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GENERALIZED MORREY REGULARITY FOR PARABOLIC EQUATIONS WITH DISCONTINUOUS DATA

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Abstract We prove continuity in generalized parabolic Morrey spaces $M_{p,\varphi}$ of sublinear operators generated by the parabolic Calderón–Zygmund operator and by the commutator of this operator with bounded mean oscillation (BMO) functions. As a consequence, we obtain a global $M_{p,\varphi}$-regularity result for the Cauchy–Dirichlet problem for linear uniformly parabolic equations with vanishing mean oscillation (VMO) coefficients.

Keywords: generalized Morrey spaces; sublinear operators; Calderón–Zygmund integrals; BMO; VMO; Cauchy–Dirichlet problem

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1. Introduction

The classical Morrey spaces $L_{p,\lambda}$ were originally introduced in [**17**] in order to prove local Hölder continuity of solutions to certain systems of partial differential equations (PDEs). A real-valued function f is said to belong to the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ with $p \in [1,\infty)$, $\lambda \in (0,n)$, provided that the norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \left(\sup_{(x,r)\in\mathbb{R}^n\times\mathbb{R}_+} \frac{1}{r^\lambda}\int_{\mathcal{B}_r(x)} |f(y)|^p \,\mathrm{d}y\right)^{1/p}.$$

is finite. The main result connected with these spaces is the following celebrated lemma: let $|Df| \in L_{p,\lambda}$ even locally, with $\lambda < p$; u is then Hölder continuous of exponent $\alpha = 1 - \lambda/p$. This result has applications in the study of the regularity of the strong solutions to elliptic and parabolic PDEs and systems. In [**5**] Chiarenza and Frasca showed boundedness of the Hardy–Littlewood maximal operator, which allowed them to prove continuity in $L_{p,\lambda}(\mathbb{R}^n)$ of some classical integral operators. These operators appear in the representation formulae of the solutions of various linear PDEs. Thus, the results in [**5**] allow us to study the regularity of the solutions of these operators in $L_{p,\lambda}$ (see [**20**, **21**, **24**] and the references therein). In [**16**] Mizuhara extended the Morrey concept of integral average over a ball with a certain growth, taking a weight function $\omega(x,r)\colon \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of r^λ. Thus he put the beginning of the study of the generalized Morrey spaces $L_{p,\omega}$, under various conditions on the weight ω. In [**18**] Nakai studied continuity of some classical integral operators in $L_{p,\omega}$ imposing the following doubling and integral conditions on ω:

$$C_1 \leqslant \frac{\omega(x,s)}{\omega(x,r)} \leqslant C_2, \quad r \leqslant s \leqslant 2r, \qquad \int_r^\infty \frac{\omega(x,s)}{s^{n+1}}\,\mathrm{d}s \leqslant C_3 \frac{\omega(x,r)}{r^n},$$

where the constants do not depend on s, r and x. Furthermore, in [**19**] Nakai extended the theory of Morrey spaces on homogeneous spaces (X,d,μ) endowed with a quasi-distance d and a non-negative measure μ. The generalized Morrey space is then defined to be the set of all $f \in L^1_{\mathrm{loc}}(X)$ such that

$$\|f\|_{L_{p,\phi}(X)} = \sup_{\mathcal{B}} \frac{1}{\phi(\mathcal{B})}\biggl(\frac{1}{\mu(\mathcal{B})}\int_{\mathcal{B}} |f(x)|^p\,\mathrm{d}\mu(x)\biggr)^{1/p},$$

where the supremo is taken over all balls $\mathcal{B} = \mathcal{B}(a,r)$ with respect to the quasi-distance d. The weight function $\phi(\mathcal{B}) = \phi(a,r)$ satisfies the integral condition

$$\int_r^\infty \frac{\phi(a,s)}{s}\,\mathrm{d}s \leqslant C\phi(a,r) \quad \forall a \in X,\ r > 0. \tag{1.1}$$

As a consequence, the boundedness of the Hardy–Littlewood maximal function and the Calderón–Zygmund singular integral operators in Morrey-type spaces on spaces of homogeneous type hold. Some applications of these operators in the regularity theory of partial differential equations are presented in [**23**, **25**, **26**]. Therein, the second author obtained global $L_{p,\omega}$-regularity for elliptic and parabolic boundary-value problems. The approach is based on explicit representation formulae for the higher-order derivatives of the solution and estimates for the above-mentioned operators.

Other generalizations of the Morrey spaces are considered in [**2**, **8**, **9**, **12**]. In these works Guliyev *et al.* studied the continuity of known integral operators acting from one Morrey-type space, $M_{p,\varphi_1}(\mathbb{R}^n)$, to another, $M_{p,\varphi_2}(\mathbb{R}^n)$, where the couple (φ_1,φ_2) satisfies the condition

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{s<\zeta<\infty} \varphi_1(x,\zeta)\zeta^{n/p}}{s^{(n+p)/p}}\,\mathrm{d}s \leqslant C\varphi_2(x,r), \quad (x,r) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{1.2}$$

The technique consists of obtaining some estimates for sublinear operators generated by various classical integral operators such as the Calderón–Zygmund operator, the commutator with bounded mean oscillation (BMO) functions, the Riesz potential, the Hardy–

Littlewood maximal operator and others. We note that the condition (1.2) is weaker than (1.1). Indeed, if (1.1) holds and $\varphi_1 = \varphi_2 = \varphi$, then

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{s<\zeta<\infty} \varphi(x,\zeta)\zeta^{n/p}}{s^{(n+p)/p}}\,\mathrm{d}s \leqslant \int_r^\infty \frac{\varphi(x,s)}{s}\,\mathrm{d}s \leqslant C\varphi(x,r).$$

The following example shows that there exist functions satisfying (1.2) but not (1.1).

Example 1.1. For $\beta \in (0, n/p)$, consider the weight function

$$\varphi(r) = r^{(\beta p-n)/p}\left|\sin\left(\max\left\{1, \frac{\pi}{r}\right\}\right)\right|.$$

Direct calculations give

$$\operatorname*{ess\,inf}_{r<\zeta<\infty} \varphi(\zeta)\zeta^{n/p} = \begin{cases} 0 & \text{if } r \in (0,\pi), \\ r^\beta \sin 1 & \text{if } r \in (\pi,\infty). \end{cases}$$

Then,

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{s<\zeta<\infty} \varphi(\zeta)\zeta^{n/p}}{s^{(n+p)/p}}\,\mathrm{d}s = \begin{cases} 0 & \text{if } r \in (0,\pi), \\ r^{(\beta p-n)/p} \sin 1 & \text{if } r \in (\pi,\infty) \end{cases} \leqslant C\varphi(r).$$

The function φ does not satisfy the condition (1.1).

In [**11**] we applied the results obtained in [**2**] to the study of the global $M_{p,\varphi}$-regularity for the Dirichlet problem for linear uniformly elliptic equations with vanishing mean oscillation (VMO) coefficients. In the present work we extend our study over singular integral operators with a parabolic-type kernel. As a by-product we obtain regularity results for strong solutions of parabolic boundary-value problems. The unique strong solvability of the problem under consideration is guaranteed by [**3**]. Furthermore, its regularity has been studied in the frameworks of the Morrey, Morrey-type and weighted Lebesgue spaces, in [**21**], [**25**] and [**10**], respectively, while in [**27**] we deal with the oblique derivative problem in $M_{p,\varphi}$ spaces. Our approach is based on estimates for sublinear operators generated by singular integrals with parabolic kernels (see § 3). The singular integral operators enter in the interior representation formula of the derivatives $D_{ij}u$ of the solution of (2.1). In § 4 we establish continuity of sublinear operators generated by non-singular integral operators. A similar result also holds for the commutators of these operators with BMO functions. The last integrals are included in the boundary representation formula for $D_{ij}u$, and allow us to obtain a local *a priori* estimate near the boundary. The global *a priori* estimate for u is obtained in § 6.

The following notation is used throughout this paper:

- $x = (x', t), y = (y', \tau) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$;
- $x = (x'', x_n, t) \in \mathbb{D}^{n+1}_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$, $\mathbb{D}^{n+1}_- = \mathbb{R}^{n-1} \times \mathbb{R}_- \times \mathbb{R}_+$;
- $|\cdot|$ is the Euclidean metric, $|x| = (\sum_{i=1}^n x_i^2 + t^2)^{1/2}$;

- $\mathcal{B}_r(x') = \{y' \in \mathbb{R}^n : |x' - y'| < r\}$, $|\mathcal{B}_r| = Cr^n$;
- $\mathcal{I}_r(x) = \{y \in \mathbb{R}^{n+1} : |x' - y'| < r,\ |t - \tau| < r^2\}$, $|\mathcal{I}_r| = Cr^{n+2}$;
- $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$;
- $D_i u = \partial u/\partial x_i$, $Du = (D_1 u, \dots, D_n u)$, $u_t = \partial u/\partial t$;
- $D_{ij} u = \partial^2 u/\partial x_i \partial x_j$, $D^2 u = \{D_{ij} u\}_{ij=1}^n$ denotes the Hessian matrix of u;
- for any $f \in L_p(A)$, $A \subset \mathbb{R}^{n+1}$, we write

$$\|f\|_{p,A} \equiv \|f\|_{L_p(A)} = \left(\int_A |f(y)|^p \,\mathrm{d}y \right)^{1/p};$$

- the standard summation convention on repeated upper and lower indices is adopted;
- the letter C is used for various positive constants and may change from one occurrence to another.

2. Definitions and statement of the problem

In the following, besides the standard parabolic metric $\varrho(x) = \max(|x'|, |t|^{1/2})$ we use the equivalent one

$$\rho(x) = \left(\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2} \right)^{1/2}$$

considered by Fabes and Riviére in [**7**] (see also [**28**]). The topology induced by $\rho(x)$ consists of the ellipsoids

$$\mathcal{E}_r(x) = \left\{ y \in \mathbb{R}^{n+1} : \frac{|x' - y'|^2}{r^2} + \frac{|t - \tau|^2}{r^4} < 1 \right\}, \qquad |\mathcal{E}_r| = Cr^{n+2}, \qquad \mathcal{E}_1(x) \equiv \mathcal{B}_1(x).$$

It is easy to see that the metrics $\rho(\cdot)$ and $\varrho(\cdot)$ are equivalent. In fact, for each $\mathcal{E}_r$ there exist parabolic cylinders $\underline{\mathcal{I}}$ and $\overline{\mathcal{I}}$ with measure comparable to r^{n+2} such that $\underline{\mathcal{I}} \subset \mathcal{E}_r \subset \overline{\mathcal{I}}$. In what follows, all estimates obtained over ellipsoids also hold true over parabolic cylinders, and we use this property without explicit references (see [**28**]).

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$-domain and $Q = \Omega \times (0, T)$, $T > 0$, be a cylinder in $\mathbb{R}^{n+1}_+$. We give the definitions of the functional spaces that we are going to use.

Definition 2.1. Let $a \in L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$ and let $a_{\mathcal{E}_r} = |\mathcal{E}_r|^{-1} \int_{\mathcal{E}_r} a(y) \,\mathrm{d}y$ be the mean integral of a. Denote

$$\eta_a(R) = \sup_{r \leqslant R} \frac{1}{|\mathcal{E}_r|} \int_{\mathcal{E}_r} |f(y) - f_{\mathcal{E}_r}| \,\mathrm{d}y \quad \text{for every } R > 0,$$

where $\mathcal{E}_r$ ranges over all ellipsoids in $\mathbb{R}^{n+1}$. We say that the following hold.

- $a \in \mathrm{BMO}$ (*bounded mean oscillation* [**13**]) provided the following is finite:

$$\|a\|_* = \sup_{R>0} \eta_a(R).$$

The quantity $\|\cdot\|_*$ is a norm in a BMO modulo constant function under which BMO is a Banach space.

- $a \in \mathrm{VMO}$ (*vanishing mean oscillation* [**22**]) if $a \in \mathrm{BMO}$ and

$$\lim_{R\to 0} \eta_a(R) = 0.$$

The quantity $\eta_a(R)$ is called the VMO-modulus of a.

For any bounded cylinder Q we define $\mathrm{BMO}(Q)$ and $\mathrm{VMO}(Q)$ taking $a \in L_1(Q)$ and $Q_r(x) = Q \cap \mathcal{E}_r(x)$, $x \in Q$, instead of $\mathcal{E}_r$ in the definition above.

According to [**1, 14**], having a function $a \in \mathrm{BMO}(Q)$ or $\mathrm{VMO}(Q)$, it is possible to extend the function in the whole of $\mathbb{R}^{n+1}$ preserving its BMO-norm or VMO-modulus, respectively. In the following we use this property without explicit references. Any bounded uniformly continuous (BUC) function f with modulus of continuity $\omega_f(R)$ belongs to VMO with $\eta_f(R) = \omega_f(R)$. Besides that, BMO and VMO also contain discontinuous functions, and the following example shows the inclusion $W_{1,n+2}(\mathbb{R}^{n+1}) \subset \mathrm{VMO} \subset \mathrm{BMO}$.

Example 2.2. We have that

$$\begin{gathered} f(x) = |\log \rho(x)| \in \mathrm{BMO} \setminus \mathrm{VMO}, \qquad \sin f(x) \in \mathrm{BMO} \cap L_\infty(\mathbb{R}^{n+1}), \\ f_\alpha(x) = |\log \rho(x)|^\alpha \in \mathrm{VMO} \quad \text{for any } \alpha \in (0,1), \\ f_\alpha \in W_{1,n+2}(\mathbb{R}^{n+1}) \qquad \text{for } \alpha \in (0, 1 - 1/(n+2)), \\ f_\alpha \notin W_{1,n+2}(\mathbb{R}^{n+1}) \qquad \text{for } \alpha \in [1 - 1/(n+2), 1). \end{gathered}$$

Definition 2.3. Let $\varphi\colon \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function and $p \in [1, \infty)$. The generalized parabolic Morrey space $M_{p,\varphi}(\mathbb{R}^{n+1})$ consists of all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^{n+1})$ such that

$$\|f\|_{p,\varphi;\mathbb{R}^{n+1}} = \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r)^{-1} \biggl(r^{-(n+2)} \int_{\mathcal{E}_r(x)} |f(y)|^p \, \mathrm{d}y \biggr)^{1/p} < \infty.$$

The space $M_{p,\varphi}(Q)$ consists of $L_p(Q)$ functions provided the following norm is finite:

$$\|f\|_{p,\varphi;Q} = \sup_{(x,r)\in Q\times\mathbb{R}_+} \varphi(x,r)^{-1} \biggl(r^{-(n+2)} \int_{Q_r(x)} |f(y)|^p \, \mathrm{d}y \biggr)^{1/p}.$$

The generalized weak parabolic Morrey space $WM_{1,\varphi}(\mathbb{R}^{n+1})$ consists of all measurable functions such that

$$\|f\|_{WM_{1,\varphi}(\mathbb{R}^{n+1})} = \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r)^{-1} r^{-(n+2)} \|f\|_{WL_1(\mathcal{E}_r(x))},$$

where WL_1 denotes the weak L_1 space.

The generalized Sobolev–Morrey space $W^{2,1}_{p,\varphi}(Q)$, $p \in [1,\infty)$, consists of all Sobolev functions $u \in W^{2,1}_p(Q)$ with distributional derivatives $D^l_t D^s_x u \in M_{p,\varphi}(Q)$, $0 \leqslant 2l+|s| \leqslant 2$, endowed by the norm

$$\|u\|_{W^{2,1}_{p,\varphi}(Q)} = \|u_t\|_{p,\varphi;Q} + \sum_{|s|\leqslant 2} \|D^s u\|_{p,\varphi;Q}.$$

We also define the space

$$\overset{\circ}{W}{}^{2,1}_{p,\varphi}(Q) = \{u \in W^{2,1}_{p,\varphi}(Q)\colon u(x) = 0,\ x \in \partial Q\}, \quad \|u\|_{\overset{\circ}{W}{}^{2,1}_{p,\varphi}(Q)} = \|u\|_{W^{2,1}_{p,\varphi}(Q)},$$

where ∂Q means the parabolic boundary $\Omega \cup (\partial\Omega \times (0,T))$.

We consider the linear Cauchy–Dirichlet problem

$$u_t - a^{ij}(x) D_{ij} u(x) = f(x) \text{ for almost all (a.a.) } x \in Q, \quad u \in \overset{\circ}{W}{}^{2,1}_{p,\varphi}(Q), \tag{2.1}$$

where the coefficient matrix $\boldsymbol{a}(x) = \{a^{ij}(x)\}^n_{i,j=1}$ satisfies

$$\left.\begin{aligned} \exists\,\Lambda > 0\colon \Lambda^{-1}|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2 \text{ for a.a. } x \in Q,\ \forall\, \xi \in \mathbb{R}^n,\\ a^{ij}(x) = a^{ji}(x),\ \text{which implies } a^{ij} \in L_\infty(Q). \end{aligned}\right\} \tag{2.2}$$

Theorem 2.4 (main result). *Let $\boldsymbol{a} \in \mathrm{VMO}(Q)$ with $\eta_{\boldsymbol{a}} = \sum^n_{i,j=1} \eta_{a_{ij}}$ satisfy (2.2), and, for each $p \in (1,\infty)$, let $u \in \overset{\circ}{W}{}^{2,1}_p(Q)$ be a strong solution of (2.1). If $f \in M_{p,\varphi}(Q)$ with $\varphi(x,r)$ being a measurable positive function satisfying*

$$\int_r^\infty \left(1 + \ln\frac{s}{r}\right) \frac{\operatorname{ess\,inf}_{s<\zeta<\infty} \varphi(x,\zeta)\zeta^{(n+2)/p}}{s^{(n+2+p)/p}}\,\mathrm{d}s \leqslant C\varphi(x,r), \quad (x,r) \in Q \times \mathbb{R}_+, \tag{2.3}$$

then $u \in \overset{\circ}{W}{}^{2,1}_{p,\varphi}(Q)$ and

$$\|u\|_{\overset{\circ}{W}{}^{2,1}_{p,\varphi}(Q)} \leqslant C\|f\|_{p,\varphi;Q} \tag{2.4}$$

with $C = C(n,p,\Lambda,\partial\Omega,T,\eta_{\boldsymbol{a}},\|\boldsymbol{a}\|_{\infty;Q})$.

3. Sublinear operators generated by parabolic singular integrals in generalized Morrey spaces

Let $f \in L_1(\mathbb{R}^{n+1})$ be a function with a compact support and $a \in \mathrm{BMO}$. For any $x \notin \operatorname{supp} f$ define the sublinear operators T and T_a such that

$$|Tf(x)| \leqslant C \int_{\mathbb{R}^{n+1}} \frac{|f(y)|}{\rho(x-y)^{n+2}}\,\mathrm{d}y, \tag{3.1}$$

$$|T_a f(x)| \leqslant C \int_{\mathbb{R}^{n+1}} |a(x) - a(y)| \frac{|f(y)|}{\rho(x-y)^{n+2}}\,\mathrm{d}y. \tag{3.2}$$

Suppose, in addition, that both the operators are bounded in $L_p(\mathbb{R}^{n+1})$ satisfying the estimates

$$\|Tf\|_{p;\mathbb{R}^{n+1}} \leqslant C\|f\|_{p;\mathbb{R}^{n+1}}, \qquad \|T_a f\|_{p;\mathbb{R}^{n+1}} \leqslant C\|a\|_*\|f\|_{p;\mathbb{R}^{n+1}} \tag{3.3}$$

with constants independent of a and f. The following known result concerns the Hardy operator $Hg(r) = (1/r)\int_0^r g(s)\,\mathrm{d}s$, $r > 0$.

Theorem 3.1 (Carro *et al.* [4]). *The inequality*

$$\operatorname*{ess\,sup}_{r>0} w(r)Hg(r) \leqslant A \operatorname*{ess\,sup}_{r>0} v(r)g(r) \tag{3.4}$$

holds for all non-increasing functions $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ *if and only if*

$$A = C \sup_{r>0} \frac{w(r)}{r} \int_0^r \frac{\mathrm{d}s}{\operatorname{ess\,sup}_{0<\zeta<s} v(\zeta)} < \infty. \tag{3.5}$$

Lemma 3.2. *Let* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^{n+1})$, $p \in [1,\infty)$, *be such that*

$$\int_r^\infty s^{-(n+2+p)/p} \|f\|_{p;\mathcal{E}_s(x_0)}\,\mathrm{d}s < \infty \quad \forall (x_0, r) \in \mathbb{R}^{n+1} \times \mathbb{R}_+, \tag{3.6}$$

and let T *be a sublinear operator satisfying (3.1).*

(i) *If* $p > 1$ *and* T *is bounded on* $L_p(\mathbb{R}^{n+1})$, *then*

$$\|Tf\|_{p;\mathcal{E}_r(x_0)} \leqslant C r^{(n+2)/p} \int_{2r}^\infty s^{-(n+2+p)/p} \|f\|_{p;\mathcal{E}_s(x_0)}\,\mathrm{d}s. \tag{3.7}$$

(ii) *If* $p = 1$ *and* T *is bounded from* $L_1(\mathbb{R}^{n+1})$ *on* $WL_1(\mathbb{R}^{n+1})$, *then*

$$\|Tf\|_{WL_1(\mathcal{E}_r(x_0))} \leqslant C r^{n+2} \int_{2r}^\infty s^{-(n+3)} \|f\|_{1,\mathcal{E}_s(x_0)}\,\mathrm{d}s, \tag{3.8}$$

where the constants are independent of r, x_0 *and* f.

Proof. (i) Fix a point $x_0 \in \mathbb{R}^{n+1}$ and consider an ellipsoid $\mathcal{E}_r(x_0)$. Define $2\mathcal{E}_r(x_0) = \mathcal{E}_{2r}(x_0)$, $\mathcal{E}_r^c(x_0) = \mathbb{R}^{n+1} \setminus \mathcal{E}_r(x_0)$ and consider the decomposition of f,

$$f = f\chi_{2\mathcal{E}_r(x_0)} + f\chi_{2\mathcal{E}_r^c(x_0)} = f_1 + f_2.$$

Because of the (p,p)-boundedness of the operator T and $f_1 \in L_p(\mathbb{R}^{n+1})$ we have that

$$\|Tf_1\|_{p;\mathcal{E}_r(x_0)} \leqslant \|Tf_1\|_{p;\mathbb{R}^{n+1}} \leqslant C\|f_1\|_{p;\mathbb{R}^{n+1}} = C\|f\|_{p;2\mathcal{E}_r(x_0)}.$$

It is easy to see that for arbitrary points $x \in \mathcal{E}_r(x_0)$ and $y \in 2\mathcal{E}_r^c(x_0)$ it holds that

$$\tfrac{1}{2}\rho(x_0 - y) \leqslant \rho(x - y) \leqslant \tfrac{3}{2}\rho(x_0 - y). \tag{3.9}$$

Applying (3.1), (3.9), the Fubini theorem and the Hölder inequality to Tf_2, we get that

$$\begin{aligned}|Tf_2(x)| &\leqslant C\int_{2\mathcal{E}_r^c(x_0)} \frac{|f(y)|}{\rho(x_0-y)^{n+2}}\,\mathrm{d}y\\ &\leqslant C\int_{2\mathcal{E}_r^c(x_0)} |f(y)|\biggl(\int_{\rho(x_0-y)}^{\infty}\frac{\mathrm{d}s}{s^{n+3}}\biggr)\mathrm{d}y\\ &\leqslant C\int_{2r}^{\infty}\biggl(\int_{2r\leqslant\rho(x_0-y)<s}|f(y)|\,\mathrm{d}y\biggr)\frac{\mathrm{d}s}{s^{n+3}}\\ &\leqslant C\int_{2r}^{\infty}\biggl(\int_{\mathcal{E}_s(x_0)}|f(y)|\,\mathrm{d}y\biggr)\frac{\mathrm{d}s}{s^{n+3}}\\ &\leqslant C\int_{2r}^{\infty}\|f\|_{p;\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{(n+2+p)/p}}.\end{aligned}$$

Direct calculations give

$$\|Tf_2\|_{p,\mathcal{E}_r(x_0)} \leqslant Cr^{(n+2)/p}\int_{2r}^{\infty}\|f\|_{p;\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{(n+2+p)/p}}, \tag{3.10}$$

which holds for all $p\in[1,\infty)$. Thus,

$$\|Tf\|_{p;\mathcal{E}_r(x_0)} \leqslant C\biggl(\|f\|_{p;2\mathcal{E}_r(x_0)} + r^{(n+2)/p}\int_{2r}^{\infty}\|f\|_{p;\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{(n+2+p)/p}}\biggr). \tag{3.11}$$

On the other hand,

$$\|f\|_{p,2\mathcal{E}_r(x_0)} \leqslant Cr^{(n+2)/p}\int_{2r}^{\infty}\|f\|_{p;\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{(n+2+p)/p}}, \tag{3.12}$$

which, unified with (3.11), gives (3.7).

(ii) Let $f\in L_1(\mathbb{R}^{n+1})$; the weak $(1,1)$-boundedness of T implies that

$$\begin{aligned}\|Tf_1\|_{WL_1(\mathcal{E}_r(x_0))} &\leqslant \|Tf_1\|_{WL_1(\mathbb{R}^{n+1})}\\ &\leqslant C\|f_1\|_{1,\mathbb{R}^{n+1}}\\ &= C\|f\|_{1,2\mathcal{E}_r(x_0)}\\ &\leqslant Cr^{n+2}\int_{2r}^{+\infty}\|f\|_{1,\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{n+3}},\end{aligned}$$

which, unified with (3.10), gives (3.8). □

Theorem 3.3. *Let $p\in[1,\infty)$, let $\varphi(x,r)$ be a measurable positive function satisfying*

$$\int_r^{\infty}\frac{\operatorname{ess\,inf}_{s<\zeta<\infty}\varphi(x,\zeta)\zeta^{(n+2)/p}}{s^{(n+2+p)/p}}\,\mathrm{d}s \leqslant C\varphi(x,r)\quad \forall (x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+, \tag{3.13}$$

and let T be a sublinear operator satisfying (3.1).

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(i) *If $p > 1$ and T is bounded on $L_p(\mathbb{R}^{n+1})$, then T is bounded on $M_{p,\varphi}(\mathbb{R}^{n+1})$, and*

$$\|Tf\|_{p,\varphi;\mathbb{R}^{n+1}} \leqslant C\|f\|_{p,\varphi;\mathbb{R}^{n+1}}. \tag{3.14}$$

(ii) *If $p = 1$ and T is bounded from $L_1(\mathbb{R}^{n+1})$ to $WL_1(\mathbb{R}^{n+1})$, then it is bounded from $M_{1,\varphi}(\mathbb{R}^{n+1})$ to $WM_{1,\varphi}(\mathbb{R}^{n+1})$, and*

$$\|Tf\|_{WM_{1,\varphi}(\mathbb{R}^{n+1})} \leqslant C\|f\|_{1,\varphi;\mathbb{R}^{n+1}} \tag{3.15}$$

with constants independent on f.

Proof. (i) By Lemma 3.2, we have that

$$\begin{aligned}\|Tf\|_{p,\varphi;\mathbb{R}^{n+1}} &\leqslant C \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r)^{-1}\int_r^\infty \|f\|_{p;\mathcal{E}_s(x)}\frac{\mathrm{d}s}{s^{(n+2+p)/p}}\\ &= C \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r)^{-1}\int_0^{r^{-(n+2)/p}} \|f\|_{p;\mathcal{E}_{s^{-p/(n+2)}}(x)}\,\mathrm{d}s\\ &= C \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r^{-p/(n+2)})^{-1}\int_0^{r} \|f\|_{p;\mathcal{E}_{s^{-p/(n+2)}}(x)}\,\mathrm{d}s.\end{aligned}$$

Applying Theorem 3.1 with

$$w(r) = v(r) = r\varphi(x, r^{-p/(n+2)})^{-1}, \qquad g(r) = \|f\|_{p;\mathcal{E}_{r^{-p/(n+2)}}(x)},$$
$$Hg(r) = r^{-1}\int_0^r \|f\|_{p;\mathcal{E}_{s^{-p/(n+2)}}(x)}\,\mathrm{d}s,$$

where the condition (3.5) is equivalent to (3.13), we obtain (3.14).

(ii) The estimate follows after using (3.8) and (3.4):

$$\begin{aligned}\|Tf\|_{WM_{1,\varphi}(\mathbb{R}^{n+1})} &\leqslant C \sup_{(x_0,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x_0,r)^{-1}\int_r^\infty \|f\|_{1,\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{n+3}}\\ &= C \sup_{(x_0,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x_0,r^{-1/(n+2)})^{-1}\int_0^r \|f\|_{1,\mathcal{E}_{s^{-1/(n+2)}}(x_0)}\,\mathrm{d}s\\ &\leqslant C \sup_{(x_0,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x_0,r^{-1/(n+2)})^{-1} r\|f\|_{1,\mathcal{E}_{r^{-1/(n+2)}}(x_0)}\\ &= C\|f\|_{1,\varphi;\mathbb{R}^{n+1}}.\end{aligned}$$

□

Our next step is to show boundedness of T_a in $M_{p,\varphi}(\mathbb{R}^{n+1})$. For this we recall some properties of the BMO functions.

Lemma 3.4 (John–Nirenberg lemma [3, Lemma 2.8]). *Let $a \in \mathrm{BMO}$ and $p \in [1,\infty)$. Then, for any $\mathcal{E}_r$,*

$$\left(\frac{1}{|\mathcal{E}_r|}\int_{\mathcal{E}_r} |a(y) - a_{\mathcal{E}_r}|^p\,\mathrm{d}y\right)^{1/p} \leqslant C(p)\|a\|_*.$$

As an immediate consequence of Lemma 3.4 we get the following property.

Corollary 3.5. *Let $a \in \mathrm{BMO}$. Then, for all $0 < 2r < s$,*

$$|a_{\mathcal{E}_r} - a_{\mathcal{E}_s}| \leqslant C(n)\left(1 + \ln\frac{s}{r}\right)\|a\|_*. \tag{3.16}$$

Proof. Since $s > 2r$, there exists $k \in \mathbb{N}$, $k \geqslant 1$, such that $2^k r < s \leqslant 2^{k+1} r$ and, hence, $k \ln 2 < \ln(s/r) \leqslant (k+1)\ln 2$. By [**3**, Lemma 2.9] we have that

$$\begin{aligned}
|a_{\mathcal{E}_s} - a_{\mathcal{E}_r}| &\leqslant |a_{2^k\mathcal{E}_r} - a_{\mathcal{E}_r}| + |a_{2^k\mathcal{E}_r} - a_{\mathcal{E}_s}| \\
&\leqslant C(n)k\|a\|_* + \frac{1}{|2^k\mathcal{E}_r|}\int_{2^k\mathcal{E}_r} |a(y) - a_{\mathcal{E}_s}|\,\mathrm{d}y \\
&\leqslant C(n)\left(k\|a\|_* + \frac{1}{|\mathcal{E}_s|}\int_{\mathcal{E}_s} |a(y) - a_{\mathcal{E}_s}|\,\mathrm{d}y\right) \\
&< C(n)\left(\ln\frac{s}{r} + 1\right)\|a\|_*.
\end{aligned}$$

□

To estimate the norm of T_a we employ the same idea that we have used in the proof of Lemma 3.2.

Lemma 3.6. *Let $a \in \mathrm{BMO}$ and let T_a be a bounded operator in $L_p(\mathbb{R}^{n+1})$, $p \in (1, \infty)$, satisfying (3.2) and (3.3). Suppose that, for any $f \in L_p^{\mathrm{loc}}(\mathbb{R}^{n+1})$,*

$$\int_r^\infty \left(1 + \ln\frac{s}{r}\right)\|f\|_{p;\mathcal{E}_s(x_0)} \frac{\mathrm{d}s}{s^{(n+2+p)/p}} < \infty \quad \forall (x_0, r) \in \mathbb{R}^{n+1} \times \mathbb{R}_+. \tag{3.17}$$

Then,

$$\|T_a f\|_{p;\mathcal{E}_r(x_0)} \leqslant C\|a\|_* r^{(n+2)/p} \int_{2r}^\infty \left(1 + \ln\frac{s}{r}\right)\|f\|_{p;\mathcal{E}_s(x_0)} \frac{\mathrm{d}s}{s^{(n+2+p)/p}}, \tag{3.18}$$

where C is independent of a, f, x_0 and r.

Proof. Fix a point $x_0 \in \mathbb{R}^{n+1}$ and consider the decomposition

$$f = f\chi_{2\mathcal{E}_r(x_0)} + f\chi_{2\mathcal{E}_r^{\mathrm{c}}(x_0)} = f_1 + f_2.$$

Hence,

$$\|T_a f\|_{p;\mathcal{E}_r(x_0)} \leqslant \|T_a f_1\|_{p;\mathcal{E}_r(x_0)} + \|T_a f_2\|_{p;\mathcal{E}_r(x_0)}$$

and by (3.3) as in Lemma 3.2 we have that

$$\|T_a f_1\|_{p;\mathcal{E}_r(x_0)} \leqslant C\|a\|_* \|f\|_{p;2\mathcal{E}_r(x_0)}. \tag{3.19}$$

On the other hand, because of (3.9) we can write

$$\begin{aligned}\|T_a f_2\|_{p;\mathcal{E}_r(x_0)} &\leqslant C\bigg(\int_{\mathcal{E}_r(x_0)}\bigg(\int_{2\mathcal{E}_r^c(x_0)}\frac{|a(x)-a(y)||f(y)|}{\rho(x_0-y)^{n+2}}\,\mathrm{d}y\bigg)^p\mathrm{d}x\bigg)^{1/p}\\ &\leqslant C\bigg(\int_{\mathcal{E}_r(x_0)}\bigg(\int_{2\mathcal{E}_r^c(x_0)}\frac{|a(y)-a_{\mathcal{E}_r(x_0)}||f(y)|}{\rho(x_0-y)^{n+2}}\,\mathrm{d}y\bigg)^p\mathrm{d}x\bigg)^{1/p}\\ &\quad+C\bigg(\int_{\mathcal{E}_r(x_0)}\bigg(\int_{2\mathcal{E}_r^c(x_0)}\frac{|a(x)-a_{\mathcal{E}_r(x_0)}||f(y)|}{\rho(x_0-y)^{n+2}}\,\mathrm{d}y\bigg)^p\mathrm{d}x\bigg)^{1/p}\\ &= I_1+I_2.\end{aligned}$$

Applying (3.2), the Fubini theorem and the Hölder inequality as in Lemmas 3.2 and 3.4, we get

$$\begin{aligned}I_1 &\leqslant Cr^{(n+2)/p}\bigg(\int_{2r}^{\infty}\int_{\mathcal{E}_s(x_0)}|a(y)-a_{\mathcal{E}_r(x_0)}||f(y)|\,\mathrm{d}y\bigg)\frac{\mathrm{d}s}{s^{n+3}}\\ &\leqslant Cr^{(n+2)/p}\bigg(\int_{2r}^{\infty}\int_{\mathcal{E}_s(x_0)}|a(y)-a_{\mathcal{E}_s(x_0)}||f(y)|\,\mathrm{d}y\bigg)\frac{\mathrm{d}s}{s^{n+3}}\\ &\quad+Cr^{(n+2)/p}\int_{2r}^{\infty}|a_{\mathcal{E}_r(x_0)}-a_{\mathcal{E}_s(x_0)}|\bigg(\int_{\mathcal{E}_s(x_0)}|f(y)|\,\mathrm{d}y\bigg)\frac{\mathrm{d}s}{s^{n+3}}\\ &\leqslant Cr^{(n+2)/p}\int_{2r}^{\infty}\bigg(\int_{\mathcal{E}_s(x_0)}|a(y)-a_{\mathcal{E}_s(x_0)}|^{p/(p-1)}\,\mathrm{d}y\bigg)^{(p-1)/p}\|f\|_{p;\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{n+3}}\\ &\quad+Cr^{(n+2)/p}\int_{2r}^{\infty}|a_{\mathcal{E}_r(x_0)}-a_{\mathcal{E}_s(x_0)}|\|f\|_{p;\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{(n+2+p)/p}}\\ &\leqslant C\|a\|_*r^{(n+2)/p}\int_{2r}^{\infty}\bigg(1+\ln\frac{s}{r}\bigg)\|f\|_{p;\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{(n+2+p)/p}}.\end{aligned}$$

In order to estimate I_2 we note that

$$I_2=\bigg(\int_{\mathcal{E}_r(x_0)}|a(x)-a_{\mathcal{E}_r(x_0)}|^p\,\mathrm{d}x\bigg)^{1/p}\int_{2\mathcal{E}_r^c(x_0)}\frac{|f(y)|}{\rho(x_0-y)^{n+2}}\,\mathrm{d}y.$$

By Lemma 3.4 and (3.10) we get that

$$\begin{aligned}I_2 &\leqslant C\|a\|_*r^{(n+2)/p}\int_{2\mathcal{E}_r^c(x_0)}\frac{|f(y)|}{\rho(x_0-y)^{n+2}}\,\mathrm{d}y\\ &\leqslant C\|a\|_*r^{(n+2)/p}\int_{2r}^{\infty}\|f\|_{p;\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{(n+2+p)/p}}.\end{aligned}$$

Summing up (3.19), I_1 and I_2 we get that

$$\|T_a f\|_{p;\mathcal{E}_r(x_0)}\leqslant C\|a\|_*\bigg(\|f\|_{p;2\mathcal{E}_r(x_0)}+r^{(n+2)/p}\int_{2r}^{\infty}\bigg(1+\ln\frac{s}{r}\bigg)\|f\|_{p;\mathcal{E}_s(x_0)}\frac{\mathrm{d}s}{s^{(n+2+p)/p}}\bigg)$$

and the statement follows after applying (3.12). □

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Theorem 3.7. *Let $p \in (1,\infty)$ and $\varphi(x,r)$ be measurable positive functions such that*

$$\int_r^\infty \left(1+\ln\frac{s}{r}\right)\frac{\operatorname{ess\,inf}_{s<\zeta<\infty}\varphi(x,\zeta)\zeta^{(n+2)/p}}{s^{(n+2+p)/p}}\,\mathrm{d}s \leqslant C\varphi(x,r) \quad \forall(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+ \tag{3.20}$$

where C is independent of x and r. Suppose that $a \in \mathrm{BMO}$ and let T_a be a sublinear operator satisfying (3.2). If T_a is bounded in $L_p(\mathbb{R}^{n+1})$, then it is bounded in $M_{p,\varphi}(\mathbb{R}^{n+1})$, and

$$\|T_a f\|_{p,\varphi;\mathbb{R}^{n+1}} \leqslant C\|a\|_*\|f\|_{p,\varphi;\mathbb{R}^{n+1}} \tag{3.21}$$

with a constant independent of a and f.

The statement of the theorem follows by Lemma 3.6 and Theorem 3.1 in the same manner as for Theorem 3.3.

Example 3.8. The functions

$$\varphi(x,r)=r^{\beta-(n+2)/p} \quad \text{and} \quad \varphi(x,r)=r^{\beta-(n+2)/p}\log^m(e+r),$$

with $0<\beta<(n+2)/p$ and $m \geqslant 1$, are weight functions satisfying the condition (3.20).

4. Sublinear operators generated by non-singular integrals in generalized Morrey spaces

For any $x \in \mathbb{D}_+^{n+1}$, define $\tilde{x}=(x'',-x_n,t)\in\mathbb{D}_-^{n+1}$ and $x^0=(x'',0,0)\in\mathbb{R}^{n-1}$. Consider the semi-ellipsoids $\mathcal{E}_r^+(x^0)=\mathcal{E}_r(x^0)\cap\mathbb{D}_+^{n+1}$. Let $f\in L_1(\mathbb{D}_+^{n+1})$, let $a\in\mathrm{BMO}(\mathbb{D}_+^{n+1})$, and let $\tilde{T}$ and $\tilde{T}_a$ be sublinear operators such that

$$|\tilde{T}f(x)| \leqslant C\int_{\mathbb{D}_+^{n+1}}\frac{|f(y)|}{\rho(\tilde{x}-y)^{n+2}}\,\mathrm{d}y, \tag{4.1}$$

$$|\tilde{T}_a f(x)| \leqslant C\int_{\mathbb{D}_+^{n+1}}|a(x)-a(y)|\frac{|f(y)|}{\rho(\tilde{x}-y)^{n+2}}\,\mathrm{d}y. \tag{4.2}$$

Suppose, in addition, that both the operators are bounded in $L_p(\mathbb{D}_+^{n+1})$, satisfying the estimates

$$\|\tilde{T}f\|_{p;\mathbb{D}_+^{n+1}} \leqslant C\|f\|_{p;\mathbb{D}_+^{n+1}}, \qquad \|\tilde{T}_a f\|_{p;\mathbb{D}_+^{n+1}} \leqslant C\|a\|_*\|f\|_{p;\mathbb{D}_+^{n+1}} \tag{4.3}$$

with constants independent of a and f. The following assertions can be proved in the same manner as in § 3.

Lemma 4.1. *Let $f\in L_p^{\mathrm{loc}}(\mathbb{D}_+^{n+1})$, $p\in(1,\infty)$, and, for all $(x^0,r)\in\mathbb{R}^{n-1}\times\mathbb{R}_+$,*

$$\int_r^\infty s^{-(n+2+p)/p}\|f\|_{p;\mathcal{E}_s^+(x^0)}\,\mathrm{d}s<\infty. \tag{4.4}$$

If $\tilde{T}$ is bounded on $L_p(\mathbb{D}_+^{n+1})$, then

$$\|\tilde{T}f\|_{p;\mathcal{E}_r^+(x^0)} \leqslant Cr^{(n+2)/p}\int_{2r}^\infty s^{-(n+2+p)/p}\|f\|_{p;\mathcal{E}_s^+(x^0)}\,\mathrm{d}s, \tag{4.5}$$

where the constant C is independent of r, x^0 and f.

Theorem 4.2. *Let φ be a weight function satisfying (3.13), and let $\tilde{T}$ be a sublinear operator satisfying (4.1) and (4.3). Then $\tilde{T}$ is bounded in $M_{p,\varphi}(\mathbb{D}^{n+1}_+)$, $p \in (1,\infty)$, and*

$$\|\tilde{T}f\|_{p,\varphi;\mathbb{D}^{n+1}_+} \leqslant C\|f\|_{p,\varphi;\mathbb{D}^{n+1}_+} \tag{4.6}$$

with a constant C independent of f.

Lemma 4.3. *Let $p \in (1,\infty)$, let $a \in \mathrm{BMO}(\mathbb{D}^{n+1}_+)$, and let $\tilde{T}_a$ satisfy (4.2) and (4.3). Suppose that, for all $f \in L^{\mathrm{loc}}_p(\mathbb{D}^{n+1}_+)$,*

$$\int_r^\infty \left(1 + \ln\frac{s}{r}\right) s^{-(n+2+p)/p}\|f\|_{p;\mathcal{E}^+_s(x^0)}\,\mathrm{d}s < \infty \quad \forall (x^0, r) \in \mathbb{R}^{n-1} \times \mathbb{R}_+. \tag{4.7}$$

Then,

$$\|\tilde{T}_a f\|_{p;\mathcal{E}^+_r(x^0)} \leqslant C\|a\|_* r^{(n+2)/p}\int_{2r}^\infty \left(1 + \ln\frac{s}{r}\right)\|f\|_{p;\mathcal{E}^+_s(x^0)}\frac{\mathrm{d}s}{s^{(n+2+p)/p}}$$

with a constant C independent of a, f, x^0 and r.

Theorem 4.4. *Let $p \in (1,\infty)$, $a \in \mathrm{BMO}(\mathbb{D}^{n+1}_+)$, let $\varphi(x^0, r)$ be a weight function satisfying (3.20) and $\tilde{T}_a$ be a sublinear operator satisfying (3.2) and (3.3). Then $\tilde{T}_a$ is bounded in $M_{p,\varphi}(\mathbb{D}^{n+1}_+)$, and*

$$\|\tilde{T}_a f\|_{p,\varphi;\mathbb{D}^{n+1}_+} \leqslant C\|a\|_*\|f\|_{p,\varphi;\mathbb{D}^{n+1}_+} \tag{4.8}$$

with a constant C independent of a and f.

5. Singular and non-singular integrals in generalized Morrey spaces

In the present section we apply the above results to Calderón–Zygmund-type operators with parabolic kernel. Since these operators are sublinear and bounded in $L_p(\mathbb{R}^{n+1})$, their continuity in $M_{p,\varphi}$ follows immediately.

Definition 5.1. A measurable function $\mathcal{K}(x,\xi)\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is called a variable parabolic Calderón–Zygmund kernel if the following hold.

(i) $\mathcal{K}(x,\cdot)$ is a parabolic Calderón–Zygmund kernel for a.a. $x \in \mathbb{R}^{n+1}$:

(a) $\mathcal{K}(x,\cdot) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$,

(b) $\mathcal{K}(x,\mu\xi) = \mu^{-(n+2)}\mathcal{K}(x,\xi)$ for all $\mu > 0$,

(c)
$$\int_{\mathbb{S}^n} \mathcal{K}(x,\xi)\,\mathrm{d}\sigma_\xi = 0, \qquad \int_{\mathbb{S}^n} |\mathcal{K}(x,\xi)|\,\mathrm{d}\sigma_\xi < +\infty.$$

(ii) $\|D^\beta_\xi \mathcal{K}\|_{L_\infty(\mathbb{R}^{n+1}\times\mathbb{S}^n)} \leqslant M(\beta) < \infty$ for every multi-index β.

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Moreover,

$$|\mathcal{K}(x,x-y)| \leqslant \rho(x-y)^{-(n+2)}\left|\mathcal{K}\left(x,\frac{x-y}{\rho(x-y)}\right)\right| \leqslant \frac{M}{\rho(x-y)^{n+2}},$$

which means that the singular integrals

$$\left.\begin{aligned} \mathfrak{K}f(x) &= \mathrm{PV}\int_{\mathbb{R}^{n+1}} \mathcal{K}(x,x-y)f(y)\,\mathrm{d}y, \\ \mathfrak{C}[a,f](x) &= \mathrm{PV}\int_{\mathbb{R}^{n+1}} \mathcal{K}(x,x-y)[a(y)-a(x)]f(y)\,\mathrm{d}y \end{aligned}\right\} \tag{5.1}$$

are sublinear and bounded in $L_p(\mathbb{R}^{n+1})$ according to the results in [**3**, **7**]. We note that any weight function φ satisfying (3.20) also satisfies (3.13) and, hence, the following holds as a simple application of the estimates proved in § 3.

Theorem 5.2. *For any $f \in M_{p,\varphi}(\mathbb{R}^{n+1})$ with (p,φ) as in Theorem 3.7 and $a \in \mathrm{BMO}$, there exist constants depending on n, p and the kernel such that*

$$\|\mathfrak{K}f\|_{p,\varphi;\mathbb{R}^{n+1}} \leqslant C\|f\|_{p,\varphi;\mathbb{R}^{n+1}}, \qquad \|\mathfrak{C}[a,f]\|_{p,\varphi;\mathbb{R}^{n+1}} \leqslant C\|a\|_*\|f\|_{p,\varphi;\mathbb{R}^{n+1}}. \tag{5.2}$$

Corollary 5.3. *Let Q be a cylinder in $\mathbb{R}^{n+1}_+$, $f \in M_{p,\varphi}(Q)$, $a \in \mathrm{BMO}(Q)$ and $\mathcal{K}(x,\xi)\colon Q\times\mathbb{R}^{n+1}_+\setminus\{0\}\to\mathbb{R}$. Then the operators (5.1) are bounded in $M_{p,\varphi}(Q)$ and*

$$\|\mathfrak{K}f\|_{p,\varphi;Q} \leqslant C\|f\|_{p,\varphi;Q}, \qquad \|\mathfrak{C}[a,f]\|_{p,\varphi;Q} \leqslant C\|a\|_*\|f\|_{p,\varphi;Q} \tag{5.3}$$

with C independent of a and f.

Proof. Define the extensions

$$\bar{\mathcal{K}}(x,\xi) = \begin{cases} \mathcal{K}(x,\xi), & (x,\xi)\in Q\times\mathbb{R}^{n+1}_+\setminus\{0\}, \\ 0 & \text{elsewhere}, \end{cases} \qquad \bar{f}(x) = \begin{cases} f(x), & x\in Q, \\ 0, & x\notin Q. \end{cases}$$

Denote by $\bar{\mathfrak{K}}f$ the singular integral with a kernel $\bar{\mathcal{K}}$ and potential $\bar{f}$. Then

$$|\mathfrak{K}f| \leqslant |\bar{\mathfrak{K}}f| \leqslant C\int_{\mathbb{R}^{n+1}} \frac{|\bar{f}(y)|}{\rho(x-y)^{n+2}}\,\mathrm{d}y$$

and

$$\|\mathfrak{K}f\|_{p,\varphi;Q} \leqslant \|\bar{\mathfrak{K}}f\|_{p,\varphi;\mathbb{R}^{n+1}} \leqslant C\|\bar{f}\|_{p,\varphi;\mathbb{R}^{n+1}} = C\|f\|_{p,\varphi;Q}.$$

The estimate for the commutator follows in a similar way. □

Corollary 5.4. *Let $a \in \mathrm{VMO}$ and (p,φ) be as in Theorem 3.7. Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon,\eta_{\boldsymbol{a}})$ such that for any $\mathcal{E}_r(x_0)$ with a radius $r \in (0,r_0)$ and all $f \in M_{p,\varphi}(\mathcal{E}_r(x_0))$*

$$\|\mathfrak{C}[a,f]\|_{p,\varphi;\mathcal{E}_r(x_0)} \leqslant C\varepsilon\|f\|_{p,\varphi;\mathcal{E}_r(x_0)} \tag{5.4}$$

where C is independent of ε, f, r and x_0.

Proof. Since any VMO function can be approximated by BUC functions (see [**6**, **22**]) for each $\varepsilon > 0$ there exists $r_0(\varepsilon, \eta_{\boldsymbol{a}})$ and $g \in \mathrm{BUC}$ with modulus of continuity $\omega_g(r_0) < \varepsilon/2$ such that $\|a - g\|_* < \varepsilon/2$. Fixing $\mathcal{E}_r(x_0)$ with $r \in (0, r_0)$ define the function

$$h(x) = \begin{cases} g(x), & x \in \mathcal{E}_r(x_0), \\ g\left(x_0 + r\dfrac{x' - x_0'}{\rho(x - x_0)}, t_0 + r^2 \dfrac{t - t_0}{\rho^2(x - x_0)}\right), & x \in \mathcal{E}_r^c(x_0), \end{cases}$$

such that $h \in \mathrm{BUC}(\mathbb{R}^{n+1})$ and $\omega_h(r_0) \leqslant \omega_g(r_0) < \varepsilon/2$. Hence,

$$\begin{aligned} \|\mathfrak{C}[a, f]\|_{p,\varphi;\mathcal{E}_r(x_0)} &\leqslant \|\mathfrak{C}[a - g, f]\|_{p,\varphi;\mathcal{E}_r(x_0)} + \|\mathfrak{C}[g, f]\|_{p,\varphi;\mathcal{E}_r(x_0)} \\ &\leqslant C\|a - g\|_* \|f\|_{p,\varphi;\mathcal{E}_r(x_0)} + \|\mathfrak{C}[h, f]\|_{p,\varphi;\mathcal{E}_r(x_0)} < C\varepsilon \|f\|_{p,\varphi;\mathcal{E}_r(x_0)}. \end{aligned}$$

□

For any $x' \in \mathbb{R}^n_+$ and any fixed $t > 0$, define the *generalized reflection*

$$\mathcal{T}(x) = (\mathcal{T}'(x), t), \qquad \mathcal{T}'(x) = x' - 2x_n \frac{\boldsymbol{a}^n(x', t)}{a^{nn}(x', t)}, \tag{5.5}$$

where $\boldsymbol{a}^n(x)$ is the last row of the coefficients matrix $\boldsymbol{a}(x)$ of (2.1). The function $\mathcal{T}'(x)$ maps $\mathbb{R}^n_+$ into $\mathbb{R}^n_-$, and the kernel $\mathcal{K}(x, \mathcal{T}(x) - y) = \mathcal{K}(x, \mathcal{T}'(x) - y', t - \tau)$ is non-singular for any $x, y \in \mathbb{D}^{n+1}_+$. Taking $\tilde{x} \in \mathbb{D}^{n+1}_-$, there exist positive constants κ_1 and κ_2 such that

$$\kappa_1 \rho(\tilde{x} - y) \leqslant \rho(\mathcal{T}(x) - y) \leqslant \kappa_2 \rho(\tilde{x} - y). \tag{5.6}$$

For any $f \in M_{p,\varphi}(\mathbb{D}^{n+1}_+)$ and $a \in \mathrm{BMO}(\mathbb{D}^{n+1}_+)$ define the non-singular integral operators

$$\left.\begin{aligned} \tilde{\mathfrak{K}} f(x) &= \int_{\mathbb{D}^{n+1}_+} \mathcal{K}(x, \mathcal{T}(x) - y) f(y)\,\mathrm{d}y, \\ \tilde{\mathfrak{C}}[a, f](x) &= \int_{\mathbb{D}^{n+1}_+} \mathcal{K}(x, \mathcal{T}(x) - y)[a(y) - a(x)] f(y)\,\mathrm{d}y. \end{aligned}\right\} \tag{5.7}$$

Since $\mathcal{K}(x, \mathcal{T}(x) - y)$ is still homogeneous and satisfies Definition 5.1 (b), we have

$$|\mathcal{K}(x, \mathcal{T}(x) - y)| \leqslant \frac{M}{\rho(\mathcal{T}(x) - y)^{n+2}} \leqslant \frac{C}{\rho(\tilde{x} - y)^{n+2}}.$$

Hence, the operators (5.7) are sublinear and bounded in $L_p(\mathbb{D}^{n+1}_+)$, $p \in (1, \infty)$ (see [**3**]). The following estimates are simple consequences of the results in § 4.

Theorem 5.5. *Let $a \in \mathrm{BMO}(\mathbb{D}^{n+1}_+)$ and $f \in M_{p,\varphi}(\mathbb{D}^{n+1}_+)$ with (p, φ) as in Theorem 3.7. Then the operators $\tilde{\mathfrak{K}} f$ and $\tilde{\mathfrak{C}}[a, f]$ are continuous in $M_{p,\varphi}(\mathbb{D}^{n+1}_+)$ and*

$$\|\tilde{\mathfrak{K}} f\|_{p,\varphi;\mathbb{D}^{n+1}_+} \leqslant C\|f\|_{p,\varphi;\mathbb{D}^{n+1}_+}, \qquad \|\tilde{\mathfrak{C}}[a, f]\|_{p,\varphi;\mathbb{D}^{n+1}_+} \leqslant C\|a\|_* \|f\|_{p,\varphi;\mathbb{D}^{n+1}_+} \tag{5.8}$$

with a constant independent of a and f.

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Corollary 5.6. *Let $a \in \mathrm{VMO}$ and (p,φ) be as above. Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon, \eta_{\boldsymbol a})$ such that for any $\mathcal{E}_r^+(x^0)$ with a radius $r \in (0, r_0)$ and all $f \in M_{p,\varphi}(\mathcal{E}_r^+(x^0))$*

$$\|\mathfrak{C}[a,f]\|_{p,\varphi;\mathcal{E}_r^+(x^0)} \leqslant C\varepsilon \|f\|_{p,\varphi;\mathcal{E}_r^+(x^0)}, \tag{5.9}$$

where C is independent of ε, f, r and x^0.

6. Proof of the main result

Consider (2.1) with $f \in M_{p,\varphi}(Q)$, (p,φ) as in Theorem 3.7. Since $M_{p,\varphi}(Q)$ is a proper subset of $L_p(Q)$, (2.1) is uniquely solvable and the solution u belongs at least to $\overset{\circ}{W}{}_p^{2,1}(Q)$. Our aim is to show that this solution also belongs to $\overset{\circ}{W}{}_{p,\varphi}^{2,1}(Q)$. For this we need an *a priori* estimate of u, which we prove in two steps.

Interior estimate. For any $x_0 \in \mathbb{R}_+^{n+1}$ define the parabolic semi-cylinders $\mathcal{C}_r(x_0) = \mathcal{B}_r(x_0') \times (t_0 - r^2, t_0)$. Let $v \in C_0^\infty(\mathcal{C}_r)$ and suppose that $v(x,t) = 0$ for $t \leqslant 0$. According to [**3**, Theorem 1.4], for any $x \in \operatorname{supp} v$ the following representation formula for the second derivatives of v holds true:

$$\begin{aligned} D_{ij}v(x) &= \mathrm{PV}\int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x, x-y)[a^{hk}(y) - a^{hk}(x)]D_{hk}v(y)\,\mathrm{d}y \\ &\quad + \mathrm{PV}\int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x,x-y)\mathcal{P}v(y)\,\mathrm{d}y + \mathcal{P}v(x)\int_{\mathbb{S}^n} \Gamma_j(x,y)\nu_i\,\mathrm{d}\sigma_y, \end{aligned} \tag{6.1}$$

where $\nu(\nu_1,\dots,\nu_{n+1})$ is the outward normal to $\mathbb{S}^n$. Here, $\Gamma(x,\xi)$ is the fundamental solution of the operator $\mathcal{P}$, and $\Gamma_{ij}(x,\xi) = \partial^2\Gamma(x,\xi)/\partial\xi_i\partial\xi_j$. Since any function $v \in W_p^{2,1}$ can be approximated by C_0^∞ functions, the representation formula (6.1) still holds for any $v \in W_p^{2,1}(\mathcal{C}_r(x_0))$. The properties of the fundamental solution (see [**3**, **15**, **24**]) imply that Γ_{ij} are variable Calderón–Zygmund kernels in the sense of Definition 5.1. Using the notation of (5.1), we can write

$$\begin{aligned} D_{ij}v(x) &= \mathfrak{C}_{ij}[a^{hk}, D_{hk}v](x) \\ &\quad + \mathfrak{K}_{ij}(\mathcal{P}v)(x) + \mathcal{P}v(x)\int_{\mathbb{S}^n}\Gamma_j(x,y)\nu_i\,\mathrm{d}\sigma_y, \quad i,j = 1,\dots,n. \end{aligned} \tag{6.2}$$

The operators $\mathfrak{K}_{ij}$ and $\mathfrak{C}_{ij}$ are defined by (5.1) with $\mathcal{K}(x,x-y) = \Gamma_{ij}(x,x-y)$. Due to Corollaries 5.3 and 5.4 and the equivalence of the metrics, we get that

$$\|D^2v\|_{p,\varphi;\mathcal{C}_r(x_0)} \leqslant C(\varepsilon\|D^2v\|_{p,\varphi;\mathcal{C}_r(x_0)} + \|\mathcal{P}u\|_{p,\varphi;\mathcal{C}_r(x_0)}) \tag{6.3}$$

for some r small enough. Moving the norm of D^2v on the left-hand side, we get that

$$\|D^2v\|_{p,\varphi;\mathcal{C}_r(x_0)} \leqslant C(n,p,\eta_{\boldsymbol a}, \|D\Gamma\|_{\infty,Q})\|\mathcal{P}v\|_{p,\varphi;\mathcal{C}_r(x_0)}.$$

Define a cut-off function $\phi(x) = \phi_1(x')\phi_2(t)$, with $\phi_1 \in C_0^\infty(\mathcal{B}_r(x_0'))$, $\phi_2 \in C_0^\infty(\mathbb{R})$ such that

$$\phi_1(x') = \begin{cases} 1, & x' \in \mathcal{B}_{\theta r}(x_0'), \\ 0, & x' \notin \mathcal{B}_{\theta' r}(x_0'), \end{cases} \qquad \phi_2(t) = \begin{cases} 1, & t \in (t_0 - (\theta r)^2, t_0], \\ 0, & t < t_0 - (\theta' r)^2, \end{cases}$$

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with $\theta \in (0,1)$, $\theta' = \theta(3-\theta)/2 > \theta$ and $|D^s\phi| \leqslant C[\theta(1-\theta)r]^{-s}$, $s = 0,1,2$, $|\phi_t| \sim |D^2\phi|$. For any solution $u \in W_p^{2,1}(Q)$ of (2.1) define $v(x) = \phi(x)u(x) \in W_p^{2,1}(\mathcal{C}_r)$. Hence,

$$\begin{aligned}\|D^2u\|_{p,\varphi;\mathcal{C}_{\theta r}(x_0)} &\leqslant \|D^2v\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} \leqslant C\|\mathcal{P}v\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} \\ &\leqslant C\biggl(\|f\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} + \frac{\|Du\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)}}{\theta(1-\theta)r} + \frac{\|u\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)}}{[\theta(1-\theta)r]^2}\biggr).\end{aligned}$$

Hence,

$$\begin{aligned}&[\theta(1-\theta)r]^2\|D^2u\|_{p,\varphi;\mathcal{C}_{\theta r}(x_0)} \\ &\quad\leqslant ([\theta(1-\theta)r]^2\|f\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} + \theta(1-\theta)r\|Du\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} + \|u\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)}) \\ &\qquad\qquad\qquad\qquad\text{(by the choice of } \theta' \text{ it follows that } \theta(1-\theta) \leqslant 2\theta'(1-\theta')) \\ &\quad\leqslant C(r^2\|f\|_{p,\varphi;Q} + \theta'(1-\theta')r\|Du\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} + \|u\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)}).\end{aligned}$$

Introducing the semi-norms

$$\Theta_s = \sup_{0<\theta<1}[\theta(1-\theta)r]^s\|D^su\|_{p,\varphi;\mathcal{C}_{\theta r}(x_0)}, \quad s = 0,1,2,$$

the above inequality becomes

$$[\theta(1-\theta)r]^2\|D^2u\|_{p,\varphi;\mathcal{C}_{\theta r}(x_0)} \leqslant \Theta_2 \leqslant C(r^2\|f\|_{p,\varphi;Q} + \Theta_1 + \Theta_0). \tag{6.4}$$

The interpolation inequality [**25**, Lemma 4.2] gives that there exists a positive constant C independent of r such that

$$\Theta_1 \leqslant \varepsilon\Theta_2 + \frac{C}{\varepsilon}\Theta_0 \quad \text{for any } \varepsilon \in (0,2).$$

Thus, (6.4) becomes

$$[\theta(1-\theta)r]^2\|D^2u\|_{p,\varphi;\mathcal{C}_{\theta r}(x_0)} \leqslant \Theta_2 \leqslant C(r^2\|f\|_{p,\varphi;Q} + \Theta_0) \quad \forall\theta \in (0,1).$$

Taking $\theta = \frac{1}{2}$ we get the Caccioppoli-type estimate

$$\|D^2u\|_{p,\varphi;\mathcal{C}_{r/2}(x_0)} \leqslant C\biggl(\|f\|_{p,\varphi;Q} + \frac{1}{r^2}\|u\|_{p,\varphi;\mathcal{C}_r(x_0)}\biggr).$$

To estimate u_t we exploit the parabolic structure of the equation and the boundedness of the coefficients

$$\begin{aligned}\|u_t\|_{p,\varphi;\mathcal{C}_{r/2}(x_0)} &\leqslant \|\boldsymbol{a}\|_{\infty;Q}\|D^2u\|_{p,\varphi;\mathcal{C}_{r/2}(x_0)} + \|f\|_{p,\varphi;\mathcal{C}_{r/2}(x_0)} \\ &\leqslant C\biggl(\|f\|_{p,\varphi;Q} + \frac{1}{r^2}\|u\|_{p,\varphi;\mathcal{C}_r(x_0)}\biggr).\end{aligned}$$

Consider the cylinders $Q' = \Omega' \times (0,T)$ and $Q'' = \Omega'' \times (0,T)$ with $\Omega' \Subset \Omega'' \Subset \Omega$; by the standard covering procedure and partition of the unity we get that

$$\|u\|_{W^{2,1}_{p,\varphi}(Q')} \leqslant C(\|f\|_{p,\varphi;Q} + \|u\|_{p,\varphi;Q''}) \tag{6.5}$$

where C depends on n, p, Λ, T, $\|D\Gamma\|_{\infty;Q}$, $\eta_{\boldsymbol{a}}$, $\|\boldsymbol{a}\|_{\infty,Q}$ and $\operatorname{dist}(\Omega', \partial\Omega'')$.

Boundary estimates. For any fixed $(x^0, r) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ define the semi-cylinders

$$\mathcal{C}_r^+(x^0) = \mathcal{B}_r^+(x^{0\prime}) \times (0, r^2) = \{|x^0 - x'| < r,\ x_n > 0,\ 0 < t < r^2\}$$

with $\mathcal{S}_r^+ = \{(x'', 0, t)\colon |x^0 - x''| < r,\ 0 < t < r^2\}$. For any solution $u \in W_p^{2,1}(\mathcal{C}_r^+(x^0))$ with $\operatorname{supp} u \in \mathcal{C}_r^+(x^0)$, the following boundary representation formula holds (see [**3**]):

$$\begin{aligned} D_{ij}u(x) = \mathfrak{C}_{ij}[a^{hk}, D_{hk}u](x) + \mathfrak{K}_{ij}(\mathcal{P}u)(x) \\ + \mathcal{P}u(x) \int_{\mathbb{S}^n} \Gamma_j(x,y)\nu_i \,\mathrm{d}\sigma_y - \mathfrak{I}_{ij}(x), \end{aligned}$$

where

$$\begin{aligned} \mathfrak{I}_{ij}(x) &= \tilde{\mathfrak{K}}_{ij}(\mathcal{P}u)(x) + \tilde{\mathfrak{C}}_{ij}[a^{hk}, D_{hk}u](x), \qquad i, j = 1, \dots, n-1, \\ \mathfrak{I}_{in}(x) &= \mathfrak{I}_{ni}(x) = \sum_{l=1}^{n} \left(\frac{\partial \mathcal{T}(x)}{\partial x_n}\right)^l [\tilde{\mathfrak{C}}_{il}[a^{hk}, D_{hk}u](x) + \tilde{\mathfrak{K}}_{il}(\mathcal{P}u)(x)], \qquad i = 1, \dots, n-1, \\ \mathfrak{I}_{nn}(x) &= \sum_{r,l=1}^{n} \left(\frac{\partial \mathcal{T}(x)}{\partial x_n}\right)^r \left(\frac{\partial \mathcal{T}(x)}{\partial x_n}\right)^l [\tilde{\mathfrak{C}}_{rl}[a^{hk}, D_{hk}u](x) + \tilde{\mathfrak{K}}_{rl}(\mathcal{P}u)(x)], \\ \frac{\partial \mathcal{T}(x)}{\partial x_n} &= \left(-2\frac{a^{n1}(x)}{a^{nn}(x)}, \dots, -2\frac{a^{nn-1}(x)}{a^{nn}(x)}, -1, 0\right). \end{aligned}$$

Here, $\tilde{\mathfrak{K}}_{ij}$ and $\tilde{\mathfrak{C}}_{ij}$ are non-singular operators defined by (5.7) with a kernel $\mathcal{K}(x, \mathcal{T}(x) - y) = \Gamma_{ij}(x, \mathcal{T}(x) - y)$. Applying the estimates (5.8) and (5.9) and having in mind that the components of the vector $\partial \mathcal{T}(x)/\partial x_n$ are bounded, we get that

$$\|D^2 u\|_{p,\varphi;\mathcal{C}_r^+(x^0)} \leqslant C(\|\mathcal{P}u\|_{p,\varphi;\mathcal{C}_r^+(x^0)} + \|u\|_{p,\varphi;\mathcal{C}_r^+(x^0)}).$$

The Jensen inequality applied to $u(x) = \int_0^t u_s(x', s)\,\mathrm{d}s$ and the parabolic structure of the equation give that

$$\|u\|_{p,\varphi;\mathcal{C}_r^+(x^0)} \leqslant Cr^2 \|u_t\|_{p,\varphi;\mathcal{C}_r^+(x^0)} \leqslant C(\|f\|_{p,\varphi;Q} + r^2 \|u\|_{p,\varphi;\mathcal{C}_r^+(x^0)}).$$

Taking r small enough we can move the norm of u on the left-hand side, obtaining that

$$\|u\|_{p,\varphi;\mathcal{C}_r^+} \leqslant C\|f\|_{p,\varphi;Q}$$

with a constant C depending on n, p, Λ, T, $\eta_{\boldsymbol{a}}$, $\|\boldsymbol{a}\|_{\infty,Q}$. By covering the boundary with small cylinders, partitioning of the unit subordinated by that covering and local flattening of $\partial\Omega$ we get that

$$\|u\|_{W^{2,1}_{p,\varphi}(Q\setminus Q')} \leqslant C\|f\|_{p,\varphi;Q}. \tag{6.6}$$

Unifying (6.5) and (6.6), we get (2.4).

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