More characterisations of parallelograms

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1. *Introduction*

Several years ago, Professor Martin Josefsson told the first-named author, in a private communication, that his 'characterisations of' and 'properties of' series of papers that comprised tangential, extangential, bicentric, orthodiagonal, equidiagonal, and bisect-diagonal quadrilaterals, and rhombi and trapezoids, did not include parallelograms because the many characterisations of these figures are well known and are easily accessed via the net, and that nothing interesting can be added. In this Article, we present some characterisations of parallelograms that are very likely unknown and that will hopefully appeal to the readers of the *Gazette*. Other interesting characterisations that we have obtained and that could not find their place here are intended to form the material of another Article.

2. *Bimedians bisecting area and perimeter*

A bimedian of a quadrilateral is the line segment that joins the midpoints of two opposite sides. One of the properties of the bimedians appears as Problem 7.1.33 (p. 191) in [1], and is reproduced as Part (a) of the next theorem. Part (b) is a characterisation of parallelograms that follows from (a) immediately.

Theorem 1: Let $ABCD$ be a convex quadrilateral, and let E, F, G and H be the midpoints of sides AB, CD, BC and AD, respectively; see Figure 2. Then

(a) EF bisects the area of $ABCD$ if, and only if, $AB \parallel CD$,

(b) $ABCD$ is a parallelogram if, and only if, each of EF and GH bisects the area of $ABCD$.

Proof: (a) Referring to Figure 1, we first assume that $AB \parallel DC$, and we show that $[EFCB] = [EFDA]$. Since triangles BCF and ADF have bases CF and DF of equal length, and since their altitudes from B and A are equal (because $AB \parallel DC$), it follows that $[BCF] = [ADF]$. Similarly, $[FBE] = [FAE]$. Adding, we obtain $[EFCB] = [EFDA]$, as desired.

Conversely, we suppose that $[EFCB] = [EFDA]$, and we show that $AB \parallel CD$. We join AF and BF as shown in Figure 1. Since E is the midpoint of AB, it follows that $[EFA] = [EFB]$, and hence $[BFC] = [AFD]$. But $FC = FD$. Therefore the altitudes of BFC and AFD from B and A, respectively, are equal, i.e. A and B are equidistant from the line CD (and lie on the same side of it). Therefore $AB \parallel CD$, as desired. This proves (a).

(b) It follows that if G and H are the midpoints of BC and AD, respectively, as shown in Figure 2, then both bimedians EF and GH bisect the area of $ABCD$ if, and only if, $AB \parallel DC$ and $BC \parallel AD$, i.e. $ABCD$ is a parallelogram.

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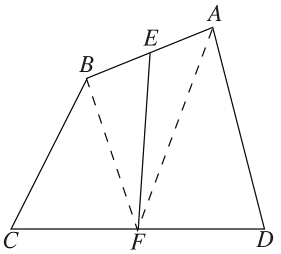

FIGURE 1

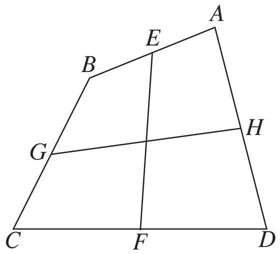

FIGURE 2

The next theorem provides a 'perimeter' analogue to Theorem 1.

Theorem 2: Let $ABCD$ be a convex quadrilateral, and let E, F, G and H be the midpoints of sides AB, CD, BC and AD, respectively. Then

(a) EF bisects the perimeter of $ABCD$ if, and only if, $BC = AD$,

(b) $ABCD$ is a parallelogram if, and only if, each of EF and GH bisects the perimeter of $ABCD$.

Proof: Referring to Figure 1 again, we see that

$$\begin{aligned} & EF \text{ bisects the perimeter of } ABCD \\ \Leftrightarrow \quad & EB + BC + CF + FE = EA + AD + DF + FE \\ \Leftrightarrow \quad & BC = AD. \end{aligned}$$

Thus

$$\begin{aligned} & \text{both } EF \text{ and } GH \text{ bisect the perimeter of } ABCD \\ \Leftrightarrow \quad & BC = AD \text{ and } BA = CD \\ \Leftrightarrow \quad & ABCD \text{ is a parallelogram.} \end{aligned}$$

3. *Points through which every line bisects the area*

If $ABCD$ is a parallelogram, and if M is the point of intersection of its diagonals, then it is clear that any line through M partitions $ABCD$ into two quadrilaterals that are congruent, and in particular have equal areas and equal perimeters. One wonders whether other quadrilaterals can have a special point like M. Theorem 3 below deals with the statement pertaining to the area, and it appears, with a detailed analysis, in [2, pp. 169-171]. A closely related result appears as Problem 2 of the 2000 Mathematical Olympiads of the Czech and Slovak Republics and reproduced below as Theorem 4. A complete solution appears in [3, p. 35]. We do not know of any similar results that pertain to the perimeter.

Theorem 3: Let $ABCD$ be a convex quadrilateral. Then $ABCD$ is a parallelogram if, and only if, there exists a point P in its plane such that every line through P bisects its area.

Theorem 4: Let $ABCD$ be a convex quadrilateral. Then there exists a point P inside $ABCD$ with the property that any line through P that intersects sides AB and CD bisects the area of $ABCD$ if, and only if, AB is parallel to CD.

4. *A characterisation pertaining to the areas of the subtriangles of a quadrilateral formed by its diagonals*

A characterisation of parallelograms that we have spotted in a journal (that we did not take note of) states that if the diagonals of a convex quadrilateral $ABCD$ intersect at M, then $ABCD$ is a parallelogram if, and only if,

$$[AMD] = [BMC] = \tfrac{1}{4}[ABCD]. \tag{1}$$

Clearly (1) holds for parallelograms. So it remains to assume that (1) holds and prove that $ABCD$ is a parallelogram. Since

$$[AMD] = [BMC] \Leftrightarrow [ACD] = [BCD] \Leftrightarrow AB /\!/ CD,$$

it follows that the statement above is equivalent to the following theorem.

Theorem 5: Let $ABCD$ be a convex quadrilateral in which $AB /\!/ CD$. Then $ABCD$ is a parallelogram if, and only if,

$$[AMD] + [BMC] = [AMB] + [CMD], \tag{2}$$

Proof: One of the implications is trivial. Thus we suppose that AB is parallel to CD, as in Figure 3, and that (2) holds, and we show that $ABCD$ is a parallelogram.

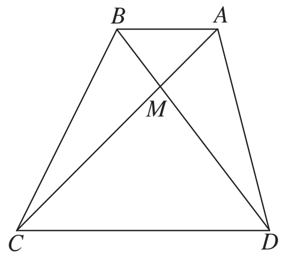

FIGURE 3

Let

$$\theta = \angle AMB,\ t = \sin\theta,\ AM = a,\ BM = b.$$

By the similarity of triangles AMB and CMD, we see that $CM = ka$ and $DM = kb$ for some $k > 0$. Thus,

$$\begin{aligned}
(2) \quad &\Leftrightarrow \quad \tfrac{1}{2}(a)(kb)t + \tfrac{1}{2}(b)(ka)t = \tfrac{1}{2}(a)(b) + \tfrac{1}{2}(kb)(ka)t \\
&\Leftrightarrow \quad 2k = 1 + k^2 \Leftrightarrow (k-1)^2 = 0 \Leftrightarrow k = 1 \\
&\Leftrightarrow \quad \text{the diagonals bisect each other} \\
&\Leftrightarrow \quad ABCD \text{ is a parallelogram.}
\end{aligned}$$

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The theorem above is related to Theorem 2 in [4], which in turn is notching but Problem 7.1.26 (p. 189) of [1]. Both say that if *ABCD* is a convex quadrilateral whose diagonals intersect at *P*, then

$$\text{one of the diagonals of } ABCD \text{ bisects the other}$$
$$\Leftrightarrow\ [PAD] + [PBC] = [PAB] + [PCD]\,. \tag{3}$$

Convex quadrilaterals in which one of the diagonals bisects the other are studied in [4], where they are called *bisect-diagonal*.

In this respect, it is tempting to mention another characterisation of bisect-diagonal quadrilaterals that appeared in [5] and that has some relation to (3), namely

$$\begin{aligned} &\text{a convex quadrilateral } ABCD \text{ is bisect-diagonal} \\ \Leftrightarrow\ &\text{there exists a point } P \text{ in its plane such that} \\ &[PAD] = [PBC] = [PAB] = [PCD]\,. \end{aligned}$$

Again in the context of (3), it is worth mentioning that a theorem, attributed to Léon Anne in [6, Morsel 32, pp. 174-175], gives the following related characterisation of parallelograms, together with a short elegant proof.

Theorem 6: Let *ABCD* be a convex quadrilateral, and let Ω be the set of all points *P* in its plane for which

$$[PAD] + [PBC] = [PAB] + [PCD]\,. \tag{4}$$

If *ABCD* is not a parallelogram, then Ω is a straight line.

It is clear that if *ABCD* is a parallelogram, then contains all the points inside *ABCD*. This shows that Theorem 6 above is a *characterisation* of parallelograms, or rather of non-parallelograms. We mention also that Theorem 6 above appears, in a more general form, as Lemma 5.2.1 (p. 143) in [1], where it is shown that it still holds if (4) is replaced by

$$[PAD] + [PBC] = k\big([PAB] + [PCD]\big) \tag{5}$$

for any $k > 0$. Thus Theorem 6 is the special case $k = 1$ of (5). In this special case, it is easy to see that the straight line Ω passes through the midpoints of the diagonals of *ABCD*, and thus it is what is known as the the Newton line of *ABCD*; see [7, Problem 49, p. 217].

5. *Medians trisecting the diagonals*

We say that a line *XY* divides a directed line segment *ZW* in the ratio $t : 1 - t, 0 \leqslant t \leqslant 1$, if *XY* crosses *ZW* at *P* with $ZP : PW = t : 1 - t$; see Figure 4. We say that *XY* bisects or trisects *ZW* if $t = \frac{1}{2}$ or $t = \frac{1}{3}$, respectively. In contexts of dividing a line segment in a certain ratio, it is to be understood that the line segment is directed.

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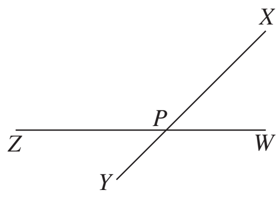

FIGURE 4

Figure 5 below shows a convex quadrilateral *ABCD* and the midpoints *E* and *F* of its sides *CD* and *AB*. The line segments *AE* and *CF* are called medians from *A* and *C*, respectively. Figure 6 shows the same configuration with the additional feature that *ABCD* is a parallelogram. Problem 39 (p. 19) of [8] refers to Figure 6, and asks readers to prove that *AE* trisects *DB* and *CF* trisects *BD*. Clearly these statements replicate each other by symmetry. In fact we shall see below that the equivalence

$$AE \text{ trisects } DB \qquad \Leftrightarrow \qquad CF \text{ trisects } BD \tag{6}$$

holds in Figure 5 also, i.e. for all quadrilaterals.

Figure 7 below shows a convex quadrilateral *ABCD* with medians *AE* and *AG*, and Figure 8 shows the special case when *ABCD* is a parallelogram. Problem 8.3.3 (p. 303) of [9] refers to Figure 8 and asks for a proof that *AE* trisects *DB* and that *AG* trisects *BD*. We shall prove below that the converse is also true, i.e. if *AE* trisects *DB* and *AG* trisects *BD* in Figure 7, then *ABCD* is a parallelogram.

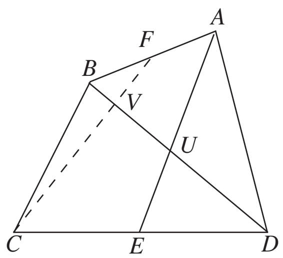

FIGURE 5

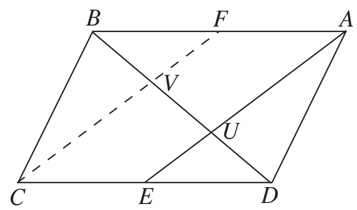

FIGURE 6

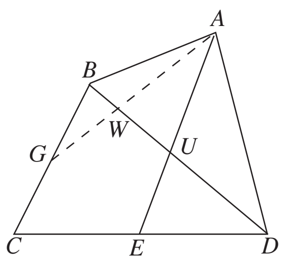

FIGURE 7

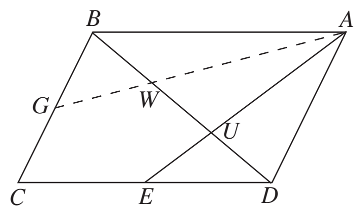

FIGURE 8

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The treatment is fairly general and may be used to produce other problems for mathematical contests.

So we start with a convex quadrilateral $ABCD$ placed in the (Euclidean) plane in such a way that the diagonals of $ABCD$ intersect at the origin O. We treat points in the plane as position vectors with O as the zero vector. Thus there exist c and d such that

$$0 < c, d < 1, \qquad (1 - c)A + cC = (1 - d)B + dD = O.$$

We may find it convenient to set

$$a = 1 - c, \qquad b = 1 - d.$$

Thus the equations above take the symmetric form

$$aA + cC = bB + dD = O.$$

We remark that these are the only dependence relations among A, B, C and D. Thus

$$\alpha A + \beta B + \gamma C + \delta D = O \Leftrightarrow \alpha A + \gamma C = \beta B + \delta D = O \tag{7}$$

$$\Leftrightarrow \alpha : \gamma = a : c \text{ and } \beta : \delta = b : d. \tag{8}$$

Let E be the midpoint of CD, as in Figure 5 or Figure 7. We shall prove that AE divides DB in the ratio $u : 1 - u$, where

$$u = \frac{1 - d}{1 + c}. \tag{9}$$

More precisely, if AE crosses DB at U, then $DU : BU = u : 1 - u$.

For this, we assume that U divides AE in the ratio $t : 1 - t$. Thus we have

$$U = tE + (1 - t)A, \qquad U = (1 - u)D + uB. \tag{10}$$

Since $E = \frac{1}{2}(C + D)$, it follows that

$$(10) \Leftrightarrow (1 - t)A + \frac{t}{2}C = uB + \left(1 - u - \frac{t}{2}\right)D = O$$

$$\Leftrightarrow t = \frac{2c}{1 + c}, \; t = \frac{-2(u + d - 1)}{1 - d} \text{ by (7) and (8)}$$

$$\Leftrightarrow u = \frac{1 - d}{1 + c},$$

as desired in (9).

By analogy, if F is the midpoint of AB, as in Figure 5, then CF divides BD in the ratio $v : 1 - v$, where

$$v = \frac{1 - b}{1 + a} = \frac{d}{2 - c}.$$

Also, if G is the midpoint of BC, as in Figure 7, then AG divides BD in

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the ratio $w : 1 - w$, where

$$w = \frac{1 - b}{1 + c} = \frac{d}{1 + c}.$$

It follows that

$$u = \frac{1}{3} \Leftrightarrow \frac{1 - d}{1 + c} = \frac{1}{3} \Leftrightarrow 3d + c = 2, \tag{11}$$

$$v = \frac{1}{3} \Leftrightarrow \frac{d}{2 - c} = \frac{1}{3} \Leftrightarrow 3d + c = 2,$$

$$w = \frac{1}{3} \Leftrightarrow \frac{d}{1 + c} = \frac{1}{3} \Leftrightarrow 3d - c = 1, \tag{12}$$

$$u = v \Leftrightarrow \frac{1 - d}{1 + c} = \frac{d}{2 - c} \Leftrightarrow 3d + c = 2.$$

In particular,

$$u = \frac{1}{3} \Leftrightarrow v = \frac{1}{3} \Leftrightarrow u = v.$$

This implies that

$$AE \text{ trisects } DB \Leftrightarrow CF \text{ trisects } BD,$$

as claimed in (6). Actually, it proves that

$$\begin{aligned} & AE \text{ divides } DB \text{ in the same ratio as } CF \text{ does } DB \\ \Leftrightarrow\ & AE \text{ trisects } DB \\ \Leftrightarrow\ & CF \text{ trisects } BD. \end{aligned}$$

It also follows that

$$AE \text{ trisects } DB \text{ and } AG \text{ trisects } BD \Leftrightarrow u = w = \tfrac{1}{3}$$

$$\Leftrightarrow 3d + c = 2 \text{ and } 3d = c + 1 \text{ by (11) and (12)} \Leftrightarrow c = d = \tfrac{1}{2}$$

$$\Leftrightarrow AC \text{ and } BD \text{ bisect each other} \Leftrightarrow ABCD \text{ is a parallelogram.}$$

Finally, we have

$$u = w \Leftrightarrow \frac{1 - d}{1 + c} = \frac{d}{1 + c} \Leftrightarrow d = \frac{1}{2} \Leftrightarrow AC \text{ bisects } BD.$$

The statement above can be added to the list of characterisations of bisect-diagonal quadrilaterals compiled in [4].

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