points are negative, then by reversing all orientations in the omniorientation, we obtain a positive omniorientation which produces the same complex structure in the tangent spaces of the fixed points. Hence, it suffices to show that the above determinants agree for all fixed points. The orientation on P provided by e, e' agrees with that of $e'', \overline{e}$. We compute

$$\det(\alpha(e''), \alpha(\overline{e})) = -\det(\alpha(e''), \alpha(e)) = \det(\alpha(e), \alpha(e'')) = \det(\alpha(e), \alpha(e')),$$

where the last equation is again due to the congruence $\alpha(e'') \equiv \alpha(e') \mod \alpha(e)$ provided by the compatible connection. $\qquad\square$

5.2 A vector bundle over the one-skeleton

Having realized the 2-regular graph B by a 4-dimensional manifold X, it is now our intermediate goal to construct an equivariant complex vector bundle over X_1 such that the set of 0- and 1-dimensional orbits in its projectivization is precisely the T^2-space encoded in the graph Γ. In this section, we will use the theory of cohomogeneity one actions [Mos57, AA93], see also [GZ18], i.e. actions of compact Lie groups on closed manifolds whose principal orbits have codimension one. Let us briefly recall the relevant ingredients from this theory. A *group diagram* is a quadruple (G, K^+, K^-, H) of compact Lie groups $H \subset K^\pm \subset G$ such that K^+/H and K^-/H are spheres. Given a cohomogeneity one G-action on M with orbit space homeomorphic to the closed interval $[0, 1]$, we can (non-canonically) associate a group diagram to it in the following way: we choose an auxiliary G-invariant Riemannian metric, as well as a geodesic $\gamma : [0, 1] \to M$ orthogonal to the G-orbits, which meets every G-orbit exactly once. Then $G \cdot \gamma(0)$ and $G \cdot \gamma(1)$ are the unique non-regular orbits of the action, and their stabilizers $K^+ = G_{\gamma(1)}$ and $K^- = G_{\gamma(0)}$, together with the pointwise stabilizer H of the geodesic γ yields the group diagram (G, K^+, K^-, H).

Conversely, a group diagram (G, K^+, K^-, H) yields a cohomogeneity one action, as we now explain. By [Bre72, Theorem 10.1, Chapter II], any transitive Lie group action on a sphere is linear; in particular, this is the case for the K^+- and K^--actions on K^+/H and K^-/H. Hence, we find orthogonal $K^\pm$-representations on Euclidean vector spaces $V^\pm$ with principal isotropy group H. Denoting the unit disks in $V^\pm$ by $D^\pm$, we obtain a cohomogeneity one manifold by gluing the disk bundles $G \times_{K^\pm} D^\pm$ along their boundaries $G \times_{K^+} K^+/H = G/H = G \times_{K^-} K^-/H$. As emphasized above, the group diagram associated to a cohomogeneity one manifold is not unique, as it depends on a certain choice. However, two cohomogeneity one manifolds with the same group diagram are equivariantly diffeomorphic.

We make use of the notation from Remark 4.6. We also return to the notation from the proof of Proposition 4.5 which we shall recall here: choose an orientation for the edge in the fiber over v_1 with associated weight α_1. Using a compatible connection, this inductively defines

orientations on the edges in the fibers over all the v_i such that $\alpha_i \equiv \alpha_{i+1} \mod \gamma_i$. We extend this notation with this property to all $i \in \mathbb{Z}$. Transporting an edge around a lift of the path which goes around the base graph once reverses its orientation if Γ is of twisted type and preserves the orientation if it is of product type. Hence, we have $\alpha_i = (-1)^\eta \alpha_{i+n}$, where $\eta = 1$ if Γ is of twisted type and $\eta = 0$ if it is of product type. Furthermore, let $k_i \in \mathbb{Z}$ be as in Remark 4.6, i.e. the numbers uniquely determined by the equation

$$\alpha_i = k_i \gamma_{i-1} - k_{i-1} \gamma_i.$$

We now come to the construction of the vector bundle: for each invariant 2-sphere in X corresponding to some edge e_i $(i = 1, \ldots, n)$ we want to construct a T-equivariant U(2)-principal bundle $P_i \to S^2$ such that the projectivization of the associated $\mathbb{C}^2$-bundle $P_i \times_{\mathrm{U}(2)} \mathbb{C}^2 \to S^2$ with respect to the standard representation of U(2) on $\mathbb{C}^2$ has as GKM graph exactly $\pi^{-1}(e_i)$, as depicted in Lemma 5.4 below. We will construct P_i as a $(T \times \mathrm{U}(2))$-cohomogeneity one manifold such that the U(2)-subaction is free. In this situation, we can divide out the U(2)-action and obtain a smooth manifold, to which the T-subaction on P_i descends. This quotient action will again be of cohomogeneity one. Using the observation recalled above that a group diagram of a cohomogeneity one action determines it up to equivariant diffeomorphism, this will allow us to identify the quotient as S^2 with the correct action. In this way, P_i will be seen to be a T-equivariant U(2)-principal bundle. For some arbitrary integers a_i, b_i, c_i (which we will specify below), define the homomorphisms

$$A_i^+ : T \to \mathrm{U}(2), \quad t \mapsto \begin{pmatrix} a_i \gamma_{i-1}(t) - b_i \gamma_i(t) & \\ & (a_i - k_i)\gamma_{i-1}(t) + (k_{i-1} - b_i)\gamma_i(t) \end{pmatrix}$$

and

$$A_i^- : T \to \mathrm{U}(2), \quad t \mapsto \begin{pmatrix} c_i \gamma_i(t) - a_i \gamma_{i+1}(t) & \\ & (c_i - k_{i+1})\gamma_i(t) + (k_i - a_i)\gamma_{i+1}(t) \end{pmatrix},$$

where the entries of the matrices are to be interpreted as the homomorphisms $T \to S^1$ corresponding to the respective elements of $\mathbb{Z}_{\mathfrak{t}}^*$.

Let P_i be the cohomogeneity one manifold defined by the group diagram (G, K_i^+, K_i^-, H_i); here,

$$\begin{aligned} G &= T \times \mathrm{U}(2) \\ K_i^+ &= \{(t, A_i^+(t)) \mid t \in T\}, \\ K_i^- &= \{(t, A_i^-(t)) \mid t \in T\}, \\ H_i &= \{(t, A_i^+(t)) \mid t \in \ker \gamma_i\} = \{(t, A_i^-(t)) \mid t \in \ker \gamma_i\}. \end{aligned}$$

The last equation holds since, by definition, $\gamma_{i-1} = -\gamma_{i+1} \mod \gamma_i$. Consequently H_i is contained both in K_i^+ and in K_i^-. Furthermore, $K_i^\pm / H_i \cong S^1$, so that this really defines a valid group diagram. We fix identifications of the outer (non-principal) orbits with G/K_i^+ and G/K_i^-.

We observe that $\mathrm{U}(2) \cong \{e\} \times \mathrm{U}(2)$ intersects $K_i^\pm$ and all its G-conjugates trivially, so that the U(2)-subaction on P_i is free; dividing out this subaction we obtain a T-manifold of cohomogeneity one. Let us identify the group diagram of this action. To this end, we choose a G-invariant Riemannian metric on P_i as well as a geodesic perpendicular to the orbits, which induce the above group diagram. We introduce on the quotient the Riemannian metric turning the projection map into a Riemannian submersion. Then the geodesic projects onto a geodesic perpendicular to the orbits. The corresponding group diagram is given by the projection of the respective subgroups

to the T-factor, i.e. $(T, T, T, \ker \gamma_i)$. Thus, the quotient manifold is U-equivariantly diffeomorphic to the $S^2 \subset X$ corresponding to the edge e_i.

Associated to $P_i \to S^2$, we obtain an associated T-equivariant $\mathbb{C}^2$-bundle

$$E_i := P_i \times_{\mathrm{U}(2)} \mathbb{C}^2 \longrightarrow S^2,$$

where $\mathrm{U}(2)$ acts on $\mathbb{C}^2$ by the standard representation. Let us compute the T^2-representation on the fibers over the fixed points in S^2 through the identifications
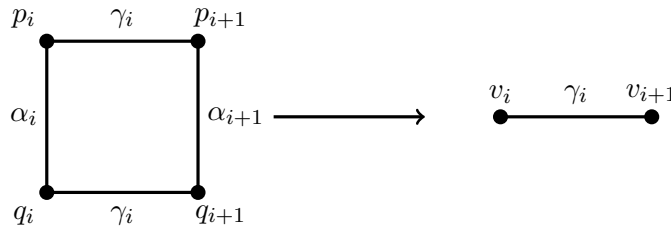
$$\mathbb{C}^2 \cong ((T \times \mathrm{U}(2))/K_i^{\pm}) \times_{\mathrm{U}(2)} \mathbb{C}^2, \quad v \mapsto [(e, I_2)K_i^{\pm}, v].$$

It is given by pulling back the standard $\mathrm{U}(2)$-representation along $A_i^{\pm}$ since

$$t \cdot [(e, I_2)K_i^{\pm}, v] = [(t, I_2)K_i^{\pm}, v] = [(e, (A_i^{\pm}(t))^{-1})K_i^{\pm}, v] = [(e, I_2)K_i^{\pm}, A_i^{\pm}(t)v].$$

Next we pass to the projectivized bundle $\mathbb{P}(E_i) \to S^2$, which is a T-equivariant $\mathbb{CP}^1$-bundle over S^2.

LEMMA 5.4. *The projectivization $\mathbb{P}(E_i) \to S^2$ is a fibration of GKM manifolds. On (unsigned) GKM graphs it is given by*

where (with respect to the identifications above) p_i and p_{i+1} correspond to $[1 : 0]$ while q_i and q_{i+1} correspond to $[0 : 1]$.

Proof. We have a fibration

$$\mathbb{CP}^1 \to \mathbb{P}(E_i) \to S^2$$

with structure group in $\mathrm{U}(2)$. Thus, fixing the standard complex structure on $\mathbb{CP}^1$ and some T^2-invariant complex structure on S^2, we obtain an almost complex structure on $\mathbb{P}(E_i)$ such that $\mathbb{P}(E_i) \to S^2$ respects almost complex structures. We argue via the corresponding signed GKM graph.

The identifications and orientations are such that the weights at q_i are given by $\pm\gamma_i$ (the sign depending on the chosen almost complex structure on S^2) and the upper left entry of A_i^+ minus the lower right entry of A_i^+ (coming from the fiber over v_i). By the definition of A_i^+, the latter difference is precisely $k_i\gamma_{i-1} - k_{i-1}\gamma_i = \alpha_i$. Similarly, one computes the weights at q_{i+1} to be $\pm\gamma_i$ and α_{i+1}. As $\alpha_i = \alpha_{i+1} \mod \gamma_i$, the existence of a connection that is compatible with the signed structure implies that q_i and q_{i+1} are adjacent. Thus, in particular, the unsigned GKM graph has the form as claimed in the lemma. $\square$

Having constructed the squares in Γ over every single edge e_i it remains to glue the E_i in an appropriate and T-equivariant manner. We start by gluing E_i to E_{i+1} for $i = 1, \ldots, n-1$, by identifying the fibers over v_{i+1}. From the side of E_i this is the representation of $\mathbb{C}^2$ defined by the homomorphism A_i^-, while from the side of E_{i+1} it is defined by A_{i+1}^+. Recall that there are unspecified parameters $a_i, b_i, c_i \in \mathbb{Z}$ in the construction. If we choose them such that $a_{i+1} = c_i$ and $b_{i+1} = a_i$, then $A_i^- = A_{i+1}^+$ and we glue the two fibers in the canonical way.

At this point the one-skeleton of the action on the projectivization is a ladder formed by gluing the individual squares from the lemma above in the obvious way:
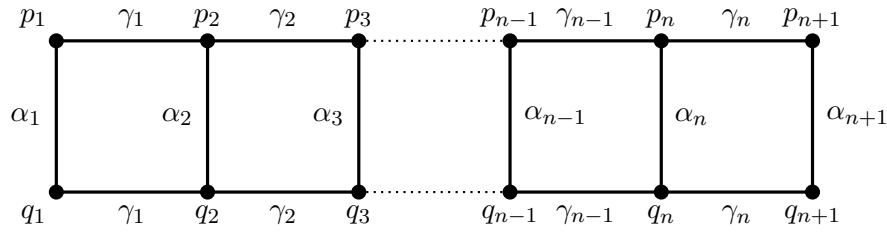
In order to obtain the graph Γ it remains to glue E_{n+1} to E_1. This needs to be done with respect to the graph theoretical structure of Γ. If Γ is of product type, we obtain Γ from the ladder by identifying $p_{n+1} = p_1$ as well as $q_{n+1} = q_1$. In the twisted case we need to identify $p_{n+1} = q_1$ and $q_{n+1} = p_1$.
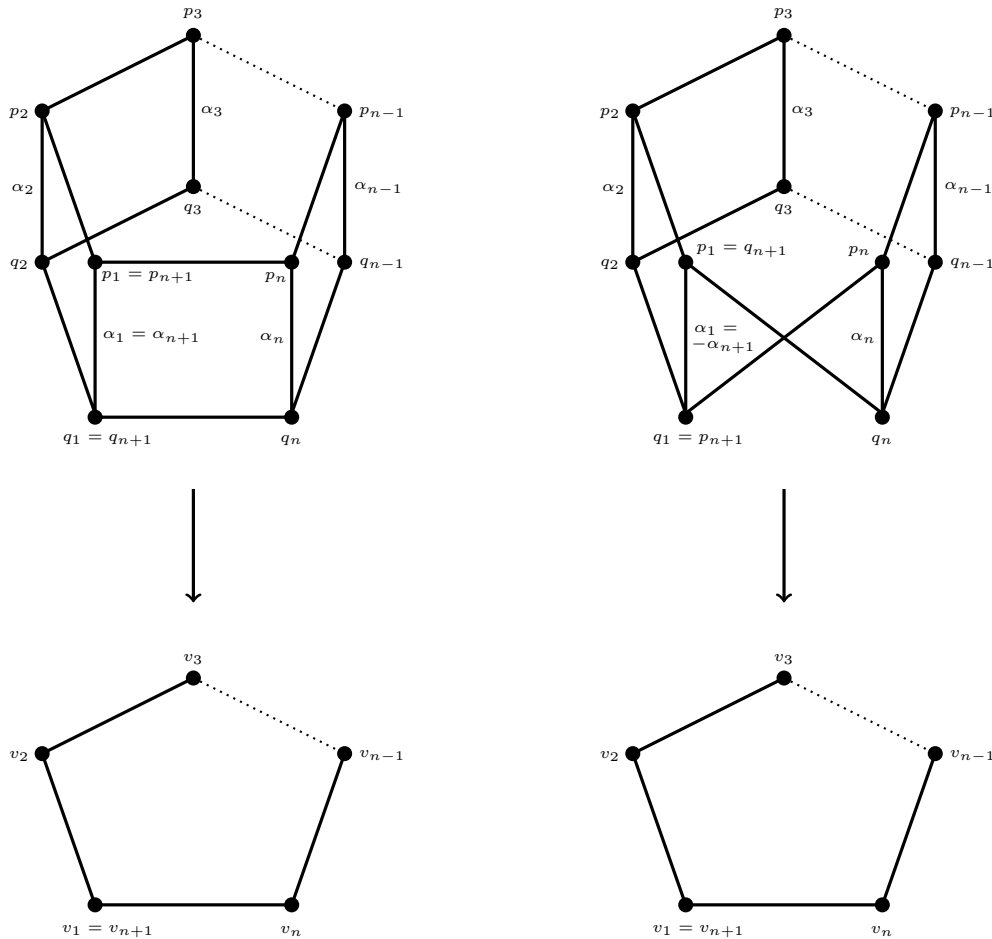
In any case we wish to identify the representations defined by A_{n+1}^- and A_1^+. Recall that $\gamma_0 = (-1)^{\varepsilon_0}\gamma_n$, $\gamma_1 = (-1)^{\varepsilon_1}\gamma_{n+1}$, and $\alpha_1 = (-1)^\eta \alpha_{n+1}$ for $\varepsilon_i, \eta \in \{0, 1\}$. We have

$$k_1\gamma_0 - k_0\gamma_1 = (-1)^\eta(k_{n+1}\gamma_n - k_n\gamma_{n+1}) = (-1)^{\eta+\varepsilon_0}k_{n+1}\gamma_0 - (-1)^{\eta+\varepsilon_1}k_n\gamma_1$$

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which implies $k_0 = (-1)^{\eta+\varepsilon_1} k_n$ and $k_1 = (-1)^{\eta+\varepsilon_0} k_{n+1}$. Consequently the matrices defining A_{n+1}^- and A_1^+ can be written as

$$\begin{pmatrix} (-1)^{\varepsilon_0} c_n \gamma_0 - (-1)^{\varepsilon_1} a_n \gamma_1 & \\ & ((-1)^{\varepsilon_0} c_n - (-1)^{\eta} k_1)\gamma_0 + ((-1)^{\eta} k_0 - (-1)^{\varepsilon_1} a_n)\gamma_1 \end{pmatrix},$$

and

$$\begin{pmatrix} a_1 \gamma_0 - b_1 \gamma_1 & \\ & (a_1 - k_1)\gamma_0 + (k_0 - b_1)\gamma_1 \end{pmatrix}.$$

Thus, in the product case, if we have $a_1 = (-1)^{\varepsilon_0} c_n$ and $b_1 = (-1)^{\varepsilon_1} a_n$, then $A_{n+1}^- = A_1^+$ and the two naturally glue, finishing the construction. Observe that an arbitrary choice of either the a_i, the b_i, or the c_i for $i = 1, \ldots, n$ uniquely defines the respective other coefficients.

In the twisted case we set $c_n = (-1)^{\varepsilon_0}(a_1 - k_1)$ and $a_n = (-1)^{\varepsilon_1}(b_1 - k_0)$. Again, an arbitrary choice of the a_i, the b_i, or the c_i defines the other coefficients uniquely such that this and the previous gluing conditions hold. Now A_{n+1}^- and A_1^+ do not agree but arise from one another by swapping the diagonal entries. It follows that the automorphism of $\mathbb{C}^2$ which swaps both factors is equivariant with respect to the actions defined by A_{n+1}^- and A_1^+. We use this automorphism to glue E_n to E_1 along the fiber over v_1. Since the induced automorphism of $\mathbb{CP}^1$ swaps the fixed points $[0:1]$ and $[1:0]$ it follows that in the graph encoding the one-skeleton of the projectivization, p_{n+1} gets glued to q_1 and q_{n+1} gets glued to p_1 and is, thus, precisely Γ.

5.3 Obstruction theory for quasitoric manifolds

Let X be a quasitoric manifold of dimension $4k$. Denote by $\pi\colon X \to P$ the projection to the associated simple $2k$-polytope P. The preimage under π of the interior of P is diffeomorphic to $T^{2k} \times \mathring{D}^{2k}$ where $\mathring{D}^{2k}$ denotes the interior of the unit closed ball in $\mathbb{R}^{2k}$. Furthermore, this space can be regarded as the interior of the equivariant free top cell of X: recall from the canonical model of a quasitoric manifold (see the beginning of § 5.1) that X is equivariantly homeomorphic to a quotient $T^{2k} \times P/\sim$. Identifying P with D^{2k}, the natural projection $T^{2k} \times D^{2k} \to T^{2k} \times P/\sim$ becomes a characteristic map for a relative CW structure on (X, A), where A is the preimage of the boundary of P under π.

Note that, although it is not quasitoric, we have the same kind of cellular structure for the T^2-action on $X = S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}$ which acts in standard fashion on the $\mathbb{C}^2$ factor. The space X arises from $A = X_1 = \{(v, w, z) \in S^4 \mid v = 0 \text{ or } w = 0\}$ by attaching a single free cell. To see this consider the space $D = \{(x, y, z) \in S^2 \mid x, y \geq 0\}$, which is a 2-disk. The map $D \times T^2 \to S^4$, defined as the equivariant extension of the map $D \times \{e\} \cong D \to S^4$ induced by the inclusion $\mathbb{R} \to \mathbb{C}$ on the first two components, is a characteristic map for the relative CW structure. Of course, this CW decomposition induces an analogous decomposition for any pullback of the above action along a group automorphism.

LEMMA 5.5. *Suppose (X, A) is the equivariant relative T^{2k}-CW complex defined above and $E \to A$ is an equivariant complex vector bundle of rank r with $r > k - 1$. Then E can be extended to an equivariant vector bundle over X.*

Proof. We will use equivariant obstruction theory to prove this lemma, cf. [tDi87, Chapter 2, § 3]. Since $E \to A$ is a T^{2k}-equivariant complex vector bundle, we have a map $\varepsilon\colon A \to B(T^{2k}, U(r))$, where $B(T^{2k}, U(r))$ is the classifying space of T^{2k}-equivariant $U(r)$-principal bundles, see [tDi69, § 3.1]. We would like to extend ε over the top cell $e_{2k}^{T^{2k}}$ of X.

Now ε has an equivariant extension to $X = A \cup_\varphi e_{2k}^T$ if the composition of maps

$$S^{2k-1} \longrightarrow T^{2k} \times S^{2k-1} \xrightarrow{\varphi} A \xrightarrow{\varepsilon} B(T^{2k}, U(r))$$

is nullhomotopic, see [tDi87, p. 115]. The composition gives an element of $\pi_{2k-1}(B(T^{2k}, U(r)))$, but the classifying space $B(T^{2k}, U(r))$ is homotopy equivalent to $BU(r)$ (see [tDi69, p. 142]) and since $r > k - 1$ we have by Bott periodicity $\pi_{2k-1}(B(T^{2k}, U(r))) = 0$. This implies that ε can be extended equivariantly to a map $\varepsilon \colon X \to B(T^{2k}, U(r))$. $\square$

Remark 5.6. As X is smooth, the complex bundle $E \to X$ in the previous lemma can be constructed in a smooth fashion as well. To see this, note that E admits an equivariant, fiberwise injective and linear map $f \colon E \to W$ to some finite-dimensional complex T^{2k}-representation (as in the proof of [tDi87, Chapter I, Proposition 9.7]). Denote by $G_r(W)$ the Grassmannian of complex r-planes (where r is the rank of E) and by $E_r \to G_r(W)$ the tautological bundle. The action on W extends to an action on $G_r(W)$ and E_r, making the latter an equivariant vector bundle. The map f induces a bundle map $g \colon E \to E_r$ which maps each fiber E_x isomorphically to the fiber over $f(E_x) \in G_r(W)$ via f. Thus, the underlying equivariant map $\bar{g} \colon X \to G_r(W)$ has the property that $\bar{g}^*(E_r) \cong E$. By [Was69, Corollary 1.12] we may equivariantly homotope $\bar{g}$ to a smooth map. The pullback along the latter is naturally smooth and still equivariantly equivalent to E (see [tDi87, Theorem 8.15]).

Applying Lemma 5.5 and Remark 5.6 to the vector bundle constructed in § 5.2 we arrive at the following theorem, which is an intermediate step in the proof of Theorem 5.1.

THEOREM 5.7. *Let $\Gamma \to B$ be a GKM fibration as in Theorem 5.1 and X be the realization of B from Proposition 5.3. We fix the notation from Remark 4.6. For any $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ there exists a smooth T^2-equivariant complex vector bundle $E \to X$ of rank 2 with the following properties.*

(i) *The isotropy representation of E over v_i, $i = 1, \ldots, n$, is isomorphic to the pullback of the standard action on $\mathbb{C}^2$ along the homomorphism $T \to U(2)$,*

$$t \mapsto \begin{pmatrix} a_i \gamma_{i-1}(t) - a_{i-1} \gamma_i(t) & \\ & (a_i - k_i)\gamma_{i-1}(t) + (-a_{i-1} + k_{i-1})\gamma_i(t) \end{pmatrix},$$

where $([k_1, \ldots, k_n], \eta)$ is defined by the fibration in the sense of correspondence 4.5 and we set $a_0 = (-1)^{\varepsilon_1} a_n + \eta k_0$ and $k_0 = (-1)^{\varepsilon_1 + \eta} k_n$.

(ii) *The fibration $\mathbb{P}(E) \to X$ is one of GKM manifolds and realizes the GKM fibration $\Gamma \to B$.*

We conclude this section with the proof of the statement on the isotropy groups of the T-action on $\mathbb{P}(E)$ in part (i) of Theorem 5.1. Recall first that the weights of the isotropy representation at any of the two fixed points over a vertex v_i of B are (up to sign) γ_{i-1}, γ_i, and $\alpha_i = k_i \gamma_{i-1} - k_{i-1} \gamma_i$. If any $k_j \neq \pm 1$, we can therefore find a fixed point in $\mathbb{P}(E)$ such that two of the adjacent weights, considered as characters $T \to S^1$, have a non-trivial common kernel, which then occurs as an isotropy group near the fixed point. Conversely, we assume that all $k_i = \pm 1$. As $\mathbb{P}(E)$ fibers equivariantly over the T-manifold X, and in X non-trivial stabilizers occur only in the one-skeleton X_1, any non-trivial stabilizer in $\mathbb{P}(E)$ necessarily occurs in $\pi^{-1}(X_1)$. However, this space is, in the notation of § 5.2, the union of 4-dimensional T^2-manifolds $\mathbb{P}(E_i)$ that fiber over S^2. By assumption on the k_i the T-action on $\mathbb{P}(E_i)$ is effective. Hence, the $\mathbb{P}(E_i)$ are torus manifolds with vanishing odd-degree cohomology, which by [MP06, Theorem 4.1] have only connected isotropy groups. See Remark 9.6 for an example of a linear realization of a total space of a GKM fibration where not all $k_i = \pm 1$.

6. Geometric structures on the realization

6.1 (Stable) almost complex structures

The realization

$$\mathbb{CP}^1 \longrightarrow \mathbb{P}(E) \longrightarrow X$$

constructed in the previous sections is a fibration with structure group U(2). Let $V_F \subset T\mathbb{P}(E)$ be the subbundle consisting of the tangent spaces of all the fibers. Since each fiber can be identified with $\mathbb{CP}^1$ uniquely up to elements from U(2), we see that a U(2)-invariant almost complex structure on $\mathbb{CP}^1$ induces on V_F the structure of a complex T-vector bundle. A complement of V_F in $\mathbb{P}(E)$ with respect to a T-invariant metric can be identified with the pullback V_X of the tangent bundle of X along the projection $\mathbb{P}(E) \to X$. As X is quasitoric or S^4, it has a T-invariant stable almost complex structure (see [BP15, Corollary 7.3.15]). It follows that $T\mathbb{P}(E) = V_F \oplus V_X$ carries a T-invariant stable almost complex structure. This finishes the proof of part (i) of Theorem 5.1.

In case the graph fibration is one of the signed graphs, X carries a T-invariant almost complex structure by Proposition 5.3. As above, it follows that $\mathbb{P}(E)$ carries an invariant almost complex structure compatible with the map to X. For the proof of part (ii) of Theorem 5.1, it remains to check that the signed GKM graphs of $\mathbb{P}(E) \to X$ with respect to the almost complex structures agree with the given fibration of signed graphs $\Gamma \to B$. The construction of X and its almost complex structure (see Proposition 5.3) can be carried out such that its signed GKM graph is precisely B. Since $\mathbb{P}(E) \to X$ is compatible with the complex structures, it follows that the given labelling function on oriented edges of Γ agrees with that induced by the almost complex structure on $\mathbb{P}(E)$ when applied to oriented horizontal edges. On (oriented) vertical edges they *a priori* only agree up to sign. However, we observe that, since both labelling functions on oriented edges admit a compatible connection, the signs in the vertical edges either agree everywhere or nowhere. In the first case, we are done. In the second case, we may apply the automorphism of Γ that interchanges the vertices in each fiber. This transforms one labelling function into the other so the proof of part (ii) is complete.

6.2 Symplectic and Kähler structures

Suppose $E \to X$ is a complex vector bundle of rank 2 and X a toric 4-manifold. First note that X is a smooth projective variety [BP15, Proposition 5.2.2] and, thus, X admits in particular a Kähler structure. We denote by ω_X the corresponding Kähler form. Since X is projective we deduce from [Sch61, Theorem 9] that E is algebraic if the determinant bundle $\det E$ is algebraic. However, since X is toric, $H^2(X) = H^{1,1}(X)$, so that $\det E$ has a holomorphic structure by the Lefschetz theorem on $(1,1)$-classes, cf. [Lef50]. With the GAGA principle [Ser55] we infer that $\det E$ is algebraic, hence E. Since E is algebraic, the manifold $\mathbb{P}(E)$ is a smooth projective variety.

From the discussion above we obtain the following result.

PROPOSITION 6.1. *For every rank-2 complex vector bundle $E \to X$ the projectivization $\mathbb{P}(E)$ is a smooth projective variety such that $\pi \colon \mathbb{P}(E) \to X$ is a holomorphic map.*

In the remaining part of this section we will construct the symplectic and Kähler structure from Theorem 5.1.

Now let $E \to X$ be a complex, equivariant T-bundle. Furthermore let h be a T-invariant Hermitian metric on E and let ω_X be the T invariant Kähler form on X. Let $L \to \mathbb{P}(E)$ be the tautological bundle (i.e. L restricted to each fiber of $\mathbb{P}(E)$ is the tautological bundle of $\mathbb{CP}^1$).

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Since, by construction, L is a subbundle of $\pi^*(E)$, the metric h can be restricted to L and it also induces a metric h^* on the dual bundle $\overline{L}$. The Chern form ω_V of the Hermitian line bundle $(\overline{L}, h^*)$ has the property that it restricts to the Fubini–Study form (associated to the metric h) on every fiber of $\mathbb{P}(E) \to X$ and that, furthermore,

$$\omega_K := \omega_V + C \cdot \pi^*(\omega_X)$$

is a Kähler form on $\mathbb{P}(E)$ for sufficiently large C (see [Voi07, § 3.3.2]).

The pullback $\pi^*(\omega_X)$ is T-invariant but ω_V and, hence, also ω_K in general are not. However, since T is compact, we can average ω_V over T, obtaining another form ω_F which is T-invariant. As h is T-invariant, the T action preserves the Fubini–Study form on the fibers, so for any fiber of $\mathbb{P}(E) \to X$ and $t \in T$ we have $i^*(t^*\omega_V) = i^*\omega_V$, where i is the fiber inclusion. This implies that also the averaged form ω_F restricts to the Fubini–Study form on every fiber. The latter is, in particular, symplectic, thus after possibly enlarging the constant C, the form

$$\omega_S := \omega_F + C \cdot \pi^*(\omega_X)$$

is a T-invariant symplectic form on $\mathbb{P}(E)$. Having chosen a fixed sufficiently large C, we note that by the same reasoning $t\omega_F + C \cdot \pi^*(\omega_X)$ is symplectic for $t \in (0,1]$. On the symplectic complement V of a fiber, this defines a homotopy between $\omega_S|_V$ and $\pi^*(\omega_X)|_V$ through non-degenerate 2-forms on V. Hence, the symplectic structures $(\mathbb{P}(E), \omega_S)$ and (X, ω_X) are compatible with the fiber bundle in the sense of the definition above Theorem 5.1. Since $H^1(\mathbb{P}(E)) = 0$ we infer from [GS84, Addendum to Theorem 28.1] that the action is, in fact, Hamiltonian.

PROPOSITION 6.2. *The Kähler form ω_K and the invariant symplectic form ω_S are symplectomorphic.*

Proof. Since averaging a closed form over a compact group does not change its de Rham class there is a 1-form η such that $\omega_F = \omega_V + d\eta$. Because ω_V and ω_F agree on the fibers of $\mathbb{P}(E) \to X$ it follows that $d\eta$ restricts to 0. In particular, for any $t \in \mathbb{R}$ the forms $\omega_V + t d\eta$ all restrict to symplectic forms on the fibers. Therefore, the forms $\omega_t := \omega_V + t \, d\eta + C \cdot \pi^*(\omega_X)$ are symplectic for $t \in [0,1]$ with an eventually bigger constant C. Thus, ω_K and ω_S are joined by a path of symplectic forms in the same de Rham class and by Moser's trick they are symplectomorphic. $\square$

This completes the proof of Theorem 5.1. In the next section, we will show that GKM fibrations of product type always admits a toric symplectic realization, as announced in Remark 5.2.

6.3 Fibrations of product type: the toric case

In this section we consider a signed GKM fibration $\pi : \Gamma \to B$ of product type, where B is the boundary of a 2-dimensional Delzant polytope $P \subset \mathfrak{t}^*$. Let X be the 4-dimensional toric manifold with Delzant polytope P.

In this setting, we can refine the construction in § 5.2. The T-equivariant complex rank-2 vector bundle $E \to X_1$ over the one-skeleton of X constructed there is, because the graph Γ is of product type, the Whitney sum of two T-equivariant complex line bundles $L_i \to X_1$, $i = 1, 2$. By Lemma 5.5 we can extend not only the bundle E equivariantly to all of X, but also the bundles L_i. In this way, we obtain an extension $E \to X$ which is globally the sum of two equivariant line bundles.

Now, any complex line bundle admits a canonical circle action, so that E admits a canonical auxiliary action of a 2-dimensional torus, which commutes with the T-action. Note that the diagonal action becomes trivial after passing to the projectivization $\mathbb{P}(E)$, but the fiberwise

circle action given in homogeneous coordinates by $t \cdot [z : w] = [tz : w]$ defines an effective circle action on $\mathbb{P}(E)$ commuting with the T-action. In total, we obtain a T^3-action on $\mathbb{P}(E)$.

As explained in § 6.2, the projectivization $\mathbb{P}(E)$ admits a T-invariant symplectic structure. We can modify this construction to make the symplectic form invariant also under the T^3-action. We, thus, see that $\mathbb{P}(E)$ admits the structure of a toric symplectic manifold. Note that toric structures on projectivizations of sums of circle bundles have been considered before; see, e.g., [CP16].

Remark 6.3. One can construct a 6-dimensional toric manifold such that the restriction of the action to a 2-dimensional subtorus has the prescribed GKM graph Γ via Delzant's theorem as follows.

Let $T^3 = T^2 \times S^1$, with Lie algebra $\mathfrak{t}^3 = \mathfrak{t} \oplus \mathbb{R}$ with integral lattice $\mathbb{Z}^2 \times \mathbb{Z}$, and dual $(\mathfrak{t}^3)^* \cong \mathfrak{t}^* \oplus \mathbb{R} \cong \mathbb{R}^3$. We construct a Delzant polytope $Q \subset (\mathfrak{t}^3)^*$ as follows: in the hyperplane $z = 0$ it contains the polytope $P \subset \mathfrak{t}^* \cong \mathfrak{t}^* \oplus \{0\} \subset (\mathfrak{t}^3)^*$. As in § 5.2, we let $v_1, \ldots, v_n$ be the vertices of P, and γ_i the label of the edge e_i from v_i to v_{i+1}. The only other vertices of Q are the elements $w_i := v_i + (\alpha_i, 1)$, all contained in the hyperplane $z = 1$.

Then Q is a Delzant polytope: the slopes of the edges emerging from the vertex v_i are $-\gamma_{i-1}, \gamma_i$, and $(\alpha_i, 1)$. These three vectors form a basis of $\mathbb{Z}^3$ because the γ_{i-1} and γ_i form a basis of $\mathbb{Z}^2$. The edge from w_i to w_{i+1} points in direction $v_{i+1} - v_i + \alpha_{i+1} - \alpha_i$, which is a multiple of γ_i; this shows that the Delzant condition is also satisfied at the vertices w_i.

By construction, the 6-dimensional toric symplectic manifold with Delzant polytope Q satisfies that the restriction of the T^3-action to $T^2 \cong T^2 \times \{1\}$ is GKM, with (signed) GKM graph Γ.

7. Non-Kähler GKM graphs

The goal of this section is to show that certain GKM actions do not admit compatible Kähler structures. The necessary criteria to show that signed GKM structures can not come from such an action were already described in § 2. However, in order to show the non-existence of such structures for a given action, there is more work to do. The results of this section are collected in the following.

THEOREM 7.1. *Let $\Gamma \to B$ be a signed fibration of GKM graphs of twisted type such that Γ is 3-valent and B is the boundary of a 2-dimensional Delzant polytope with n vertices (labels of the edges are with respect to a rank-2 lattice $\mathbb{Z}_{\mathfrak{t}}^*$). We assume that Γ has $n - 1$ interior vertices (cf. Definition 4.9 and Corollary 4.11). Then, up to isomorphism, there are up to three structures of signed GKM graphs which are compatible with the underlying unsigned GKM graph of Γ. They have the following properties.*

- (I) *The signed graph Γ itself. This signed structure cannot be realized by a complex structure compatible with an invariant Kähler form.*
- (II) *The signed structure which arises from Γ by changing the sign of the weights of every second pair of horizontal edges. It exists only if n is even. If, furthermore, $n \neq 4$, then this signed structure is not realized by a Hamiltonian action.*
- (III) *A unique signed structure where no two horizontal edges over the same edge have the same weights. It exists only if $k_i = \pm 1$ for $i = 1, \ldots, n$ (as in § 4) and is not realized by a Hamiltonian action.*

COROLLARY 7.2. *A GKM action whose unsigned GKM graph is that of Γ as above with $n \neq 4$ does not admit an invariant Kähler structure.*

Remark 7.3. We have seen that case (I) is always realized as a Hamiltonian action. For even n, case (II) is realizable as an equivariant fibration of almost complex manifolds since it is a signed fibration over the signed base graph which arises from B by changing the sign of every second edge. Note that by the discussion in § 6 we may use the same underlying T^2-manifold for both cases and only need to vary the almost complex structures. We do not know about the realizability of case (III).

7.1 The winding number

In order to argue that certain GKM graphs do not come from invariant Kähler actions, we need to argue that certain subgraphs are not of polytope type. We will make use of the following concept.

DEFINITION 7.4. Let $(w_1, \ldots, w_n)$ be a sequence of non-zero vectors in $\mathbb{R}^2$ and let $\varepsilon = \pm 1$. The *winding number* of the sequence with respect to the orientation ε is given by

$$\sigma(w_1, \ldots, w_n, \varepsilon) = \frac{1}{2\pi} \sum_{i=1}^n |\eta_i|,$$

where η_i is the angle between w_i and w_{i+1} with representative chosen in $[0, 2\pi)$ if $\varepsilon = 1$ and in $(-2\pi, 0]$ if $\varepsilon = -1$ (where we set $w_{n+1} = w_1$).

LEMMA 7.5.

(i) *If $\gamma_i \in \mathbb{Z}_{\mathfrak{t}}^* \cong \mathbb{Z}^2$ are the weights along a path around a 2-valent signed GKM graph, then the angle between γ_i and γ_{i+1} is, for all i, either always represented in $(0, \pi)$ or always represented in $(-\pi, 0)$ (we speak of a locally convex sequence). Choosing ε such that the angles get measured in the respective interval minimizes the winding number. We call this the preferred orientation for $(\gamma_1, \ldots, \gamma_n)$.*

(ii) *If $(\gamma_1, \ldots, \gamma_n)$ is a locally convex sequence with preferred orientation ε and n is even, then*

$$\sigma(\gamma_1, \gamma_3, \ldots, \gamma_{n-1}, \varepsilon) = \sigma(\gamma_1, \gamma_2, \ldots, \gamma_n, \varepsilon).$$

(iii) *Let $\gamma_1, \ldots, \gamma_n$ be the vectors along the boundary of a convex polytope in $\mathbb{R}^2$ with n vertices. We have $\sigma(\gamma_1, \ldots, \gamma_n, \varepsilon) = 1$ with respect to the preferred orientation ε. If n is even, then the sequence $(\gamma_1, -\gamma_2, \ldots, \gamma_{n-1}, -\gamma_n)$ is locally convex with preferred orientation $-\varepsilon$ and*

$$\sigma(\gamma_1, -\gamma_2, \ldots, \gamma_{n-1}, -\gamma_n, -\varepsilon) = \frac{n-2}{2}.$$

Proof. For the proof of part (i) we observe that the connection of the signed graph has to transport $\overline{e_i}$ onto e_{i+2} along e_{i+1} so

$$\gamma_i \equiv -\gamma_{i+2} \mod \gamma_{i+1}.$$

We conclude that γ_i and γ_{i+2} lie on opposite sides of the ray defined by γ_{i+1} as they cannot lie on the ray due to the condition of adjacent weights being linearly independent. Thus, if the angle between γ_i and γ_{i+1} is represented in $(0, \pi)$ (respectively, in $(-\pi, 0)$), then the same holds for the angle between γ_{i+1} and γ_{i+2}.

Assertion (ii) follows from the fact that two consecutive angles never add up to a full rotation and so nothing is lost by skipping every second vector.

For the proof of part (iii) we assume for simplicity that the preferred orientation is $\varepsilon = 1$, i.e. $\eta_i \in (0, \pi)$ for $i = 1 \ldots, n$, where η_i denotes the angle between γ_i and γ_{i+1} (and η_n is the angle

between γ_n and γ_1). The remaining case is proved analogously. We have

$$\sum_{i=1}^{n} \eta_i = 2\pi.$$

Now note that the angle between γ_i and $-\gamma_{i+1}$ as well as from $-\gamma_i$ to γ_{i+1} is represented in $(-\pi, 0)$. It follows that $(\gamma_1, -\gamma_2, \ldots, \gamma_{n-1}, -\gamma_n)$ is indeed locally convex with the opposite preferred orientation. Now by part (ii) we have

$$\sigma(\gamma_1, -\gamma_2, \ldots, \gamma_{n-1}, -\gamma_n, -1) = \sigma(\gamma_1, \gamma_3, \ldots, \gamma_{n-1}, -1).$$

The angle between γ_i and γ_{i+2} with respect to this orientation is exactly $\eta_i + \eta_{i+1} - 2\pi$ so we obtain

$$\sigma(\gamma_1, \gamma_3, \ldots, \gamma_{n-1}, -1) = \frac{1}{2\pi} \sum_{i=1}^{n/2} 2\pi - \eta_{2i} - \eta_{2i+1} = \frac{n-2}{2}$$

as claimed. $\qquad\square$

7.2 Case (I)

We show that the signed structure Γ is indeed not realized by an invariant Kähler form.

LEMMA 7.6. *If two fibers in Γ are connected by a horizontal edge and both contain an interior vertex, then the interior vertices are connected by a horizontal edge.*

Proof. Let p be an interior vertex of Γ. Let e be a horizontal edge emanating from p with endpoint q and weight γ. In addition, let γ' and γ'' be the weights of the other horizontal edges emanating from p and q and let α, α' be the weights of the vertical edges emanating from p and q. Through the connection we obtain $\alpha = k\gamma + l\gamma'$ and $\alpha' = l\gamma'' + m\gamma$ for some $k, l, m \in \mathbb{Z}$. As p is an interior vertex we have $k, l < 0$. This implies that the second vertex q' in the fiber containing q has emanating weights γ'', $-\gamma$, and $-l\gamma'' - m\gamma$. Since $-l > 0$ we deduce that q' is an exterior vertex. We conclude that if the fiber contains an interior vertex, then q is interior. $\qquad\square$

LEMMA 7.7. *Let p be an interior vertex of Γ. Further let e, e' be the oriented horizontal edges emanating from p, let q be the endpoint of e' and e_F be the vertical edge emanating from q. Then there is no 2-valent GKM subgraph of Γ that contains e, e', and e_F.*

Proof. We assume the existence of a compatible connection on such a subgraph which implies that the weights γ, γ', and α of e, e', and e_F satisfy $\gamma \equiv \alpha \mod \gamma'$. Let β be the weight of the vertical edge emanating from p and observe that

$$\beta \equiv \alpha \equiv \gamma \mod \gamma'.$$

Since the weights of the edges emanating from p are γ, γ', and β, this implies that p is not an interior vertex. $\qquad\square$

PROPOSITION 7.8. *The signed graph Γ does not satisfy the criterion from Corollary 2.13. In particular, it is not realized by a Hamiltonian action on a Kähler manifold.*

Proof. Let p be an interior fixed point and consider the two horizontal edges emanating from p. Suppose that we find a polytope-type 2-valent GKM subgraph of Γ (i.e. a closed convex loop) containing those horizontal edges. Moving along such a path, starting with a horizontal edge at p, there are two possible choices for continuing the path: the horizontal edge and the vertical edge. However, by Lemma 7.7 the choice has to be the horizontal edge. Consequently, in both directions we need to move along horizontal edges until we reach exterior vertices. By Lemma 7.6, moving

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from an interior vertex into a fiber containing an interior vertex will end in the interior vertex. By the assumption that Γ has $n-1$ interior vertices there is only one fiber $F \subset \Gamma$ which does not contain an interior fixed point, it follows that the desired 2-valent polytope-type subgraph would need to contain a lift of the closed path running around B once, which starts and ends in the fiber F. Since Γ is of twisted type this path is not closed. Again by Lemma 7.7 it follows that the subgraph does not contain the vertical edge in F so the only way to close the loop is to continue with the other lift of the loop around B until we reach the starting point. Thus, the only possible choice of 2-valent subgraph is Γ with all the vertical edges removed. This is indeed a signed GKM subgraph but globally it is not a polytope-type graph. This follows from Lemma 7.5 as the winding number of the subgraph in question is 2. $\qquad\square$

7.3 Case (II)

Assume now that we have another signed GKM structure Γ' on the underlying graph of Γ such that the induced unsigned GKM graph agrees with that of Γ. Let e_i denote the oriented edges of B as in §4 and let f_i, h_i be the edges in Γ over e_i such that f_i and f_{i+1} are adjacent for $i = 1, \ldots, n-1$, and the same holds for f_n and h_1. We also denote by g_i the directed edge in the fiber over v_i whose starting point is the same as that of f_i. Occasionally we extend this notation mod n such that, e.g., $f_{n+1} = h_1$ and $f_0 = h_n$.

In case (II), we furthermore make the assumption that in the signed structure of Γ' the labels of f_i and h_i agree (where in full generality they might differ by a sign). We do not assume that Γ' admits a signed fibration over B, but as the underlying graphs are unchanged we may still speak about horizontal and vertical edges in Γ'.

LEMMA 7.9. *The weights of the vertical edges of Γ' agree with those of Γ up to a global sign.*

Proof. Let α_i' be the weight associated to g_i in Γ' and let α_i be the weight associated to g_i in Γ. Let ∇' denote a compatible connection of Γ'. For $i = 1, \ldots n-1$, we claim that $\nabla'_{f_i} g_i = g_{i+1}$ or $\nabla'_{h_i} \overline{g_i} = \overline{g_{i+1}}$. If this were false, then we would have $\nabla'_{f_i} g_i = f_{i+1}$ and $\nabla'_{h_i} \overline{g_i} = h_{i+1}$. However, by assumption, f_{i+1} and h_{i+1} have the same weight $\pm\gamma_{i+1}$ so

$$\alpha_i' \equiv \pm\gamma_{i+1} \equiv -\alpha_i' \mod \gamma_i$$

which is a contradiction. Thus, the claim holds, which implies that if $\alpha_i = \alpha_i'$, then also $\alpha_{i+1} = \alpha_{i+1}'$ because

$$\alpha_{i+1}' \equiv \alpha_i' \equiv \alpha_i \equiv \alpha_{i+1} \mod \gamma_i.$$

Analogously $\alpha_i = -\alpha_i'$ implies $\alpha_{i+1} = -\alpha_{i+1}'$. Thus, the signs of the α_i and α_i' either globally agree or globally disagree. $\qquad\square$

LEMMA 7.10. *If n is odd, then $\Gamma \cong \Gamma'$. If n is even, then either $\Gamma \cong \Gamma'$ or Γ' is isomorphic to the signed structure that arises from Γ by changing the signs of the labels of f_{2i} and h_{2i} for $i = 1, \ldots, n/2$ where a compatible connection can be chosen identical to that of Γ.*

Proof. Let γ_i' be the weight of f_i and h_i in Γ'. Assume that for some $i = 1, \ldots, n-2$ we have $\gamma_i = \gamma_i'$ but $\gamma_{i+2} = -\gamma_{i+2}'$. Since $\gamma_i \equiv -\gamma_{i+2} \mod \gamma_{i+1}$ it follows that $\gamma_i' \not\equiv -\gamma_{i+2}' \mod \gamma_{i+1}$. As a consequence, we must have $\nabla'_{f_{i+1}} f_i = g_{i+2}$ and $\nabla'_{h_{i+1}} h_i = \overline{g_{i+2}}$ which results in the contradiction

$$\alpha_{i+2}' \equiv \gamma_i' \equiv -\alpha_{i+2}' \mod \gamma_{i+1}.$$

If we assume that $\gamma_i = -\gamma_i'$ but $\gamma_{i+2} = \gamma_{i+2}'$, then we arrive at the same contradiction. We have proved that γ_i and γ_{i+2} must either both agree or both disagree with their counterparts γ_i' and γ_{i+2}'. The same holds for the pairs γ_n, γ_2, and γ_n', γ_2'. Thus, if n is odd, the γ_i' globally agree or

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disagree with the γ_i. In each case we have $\Gamma \cong \Gamma'$ due to the previous lemma. If n is even, then the sign of the odd or the even edges may be switched independently of the other. Still, globally changing the signs of all the γ_i' or all α_i' yields isomorphic graphs so the lemma follows. $\qquad\square$

PROPOSITION 7.11. *If $n \neq 4$ is even, then the alternative signed structure Γ' as in Lemma 7.10 cannot be realized as the GKM graph of a symplectic action.*

For the proof we will make use of the following concept.

Proof of Proposition 7.11. Let B' be the signed GKM structure on the underlying graph of B where e_{2i+1} has the weight γ_{2i+1} and e_{2i} has the weight $-\gamma_{2i}$. Note that the connection of B also is a compatible connection on B' and that $\Gamma' \to B'$ is a signed GKM fibration. Let $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ be the vector (unique up to sign) corresponding to the fibration $\Gamma \to B$ in the sense of §4. Then $\Gamma' \to B'$ corresponds to $(-k_1, k_2, \ldots, -k_{n-1}, k_n)$. As Γ was assumed to have the maximal number of interior fixed points, it follows from Proposition 4.10 that Γ' has the minimal number of interior fixed points, i.e. exactly one.

Now assume that Γ' is realized by a Hamiltonian action. The moment image is a convex polytope in $\mathfrak{t}^*$, which we identify with $\mathbb{R}^2$, spanned by the images of the exterior fixed points. Note that, as part of the convexity theorem for Hamiltonian actions, the preimage of the vertices of the moment image is connected and contains only fixed points. Thus, the vertices of the polytope correspond bijectively to exterior vertices in the GKM graph Γ'. A path around the boundary of this polytope corresponds to a closed path in Γ' that runs through every exterior vertex without going through a vertex twice or going through an interior vertex. Since the sequence of weights along this path correspond exactly to the slopes of the boundary edges of the polytope, we deduce that it has winding number equal to 1 with respect to its preferred orientation.

Now assume without loss of generality that the unique interior fixed point of Γ' is the end point of g_i and that we have a path as above starting with the edge f_i. Since the path must not go through the interior fixed point the only possibility is the path

$$f_i, g_{i+1}, h_{i+1}, \overline{g_{i+2}}, \ldots, h_{i-2}, \overline{g_{i-1}}, f_{i-1},$$

which alternates between horizontal and vertical edges (excluding the fiber which contains the interior vertex). If i is odd, then the associated sequence of weights is

$$(\gamma_i, \alpha_i, -\gamma_{i+1}, -\alpha_{i+1}, \ldots, \gamma_{i-2}, -\alpha_{i-1}, -\gamma_{i-1}),$$

and then, by Lemma 7.5, its winding number with respect to the preferred orientation ε satisfies

$$\sigma(\gamma_i, \alpha_i, -\gamma_{i+1}, -\alpha_{i+1}, \ldots, -\alpha_{i-1}, -\gamma_{i-1}, \varepsilon) = \sigma(\gamma_i, -\gamma_{i+1}, \ldots, \gamma_{i-2}, -\gamma_{i-1}, \varepsilon) \geq \frac{n-2}{2},$$

with equality on the right if ε is the preferred orientation for the central expression. If $n > 4$, then this is in any case not equal to 1 which is a contradiction. If i is even, then in the weight sequences the signs of all γ_k are reversed, whence we arrive at the same contradiction. $\qquad\square$

7.4 Case (III)

Let f_i, h_i, and g_i denote the horizontal and vertical edges of the underlying graph of Γ as in the previous section. It remains to treat the case where the weights of f_i and h_i do not agree for some i. Let Γ' be a signed GKM structure on the unsigned GKM graph of Γ satisfying the above property.

Lemma 7.12.

(i) *The weights of f_i and h_i do not agree for any $i = 1, \ldots, n$.*

(ii) *The vector $(k_1, \ldots, k_n)$ corresponding to the original signed fibration $\Gamma \to B$ in the sense of §4 satisfies $k_i = \pm 1$ for $i = 1, \ldots, n$.*

(iii) *If k_i and k_{i+1} have the same sign, then the weights of f_i and f_{i+1} in the signed structure Γ' both agree or both disagree with the weights in Γ.*

Proof. Let ∇' be a connection that is compatible with the signed structure Γ' and let $\gamma_i', \delta_i' = \pm \gamma_i$ be the weights of f_i and h_i. If $\gamma_i' = -\delta_i'$, then necessarily $\nabla'_{g_{i+1}} \overline{f_i} = h_{i+1}$ and $\nabla'_{\overline{g_{i+1}}} \overline{h_i} = f_{i+1}$. This implies

$$\gamma_{i+1}' \equiv -\delta_i' \equiv \gamma_i' \equiv -\delta_{i+1}' \mod \alpha_{i+1} \tag{2}$$

and assertion (i) follows.

For the original signed structure Γ, the equation $\gamma_i' \equiv -\delta_{i+1}' \mod \alpha_{i+1}$ in (2) translates to $\gamma_i = \pm \gamma_{i+1} \mod \alpha_{i+1}$. This congruence in the lattice $\mathbb{Z}_t^*$ can only be solved if up to sign we have $\alpha_{i+1} = \gamma_i \pm \gamma_{i+1}$ because γ_i and γ_{i+1} form a basis of $\mathbb{Z}_t^*$. In particular $k_i, k_{i+1} = \pm 1$ is necessary which proves part (ii).

More specifically, if k_i and k_{i+1} have the same sign then $\pm \alpha_{i+1} = \gamma_i - \gamma_{i+1}$ and the only solvable congruence of the above form is $\gamma_i = \gamma_{i+1} \mod \alpha_{i+1}$. Thus, if $\gamma_i' = \gamma_i$ then $\gamma_i' \equiv \gamma_{i+1}'$ mod α_{i+1} (see (2)) implies $\gamma_{i+1}' = \gamma_{i+1}$. Analogously $\gamma_i' = -\gamma_i$ implies $\gamma_{i+1}' = -\gamma_{i+1}$. $\qquad \square$

By assumption, Γ has the maximal number of interior fixed points in the sense of §4 and, thus, there is only one spot $j \in \{1, \ldots, n\}$ for which $k_j \neq k_{j-1}$ (setting $k_0 = -k_n$). Fixing this j, Lemma 7.12 implies that regarding horizontal edges Γ' has the form
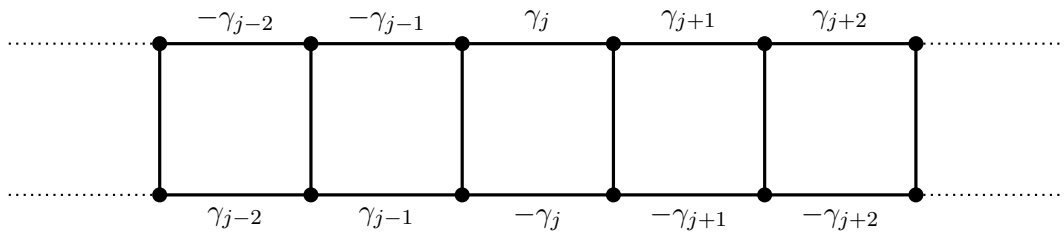
where the horizontal edges are oriented from left to right and the ends (the not depicted positions 1 and $n+1$) of the ladder are glued in a twisted fashion. Note that interchanging the top and bottom row while applying multiplication with -1 in $\mathbb{Z}_t^*$ defines an automorphism of Γ'. Consequently, we can assume that the f_i are the horizontal edges in the upper row, the h_i are the horizontal edges in the lower row, and the g_i are the vertical edges emanating from the starting point of f_i.

PROPOSITION 7.13. *In the graph above there is a unique way to define the signs of the weights of the g_i such that there exists a compatible connection. Consequently, the signed GKM structure Γ' is unique up to isomorphism. Furthermore, Γ' has only interior fixed points and is not realized by a Hamiltonian action.*

Proof. Let α_i' denote the weight of g_i in Γ'. By the choice of j we have $\pm \alpha_j' = \gamma_{j-1} + \gamma_j$ and $\pm \alpha_i' = \gamma_{i-1} - \gamma_i$ for the remaining values $i \neq j$. Observe that since $\gamma_{j-2} \equiv -\gamma_j \mod \gamma_{j-1}$ we cannot have $\nabla'_{f_{j-1}} \overline{f_{j-2}} = f_j$ due to the change of sign in Γ'. As a consequence, we have $\nabla'_{f_{j-1}} \overline{f_{j-2}} = g_j$ which implies $\alpha_j' \equiv \gamma_{j-2} \equiv -\gamma_j \mod \gamma_{j-1}$ and forces $\alpha_j' = -\gamma_{j-1} - \gamma_j$. Analogously, one has $\nabla'_{f_j} \overline{f_{j-1}} = g_{j+1}$ and it follows that $\alpha_{j+1}' \equiv \gamma_{j-1} \equiv -\gamma_{j+1} \mod \gamma_j$ forcing $\alpha_{j+1}' = \gamma_j - \gamma_{j+1}$.

We prove inductively that $\alpha_i' = \gamma_{i-1} - \gamma_i$ for $i = j+1, \ldots, n$ and $\alpha_i' = -\gamma_{i-1} + \gamma_i$ for $i = 1, \ldots, j-1$. We showed this already for $j+1$ and we assume it holds for some $i \in \{j+1, \ldots, n-1\}$. Through the congruence $\alpha_i' \equiv \gamma_{i-1} \equiv -\gamma_{i+1} \not\equiv \gamma_{i+1} \mod \gamma_i$ we see that $\nabla_{f_i}' g_i \neq f_{i+1}$. Consequently, $\nabla_{f_i} g_i = g_{i+1}$ which implies $\alpha_{i+1}' \equiv \gamma_{i-1} - \gamma_i \equiv -\gamma_{i+1} \mod \gamma_i$ and, thus, $\alpha_{i+1}' = \gamma_i - \gamma_{i+1}$. The rest of the argument is carried out analogously where one first shows that $\nabla_{f_n}' g_n = \overline{g_1}$ and, thus, $\alpha_1' \equiv -\gamma_{n-1} + \gamma_n \equiv \gamma_1 \mod \gamma_n$ due to the twist in our notation. It follows that $\alpha_1' = -\gamma_n + \gamma_1$ and from there on the induction can be continued up to $j-1$.

We have proved that Γ' is unique up to isomorphism. One easily checks that a compatible connection ∇' is indeed given as follows: ∇_{f_i}' and ∇_{h_i}' send horizontal to horizontal edges except for $i = j-1, j$ where horizontal and vertical edges are interchanged. Along the g_i, the horizontal edges $\overline{f_i}$ and f_{i+1} get transported to h_{i+1} and $\overline{h_i}$. Clearly, all vertices in Γ' are interior. In particular, it does not have a linear realization in the sense of § 2.3 and cannot, thus, come from a Hamiltonian action. □

8. Distinguishing the equivariant homotopy type

The goal of this section is to show that the equivariant homotopy type of the previous constructions in general depends on the input data. In particular, it will follow that the previously developed methods produce infinite families of pairwise not equivariantly homotopy equivalent examples.

PROPOSITION 8.1. *Let $\Gamma \to B$ and $\Gamma' \to B'$ be two fiberwise signed GKM fibrations as in § 4 such that the base graphs have at least 5 vertices. Then the GKM graphs Γ and Γ' are isomorphic if and only if the following hold.*

- *There is a GKM isomorphism $\varphi \colon B \to B'$.*
- *If we fix the data for B as in Remark 4.6 needed to define the correspondence in Proposition 4.5 and use φ to fix the corresponding choices for B', then the elements in $((\mathbb{Z} - 0)^n / \pm) \times \{0, 1\}$ associated to the fibrations coincide.*

Remark 8.2. Given $\Gamma \to B$ as above then, as stated before, the map in Proposition 4.5 depends on a fixed enumeration of the vertex set of B and on choices of signs for the first two edges. If we change this data by choosing a different sign for the first or second weight, then the fibration associated to $([k_1, \ldots, k_n], \eta)$ will now correspond to $([-k_1, k_2, \ldots, (-1)^n k_n], \eta)$. Changing the enumeration of the underlying n-gon, the k_i get permuted by the corresponding permutation of the dihedral group. Note however that on top of the permutation some additional signs will appear. We leave the details of the exact signs to the interested reader and settle for the slightly suboptimal corollary below.

COROLLARY 8.3. *Let $\Gamma \to B$ and $\Gamma' \to B$ be GKM fibrations associated to $([k_1, \ldots, k_n], \eta)$ and $([k_1', \ldots, k_n'], \eta')$. If $n \geq 5$, then in order for geometric realizations of Γ and Γ' to be equivariantly homotopy equivalent, it is necessary that $\eta = \eta'$ and $[k_1, \ldots, k_n] = [k_1', \ldots, k_n']$ up to signs and permutations from the dihedral group.*

Proof. It follows from Proposition 8.1 as well as the subsequent remark that the conditions in the corollary are necessary in order for Γ and Γ' to be isomorphic. It is shown in [FY19] that realizations of non-isomorphic graphs have non-isomorphic equivariant cohomology algebras which implies the claim. □

Proof of Proposition 8.1. If Γ and Γ' are isomorphic, then in particular they have the same number of vertices. Thus, the underlying graphs of B and B' are both n-gons for some $n \geq 5$. We argue that an isomorphism $\tilde{\varphi} \colon \Gamma \cong \Gamma'$ has to respect the decomposition into horizontal and vertical edges of the respective fibrations. To see this note that a horizontal edge e has the following property: there is another edge e' (namely the other horizontal edge over the same edge in the base) such that after removing e and e' the shortest path between $i(e)$ and $t(e)$ has length at least $n - 1 \geq 4$. On the other hand, any vertical edge e has the property that, after removing e and any other edge e', the shortest path between $i(e)$ and $t(e)$ has length 3. Both properties are respected by graph isomorphisms so φ indeed respects the decomposition of fiber and horizontal edges.

It follows that the lift of a path around the n-gon B gets mapped by $\tilde{\varphi}$ to the lift of a path around the n-gon B'. Thus, in particular, Γ and Γ' must either be both of twisted type or of product type so $\eta = \eta'$. Since this is true for both possible lifts and their images under $\tilde{\varphi}$ are connected through vertical edges it follows that $\tilde{\varphi}$ respects pairs of horizontal edges. This implies that lifting an edge from B to Γ, mapping it to Γ' and pushing it down to B' induces a well-defined graph isomorphism $\varphi \colon B \to B'$. Since $\tilde{\varphi}$ is a GKM isomorphism there is some automorphism ψ of $\mathbb{Z}_{\mathfrak{t}}^*$ such that for any edge e in Γ we have $\psi(\alpha(e)) = \alpha'(\tilde{\varphi}(e))$, where α and α' denote the axial functions of Γ and Γ'. It follows that if $\tilde{e}$ is an edge in Γ over some edge $e \in E(B)$, then

$$\psi(\alpha_B(e)) = \psi(\alpha(\tilde{e})) = \alpha'(\tilde{\varphi}(\tilde{e})) = \alpha_{B'}(\varphi(e)).$$

Thus, φ is a GKM isomorphism.

We enumerate the vertices and edges of B and choose weights γ_i as in Remark 4.6. We give B' the enumeration induced by φ and choose signs of γ_i' such that $\psi(\gamma_i) = \gamma_i'$. If the orientations of the vertical edges are chosen compatibly with $\tilde{\varphi}$ then the corresponding weights α_i and α_i' and the resulting integers k_i and k_i' as in Proposition 4.5 satisfy

$$k_i' \gamma_{i-1}' - k_{i-1}' \gamma_i' = \alpha_i' = \psi(\alpha_i) = k_i \psi(\gamma_{i-1}) - k_{i-1} \psi(\gamma_i).$$

Thus, $k_i = k_i'$ which proves one direction of the proposition. Conversely one easily checks that given $\varphi \colon B \to B'$ satisfying the conditions of the proposition, any graph isomorphism $\tilde{\varphi} \colon \Gamma \to \Gamma'$ covering φ is a GKM isomorphism. $\square$

9. Cohomology and characteristic classes

This section is devoted to computing the cohomology ring as well as the Chern classes of the realizations of the GKM fibrations. All cohomology rings in this section are with respect to integer coefficients.

First we remind the reader how the Chern classes of the total space of a projectivized bundle are computed in terms of the Chern classes of the bundle and the base. Note that usually this is done in case the bundle $E \to X$ is a holomorphic vector bundle over some complex manifold X. However, the same computations work in case of a 4-dimensional quasitoric base manifold or S^4.

Assume that X is a 4-dimensional stable almost complex manifold and $E \to X$ a complex vector bundle of rank 2, and denote by $\pi \colon \mathbb{P}(E) \to X$ the canonical projection. The vertical distribution $V \subset T\mathbb{P}(E)$ is a complex vector bundle (cf. §6.1), thus $\mathbb{P}(E)$ has a stable almost complex structure induced by the decomposition $T\mathbb{P}(E) = V \oplus \pi^*(TX)$. Let $L \to \mathbb{P}(E)$ be the relative tautological bundle, i.e. it restricts to every $\mathbb{CP}^1$-fiber of $\pi \colon \mathbb{P}(E) \to X$ to the tautological bundle over this fiber. From [BT82, p. 270 (20.7)] we have that the cohomology ring $H^*(\mathbb{P}(E))$

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is the quotient ring of the polynomial ring $H^*(X)[x]$ by the ideal I generated by

$$x^2 + c_1(E)x + c_2(E)$$

where $x := c_1(\overline{L}) \in H^*(\mathbb{P}(E))$ and $\overline{L}$ denotes the dual bundle of L. We obtain the following.

PROPOSITION 9.1. *The Chern classes of the stable almost complex structure of $\mathbb{P}(E)$ in the ring $H^*(X)[x]/I$ are given by*

$$c_1(\mathbb{P}(E)) = c_1(X) + c_1(E) + 2x$$

and

$$c_2(\mathbb{P}(E)) = c_2(X) + c_1(E)c_1(X) + 2c_1(X)x.$$

Clearly $c_3(\mathbb{P}(E))$ is determined by the Euler characteristic of $\mathbb{P}(E)$ which is equal to twice the Euler characteristic of X.

Proof. The relative Euler sequence [Huy05, Remark 2.4.5] also holds in this setting, i.e. we have a short exact sequence of complex vector bundles

$$0 \longrightarrow \mathbb{C} \longrightarrow \pi^*(E) \otimes \overline{L} \longrightarrow V \longrightarrow 0,$$

where $\mathbb{C}$ is the trivial vector bundle. Let c denote the total Chern class, then $c(\mathbb{P}(E)) = \pi^*(c(X))c(V)$ in $H^*(\mathbb{P}(E))$. From the relative Euler sequence we infer $c(V) = c(\pi^*(E) \otimes \overline{L})$ and using the splitting principle we obtain for the tensor product

$$c_1(\pi^*(E) \otimes \overline{L}) = \pi^*(c_1(E)) + 2c_1(\overline{L}),$$

as well as

$$c_2(\pi^*(E) \otimes \overline{L}) = \pi^*(c_2(E)) + \pi^*(c_1(E))c_1(\overline{L}) + c_1(\overline{L})^2.$$

Identifying now $H^*(\mathbb{P}(E))$ with $H^*(X)[x]/I$ we compute

$$c_1(\mathbb{P}(E)) = c_1(X) + c_1(E) + 2x$$

and

$$\begin{aligned}
c_2(\mathbb{P}(E)) &= c_1(E)c_1(X) + 2c_1(X)x + c_2(E) + c_1(E)x + x^2 + c_2(X) \\
&= c_1(E)c_1(X) + 2c_1(X)x + c_2(X),
\end{aligned}$$

where we used that $c_2(E) + c_1(E)x + x^2$ is zero in $H^*(X)[x]/I$. $\qquad\square$

We suppose X is a quasitoric manifold of dimension 4 or S^4 and T the 2-torus acting on X. We use the notation from Remark 4.6 for the GKM graph of X. We denote by δ_i the element of $\bigoplus_{i=1}^n H^*(BT) = H_T^*(X^T)$ which is zero, except at v_i, where it is equal to $1 \in H^*(BT)$. We extend the notation to $\delta_{i+n} = \delta_i$. From [MP06, Theorem 7.7] we infer that the equivariant cohomology of $H_T^*(X) \subset H_T^*(X^T)$ is generated by

$$\beta_i := -\gamma_{i-1}\delta_i + \gamma_{i+1}\delta_{i+1}$$

for $i = 1, \ldots, n$ (the β_i are, in the language of [MP06, §6.2], the Thom classes of the 2-dimensional submanifolds corresponding to the edges in the GKM graph of X).

LEMMA 9.2. *Let $E \to X$ be a T-equivariant complex vector bundle of rank 2 satisfying condition (i) of Theorem 5.7 (for some $(a_1, \ldots, a_n), (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $\eta \in \{0, 1\}$) and denote by $c_i^T(E) \in H_T^*(X)$ the integral ith equivariant Chern class of $E \to X$. The image of $c_1^T(E)$ under*

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the inclusion map $H_T^*(X) \to H_T^*(X^T)$ is given by

$$\sum_{i=1}^{n}(k_i - 2a_i)\beta_i$$

and that of $c_2^T(E)$ by

$$\sum_{i=1}^{n}(2a_i a_{i-1} - a_{i-1}k_i - a_i k_{i-1})\beta_{i-1}\beta_i + \sum_{i=1}^{n}(a_i^2 - a_i k_i)\beta_i^2,$$

where we set $a_0 = (-1)^{\varepsilon_1}a_n + \eta k_0$, $k_0 = (-1)^{\varepsilon_1 + \eta}k_n$, and $\beta_0 = (-1)^{\varepsilon_1}\beta_n$.

Proof. The image of the total Chern class $c^T(E)$ in $H_T^*(X^T)$ is given by (see [GKZ19, Proposition 5.3])

$$\sum_{i=1}^{n}(1 + \alpha_{i1})(1 + \alpha_{i2})\delta_i,$$

where α_{ij} are the weights of the T-representation on the fiber over the fixed point v_i. By abuse of notation, we write also $c^T(E) \in H_T^*(X^T)$ for the image of $c^T(E)$ under the inclusion homomorphism. By assumption, the weights α_{ij} over v_i are given by

$$\alpha_{i1} = a_i \gamma_{i-1} - a_{i-1}\gamma_i, \quad \text{and} \quad \alpha_{i2} = (a_i - k_i)\gamma_{i-1} + (-a_{i-1} + k_{i-1})\gamma_i$$

and, therefore,

$$c_1^T(E) = \sum_{i=1}^{n}((2a_i - k_i)\gamma_{i-1} + (-2a_{i-1} + k_{i-1})\gamma_i)\delta_i.$$

Using that $k_0 = (-1)^{\eta + \varepsilon_1}k_n$ (cf. §5.2) we compute

$$c_1^T(E) = (2a_1 - k_1)\gamma_0\delta_1 + (-2((-1)^{\varepsilon_1}a_n + \eta k_0) + k_0)\gamma_1\delta_1$$

$$+ \sum_{i=2}^{n}((2a_i - k_i)\gamma_{i-1} + (k_{i-1} - 2a_{i-1})\gamma_i)\delta_i$$

$$= (2a_1 - k_1)\gamma_0\delta_1 - (2a_n - (2\eta + (-1)^{\eta})k_n)\gamma_{n+1}\delta_{n+1}$$

$$+ \sum_{i=2}^{n}((2a_i - k_i)\gamma_{i-1} + (k_{i-1} - 2a_{i-1})\gamma_i)\delta_i$$

$$= (2a_1 - k_1)\gamma_0\delta_1 - (2a_n - k_n)\gamma_{n+1}\delta_{n+1} + (2a_n - k_n)\gamma_{n-1}\delta_n$$

$$- (2a_1 - k_1)\gamma_2\delta_2 - \sum_{i=2}^{n-1}(2a_i - k_i)\beta_i$$

$$= \sum_{i=1}^{n}(k_i - 2a_i)\beta_i.$$

The second Chern class is given by

$$c_2^T(E) = \sum_{i=1}^{n}(a_i\gamma_{i-1} - a_{i-1}\gamma_i)((a_i - k_i)\gamma_{i-1} + (-a_{i-1} + k_{i-1})\gamma_i)\delta_i$$

$$= \sum_{i=1}^{n}((a_i^2 - a_i k_i)\gamma_{i-1}^2 + (-2a_i a_{i-1} + a_{i-1}k_i + a_i k_{i-1})\gamma_{i-1}\gamma_i + (a_{i-1}^2 - a_{i-1}k_{i-1})\gamma_i^2)\delta_i.$$

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Let us examine the middle term first. We have

$$\sum_{i=1}^{n}(-2a_i a_{i-1} + a_{i-1}k_i + a_i k_{i-1})\gamma_{i-1}\gamma_i\delta_i = \sum_{i=1}^{n}(2a_i a_{i-1} - a_{i-1}k_i - a_i k_{i-1})\beta_{i-1}\beta_i,$$

where $\beta_0 := -\gamma_{-1}\delta_n + \gamma_1\delta_1 = -(-1)^{\varepsilon_1}\gamma_{n-1}\delta_n + (-1)^{\varepsilon_1}\gamma_{n+1}\delta_n = (-1)^{\varepsilon_1}\beta_n$. The remaining terms can be rearranged as follows:

$$\sum_{i=1}^{n}((a_i^2 - a_i k_i)\gamma_{i-1}^2 + (a_{i-1}^2 - a_{i-1}k_{i-1})\gamma_i^2)\delta_i$$

$$= \sum_{i=1}^{n}(a_i^2 - a_i k_i)\gamma_{i-1}^2\delta_i + \sum_{i=1}^{n-1}(a_i^2 - a_i k_i)\gamma_{i+1}^2\delta_{i+1} + (a_0^2 - a_0 k_0)\gamma_1^2\delta_1$$

$$= \sum_{i=1}^{n-1}(a_i^2 - a_i k_i)\beta_i^2 + (a_n^2 - a_n k_n)\gamma_{n-1}^2\delta_n + (a_0^2 - a_0 k_0)\gamma_1^2\delta_1$$

$$= \sum_{i=1}^{n}(a_i^2 - a_i k_i)\beta_i^2,$$

where in the last equality we used that $a_0^2 - a_0 k_0 = a_n^2 + (-1)^{\epsilon_1}a_n k_0(2\eta - 1) = a_n^2 - a_n k_n$. $\qquad\square$

The cohomology $H^*(X)$ is isomorphic to $H_T^*(X)/(H^{>0}(BT) \cdot H_T^*(X))$, see the proof of [MP06, Lemma 2.1]. We denote by $\overline{\beta}_i$ the elements in $H^*(X)$ which are the images of β_i under the projection map $H_T^*(X) \to H^*(X)$. Thus, the elements $\overline{\beta}_i$ generate $H^*(X)$.

When it comes to computing the cohomology of the projectivization of the vector bundles from Theorem 5.7 we note that the result does not depend on $a = (a_1, \ldots, a_n)$: the cohomology is completely determined by the GKM graph, on which a has no effect. Thus, we may set $a = 0$ (note that then $a_0 = \eta k_0$) and apply Lemma 9.2 to obtain the following.

COROLLARY 9.3. *Let $E \to X$ be a T-equivariant complex vector bundle of rank 2 as in Theorem 5.7. Then we have*

$$H^*(\mathbb{P}(E)) = H^*(X)[x]/\langle x^2 + \left(\sum_{i=1}^{n}k_i\overline{\beta}_i\right)x + \eta k_n k_1\overline{\beta}_1\overline{\beta}_n\rangle.$$

The above description of $H^*(\mathbb{P}(E))$ combines the combinatorial computation of the Chern classes with a topological result which exposes $H^*(\mathbb{P}(E))$ as a free $H^*(X)$-module in the sense of the Leray–Hirsch theorem together with additional multiplicative relations describing also the algebra structure. We note that a purely graph theoretic version of the Leray–Hirsch theorem has been obtained in [GZ01, Theorem 3.5]. It could be used instead to enable similar computations.

Any T-invariant (stable) almost complex structure on X induces such a structure on $\mathbb{P}(E)$, so that the decomposition $T\mathbb{P}(E) = V \oplus \pi^*(TX)$ is a decomposition of stable almost complex vector bundles, cf. Theorem 5.1 and § 6.1. Recall that at least a stable almost complex structure always exists in case X is quasitoric or S^4. We wish to compute the Chern classes of the resulting structure on $\mathbb{P}(E)$ using Proposition 9.1. Therefore, we first have to determine the Chern classes of X, which we will conduct using equivariant cohomology. For simplicity, we will restrict to the case of an almost complex structure on X and the choice of the γ_i will be assumed to be that of the resulting signed GKM structure. In this case, the equivariant Chern class of X is given by

$$c^T(X) = \sum_{i=1}^{n}(1 - \gamma_{i-1})(1 + \gamma_i)\delta_i.$$

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An easy computation shows that

$$c_1^T(X) = \sum_{i=1}^n \beta_i$$

and

$$c_2^T(X) = \sum_{i<j} \beta_i \beta_j.$$

When it comes to the vector bundles from Theorem 5.7, note that we may again assume $a = 0$ for the computation since the Chern classes depend only on the GKM graph. In total, we obtain the following result.

PROPOSITION 9.4. *If X is almost complex and $E \to X$ is as in Theorem 5.7, then the Chern classes of $\mathbb{P}(E)$ expressed in the Chow ring are given by*

$$c_1(\mathbb{P}(E)) = \sum_{i=1}^n (k_i + 1)\overline{\beta}_i + 2x$$

and

$$c_2(\mathbb{P}(E)) = \sum_{i<j} \overline{\beta}_i \overline{\beta}_j + \sum_{i,j} k_i \overline{\beta}_i \overline{\beta}_j + 2\sum_{i=1}^n \overline{\beta}_i x.$$

Finally, we would like to show in a special case, that the homotopy type of $\mathbb{P}(E)$ depends on k. We will consider the discriminant of the symmetric trilinear form given by the triple cup product on integer cohomology in dimension 2, cf. [OV95, §§ 3.1 and 5.2]: let N be a simply connected, closed, and orientable 6-manifold such that its second Betti number is equal to 2. Choose a basis (e_1, e_2) of $H^2(N)$ and an orientation homology class $[N] \in H_6(N)$. Consider the following integers using the cup product of $H^*(N)$

$$n_0 := \langle e_1^3, [N] \rangle, \quad n_1 := \langle e_1 e_2^2, [N] \rangle, \quad n_2 := \langle e_1^2 e_2, [N] \rangle, \quad n_3 := \langle e_2^3, [N] \rangle.$$

The number

$$\Delta_N := (n_0 n_3 - n_1 n_2)^2 - 4(n_0 n_2 - n_1^2)(n_1 n_3 - n_2^2)$$

is invariant under the action of $\mathrm{GL}(2, \mathbb{Z})$ on $H^2(N)$ and does not depend on the chosen orientation. Thus, it represents an invariant of the homotopy type of N.

Let us now assume $X = \mathbb{CP}^2$ with the standard action of T^2 and denote by B the GKM graph of X. Consider a signed GKM fibration $\Gamma \to B$ corresponding to (k_1, k_2, k_3, η) in the sense of 4.5 (with respect to some choice of data in B as in Remark 4.6). We assume that the fibration is of twisted type i.e. $\eta = 1$. Let E_k be a T^2-equivariant complex vector bundle as in Theorem 5.7 with $(a_1, \ldots, a_n) = 0$ such that $\mathbb{P}(E) \to X$ realizes $\Gamma \to B$ and denote by $\mathbb{P}_k$ the projectivization of E_k. Note that in $H^*(\mathbb{CP}^2)$ we have $\overline{\beta}_1 = \overline{\beta}_2 = \overline{\beta}_3$ and $\overline{\beta}_1\overline{\beta}_2 = \overline{\beta}_2\overline{\beta}_3 = \overline{\beta}_1\overline{\beta}_3$. From [OV95, Proposition 17] it follows that

$$\Delta_{\mathbb{P}_k} = c_1(E_k)^2 - 4c_2(E_k)$$

when interpreting the Chern classes in $H^*(\mathbb{CP}^2)$ as integers. Thus, using Lemma 9.2 we obtain

$$\Delta_{\mathbb{P}_k} = (k_1 + k_2 + k_3)^2 - 4\eta \cdot k_1 k_3,$$

which proves that one obtains infinitely many different homotopy types, when varying k. Note that the discussion on the non-Kählerness of the action in § 7 only cared about the signs of the k_i. Thus, we have the following.

Proposition 9.5. *There are infinitely many homotopy types among compact simply-connected 6-dimensional manifolds which carry a Hamiltonian GKM T^2-action with 6 fixed points but do not admit an invariant Kähler structure.*

Remark 9.6. Note that the stabilizers of the produced examples are connected if and only if all the k_i are ± 1. One can show that for any GKM fibration $\Gamma \to B$ of twisted type over the GKM graph B of $\mathbb{CP}^2$ with $k_i = \pm 1$ the GKM graph Γ is isomorphic to that of $SU(3)/T^2$ or of Eschenburg's twisted flag manifold $SU(3)//T^2$, cf. Examples 4.7 and 4.8. Thus, among the infinitely many examples of Hamiltonian T^2-actions we constructed for Proposition 9.5, the only one with connected stabilizers and without invariant compatible Kähler structure is the original Hamiltonian non-Kähler example due to Tolman. An example in which not all k_i are ± 1 is given as follows:
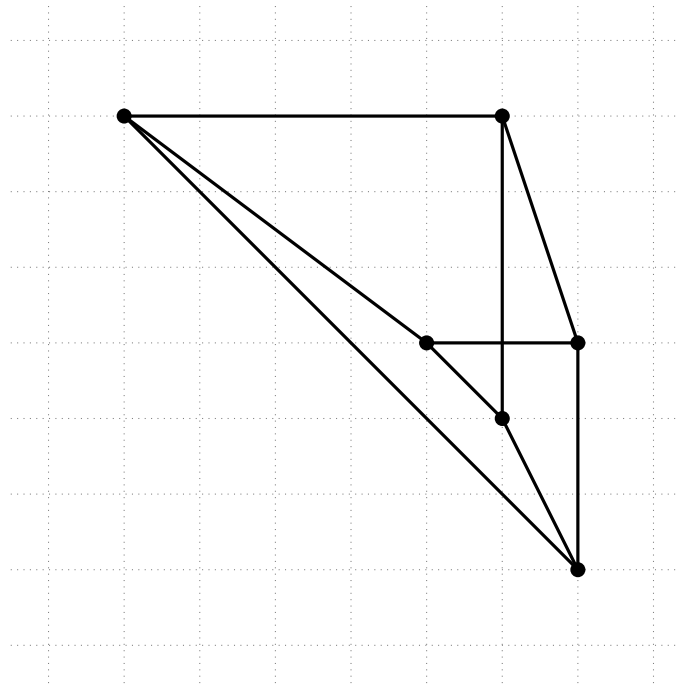
Conflicts of Interest
None.

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