Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 August 2023.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

107.A (Prithwijit De)

We say that three numbers such as 75, 84, 93 form a *reversible arithmetic progression* because both 75, 84, 93 and the numbers formed by reversing their digits, 57, 48, 39, are in arithmetic progression.

(a) What property characterises the digits of triples of 2-digit reversible arithmetic progressions?

(b) Does this property also characterise the digits of triples of 3-digit, 4-digit, … reversible arithmetic progressions?

107.B (Neil Curwen)

Points X and Y lie on edges OA and OC, respectively, of a rectangular sheet of card $OABC$. The triangular corner OXY is folded along XY so that O becomes the vertex of a pyramid with pentagonal base $XABCY$. Given that length $OA = a$ and length $OC = b$ (with $a \geqslant b$), determine the positions of X and Y that maximise the volume of the pyramid.

107.C (Toyesh Prakash Sharma)

In this problem you may assume that $\int e^x \tan x \, dx$ is not expressible in terms of elementary functions.

(a) Show that, for integers $n \geqslant 2$, $\int e^x \tan^n x \, dx$ is not expressible in terms of elementary functions.

(b) Show that, for integers $m, n \geqslant 2$ with $m \neq n$, there is a constant a (dependent on m, n) for which $\int e^x \left(\tan^m x + a \tan^n x\right) dx$ is expressible in terms of elementary functions.

107.D (George Stoica)

Let $0 < a < b < 1$ and $\varepsilon > 0$ be given. Prove the existence of positive integers m and n such that $\left(1 - b^m\right)^n < \varepsilon$ and $\left(1 - a^m\right)^n > 1 - \varepsilon$.

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Solutions and comments on **106.E**, **106.F**, **106.G**, **106.H** (July 2022).

106.E (Stan Dolan and Kieren MacMillan)

Extend the definition $T_k = \frac{1}{2}k(k+1)$ of a triangular number to all integers k. Given that $a + b + c = d + e + f = 0$, find integers x, y, z such that

$$T_x + T_y + T_z = (T_a + T_b + T_c)(T_d + T_e + T_f).$$

Most solvers generated solutions to this open-ended problem using the following method, although there was considerable variation in how the details were presented.

Since $c = -a - b$,

$$\begin{aligned} T_a + T_b + T_c &= \tfrac{1}{2}a(a+1) + \tfrac{1}{2}b(b+1) + \tfrac{1}{2}(a+b)(a+b-1) \\ &= a^2 + ab + b^2 \end{aligned}$$

so that

$$(T_a + T_b + T_c)(T_d + T_e + T_f) = (a^2 + ab + b^2)(d^2 + de + e^2).$$

We thus seek integers x, y and $z = -x - y$ such that

$$(a^2 + ab + b^2)(d^2 + de + e^2) = T_x + T_y + T_z = x^2 + xy + y^2.$$

A neat approach uses factorisation into Eisenstein integers. If $\omega = e^{2\pi i/3}$, then $a^2 + ab + b^2 = (a - \omega b)(a - \omega^2 b)$ so that

$$\begin{aligned} (a^2 + ab + b^2)(d^2 + de + e^2) &= (a - \omega b)(d - \omega e)\cdot(a - \omega^2 b)(d - \omega^2 e) \\ &= (x - \omega y)(x - \omega^2 y) \end{aligned}$$

where

$$x = ad - be, y = ae + bd + be = ae - bf, z = -x - y = af - bd. \qquad (1)$$

Also

$$\begin{aligned} (a^2 + ab + b^2)(d^2 + de + e^2) &= (a - \omega b)(d - \omega^2 e)\cdot(a - \omega^2 b)(d - \omega e) \\ &= (x - \omega y)(x - \omega^2 y) \end{aligned}$$

where

$$x = -af + be, y = -ae + bd, z = -x - y = -ad + bf. \qquad (2)$$

Since

$$T_x + T_y + T_z = \tfrac{1}{2}\left[x(x+1) + y(y+1) + z(z+1)\right] = \tfrac{1}{2}(x^2 + y^2 + z^2),$$

the respective values of x, y, z in the families of solutions (1), (2) may be permuted with ± signs affixed, provided that $x + y + z = 0$ is preserved.

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Families (1), (2) may also be derived using the general composition laws for quadratic forms noted by J. A. Mundie

$$\left(a^2 + Pab + Qb^2\right)\left(d^2 + Pde + Qe^2\right)$$
$$= (ad - Qbe)^2 + P(ad - Qbe)(ae + bd + Pbe) + Q(ae + bd + Pbe)^2.$$

They may also be obtained by just using the quadratic form $x^2 + 3y^2$, since $a^2 + ab + b^2 = \left(\frac{1}{2}a + b\right)^2 + 3\left(\frac{1}{2}a\right)^2$.

Correct solutions were received from: R. G. Bardelang, S. Bhadra, N. Curwen, M. G. Elliott, M. Hennings, J. A. Mundie, J. Siehler, C. Starr and the proposers Stan Dolan and Kieren MacMillan.

106.F (George Stoica)

If the polynomial $P(z) = a_0 + a_1 z + \dots + a_n z^n$ satisfies $|P(z)| \leqslant 1$ for all complex numbers z with $|z| \leqslant 1$, then show that $|a_i| \leqslant 1$ for $i = 0, 1, \dots, n$.

By far the most popular method of solution was to employ results from complex analysis such as the following.

For $0 \leqslant k \leqslant n$, $\dfrac{P(z)}{z^{k+1}} = \dfrac{a_0}{z^{k+1}} + \dots + \dfrac{a_k}{z} + \dots + a_n z^{n-k-1}$ so, integrating over $|z| = 1$ using the result

$$\int_{|z|=1} z^m \, dz = \begin{cases} 2\pi i, & m = -1 \\ 0, & m \neq -1 \end{cases},$$

we obtain $2\pi i a_k = \displaystyle\int_{|z|=1} \frac{P(z)}{z^{k+1}} dz.$

(This result is also an immediate consequence of Cauchy's integral formula.) Since $\left|\dfrac{P(z)}{z^{k+1}}\right| \leqslant 1$ when $|z| = 1$, $2\pi|a_k| \leqslant \displaystyle\int_{|z|=1} 1|dz| = 2\pi$ and $|a_k| \leqslant 1$, as required.

Zoltan Retkes observed that the same argument shows that if $|P(z)| \leqslant M$ whenever $|z| \leqslant R$, then $|a_k| \leqslant \dfrac{M}{R^k}$ for $0 \leqslant k \leqslant n$.

Soham Bhadra and Kee-Wai Lau proved the stronger result that $\displaystyle\sum_{k=0}^{n} |a_k|^2 \leqslant 1$. To see this, consider

$$\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^2 d\theta = \int_0^{2\pi} \left(\sum_{r=0}^{n} a_r e^{ir\theta}\right) \sum_{s=0}^{n} \left(\overline{a_s} e^{-is\theta}\right) d\theta$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{n} a_r \overline{a_s} \int_0^{2\pi} e^{i(r-s)\theta} d\theta = 2\pi \sum_{k=0}^{n} |a_k|^2,$$

since $\displaystyle\int_0^{2\pi} e^{i(r-s)\theta} d\theta = \begin{cases} 2\pi, & r = s \\ 0, & r \neq s \end{cases}.$

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Thus $2\pi \sum_{k=0}^{n} |a_k|^2 = \int_0^{2\pi} |P(e^{i\theta})|^2 \, d\theta \leqslant \int_0^{2\pi} 1 \, d\theta = 2\pi$.

Didier Pinchon gave a proof using the discrete Fourier transform and evaluation of $P(z)$ at the $(n + 1)$th roots of unity. And the proposer, George Stoica, gave a proof by induction on the degree of $P(z)$.

Correct solutions were received from: S. Bhadra, S. Dolan, M. G. Elliott, M. Hennings, P. F. Johnson, K-W Lau, D. Pinchon, Z. Retkes (2 solutions), H. Ricardo and the proposer George Stoica.

106.G (Seán Stewart)

If C_n is the nth Catalan number defined by the recurrence relation $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$ with $C_0 = 1$, prove that:

(a) $$\sum_{n=0}^{\infty} \frac{C_n}{4^n(2n + 3)} = 2 - \frac{\pi}{2},$$

(b) $$\sum_{n=0}^{\infty} \frac{C_n}{4^n(2n + 3)^2} = 2 - \frac{\pi}{4}\left(1 + \ln 4\right),$$

(c) $$\sum_{n=0}^{\infty} \frac{C_n}{4^n(2n + 3)^3} = 2 - \frac{\pi}{16}\left(2 + \ln 16 + \ln^2 4\right) - \frac{\pi^3}{48}.$$

All solvers used the generating function for the Catalan numbers, given by $\sum_{n=0}^{\infty} C_n t^n = \dfrac{1 - \sqrt{1 - 4t}}{2t}$. Most solutions then used the following approach: we give Stan Dolan's version which first isolates the key integrals, $\int_0^{\pi/2} \ln \sin x \, dx$ and $\int_0^{\pi/2} (\ln \sin x)^2 \, dx$, involved.

Define $U(x) = \int_0^x \dfrac{\sin^{-1} 2t}{t} dt$. We then have

A. $\int_0^x \sqrt{1 - 4t^2} dt = \frac{1}{4} \sin^{-1} 2x + \frac{1}{2} x\sqrt{1 - 4x^2}$.

B. $U\left(\frac{1}{2}\right) = \int_0^{1/2} \frac{\sin^{-1} 2t}{t} dt = \frac{\pi}{4} \ln 4$.

This comes from setting $u = \sin^{-1} 2t$ and then integrating by parts.

$$\int_0^{1/2} \frac{\sin^{-1} 2t}{t} = \int_0^{\pi/2} u \cot u \, du = -\int_0^{\pi/2} \ln \sin u \, du = \frac{\pi}{2} \ln 2,$$

using the standard integral $\int_0^{\pi/2} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2$.

C. $\displaystyle\int_0^{1/2} \frac{\ln t \, \sin^{-1} 2t}{t} dt = -\frac{\pi}{48}\left(\pi^2 + 9 \ln^2 4\right)$

Integrating by parts and then substituting $u = \sin^{-1} 2t$, we obtain

$$\int_0^{1/2} \frac{\ln t \, \sin^{-1} 2t}{t} dt = \left[\frac{1}{2} (\ln t)^2 \sin^{-1} 2t\right]_0^{1/2} - \int_0^{1/2} \frac{(\ln t)^2}{\sqrt{1 - 4t^2}} dt$$

$$= \frac{\pi}{4}(\ln 2)^2 - \frac{1}{2}\int_0^{\pi/2}(\ln\sin u - \ln 2)^2\,du$$

$$= -\frac{\pi}{2}(\ln 2)^2 - \frac{1}{2}\int_0^{\pi/2}(\ln\sin u)^2\,du$$

$$= -\frac{\pi^3}{48} - \frac{3\pi}{16}\ln^2 4,$$

since $\int_0^{\pi/2}(\ln\sin u)^2\,du = \frac{\pi^3}{24} + \frac{\pi}{2}(\ln 2)^2$.

For the last integral, a quick evaluation uses Fourier series.

From $\ln\left(1 - e^{i\theta}\right) = -\sum_{n=1}^{\infty}\frac{e^{in\theta}}{n}$, $0 < \theta < 2\pi$, Parseval's identity gives

$$\int_0^{2\pi}\left|\ln\left(1 - e^{i\theta}\right)\right|^2 d\theta = 2\pi\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^3}{3}.$$

But $\int_0^{2\pi}\left|\ln\left(1 - e^{i\theta}\right)\right|^2 d\theta = \int_0^{2\pi}\left(\ln 2\sin\frac{\theta}{2}\right)^2 + \left(\frac{\theta - \pi}{2}\right)^2 d\theta$

whence $\int_0^{2\pi}\left(\ln 2\sin\frac{\theta}{2}\right)^2 d\theta = \frac{\pi^3}{6}$, leading to $\int_0^{\pi}(\ln 2 + \ln\sin t)^2\,dt = \frac{\pi^3}{12}$

and $\pi(\ln 2)^2 + 4\ln 2\int_0^{\pi/2}\ln\sin t\,dt + 2\int_0^{\pi/2}(\ln\sin t)^2\,dt = \frac{\pi^3}{12}$

from which $\int_0^{\pi/2}(\ln\sin t)^2\,dt = \frac{\pi^3}{24} + \frac{\pi}{2}(\ln 2)^2$.

D. $\int_0^{1/2}\frac{U(t)}{t}dt = \frac{\pi}{48}\left(\pi^2 + 3\ln^2 4\right)$

Integrating by parts,

$$\int_0^{1/2}\frac{U(t)}{t}dt = U\left(\frac{1}{2}\right)\ln\left(\frac{1}{2}\right) - \int_0^{1/2}\frac{\ln t\,\sin^{-1}2t}{t}dt$$

$$= -\frac{\pi}{4}(\ln 4)(\ln 2) + \frac{\pi^3}{48} + \frac{9\pi}{48}\ln^2 4$$

$$= \frac{\pi}{48}\left(\pi^2 + 3\ln^2 4\right).$$

With these preliminaries completed, the answers to (a), (b), (c) quickly follow.

(a) Since $2\sum_{n=0}^{\infty}C_n t^{2n+2} = 1 - \sqrt{1 - 4t^2}$, integrating from 0 to x using (A) gives

$$2\sum_{n=0}^{\infty}\frac{C_n}{2n + 3}x^{2n+3} = x - \frac{1}{4}\sin^{-1}2x - \frac{1}{2}x\sqrt{1 - 4x^2}.$$

Setting $x = \frac{1}{2}$ gives $\frac{1}{4}\sum_{n=0}^{\infty}\frac{C_n}{4^n(2n + 3)} = \frac{1}{2} - \frac{\pi}{8}$, hence (a).

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(b) Dividing the series in (a) by x and integrating again using (A) gives

$$2\sum_{n=0}^{\infty}\frac{C_n}{(2n+3)^2}x^{2x+3} = x - \frac{1}{4}U(x) - \frac{1}{8}\sin^{-1}2x - \frac{1}{4}x\sqrt{1-4x^2}.$$

Setting $x = \frac{1}{2}$ and using (B) gives

$$\frac{1}{4}\sum\frac{C_n}{4^n(2n+3)^2} = \frac{1}{2} - \frac{\pi}{16}\ln 4 - \frac{\pi}{16}, \text{ hence (b).}$$

(c) Dividing the series in (b) by x and integrating for a third time using (A) gives

$$2\sum_{n=0}^{\infty}\frac{C_n}{(2n+3)^3}x^{2n+3} = x - \frac{1}{4}\int_0^x\frac{U(t)}{t}dt - \frac{1}{8}U(x) - \frac{1}{16}\sin^{-1}2x - \frac{1}{8}x\sqrt{1-4x^2}.$$

Setting $x = \frac{1}{2}$ and using (B), (D) gives

$$\frac{1}{4}\sum_{n=0}^{\infty}\frac{C_n}{4^n(2n+3)^3} = \frac{1}{2} - \frac{1}{4}\cdot\frac{\pi}{48}\left(\pi^2 + 3\ln^2 4\right) - \frac{1}{8}\cdot\frac{\pi}{4}\ln 4 - \frac{\pi}{32}.$$

hence (c).

Mark Hennings embedded the results of **106.G** in a more general setting. Let $F(u) = \sum_{n=0}^{\infty}\frac{C_n}{4^n}u^{2n+2} = 2\left(1-\sqrt{1-u^2}\right)$ and $A(t) = \int_0^1 u^t F(u)du$. Then, if S_1, S_2, S_3 denote the sums in (a), (b), (c), we have $S_1 = A(0)$, $S_2 = -A'(0)$, $S_3 = \frac{1}{2}A''(0)$, But

$$A(t) = 2\int_0^1 u^t - u^t\sqrt{1-u^2}\,du = 2\int_0^{\pi/2}\sin^t\theta\cos\theta\left(1-\cos\theta\right)d\theta$$

$$= B\left(\tfrac{1}{2}(t+1),1\right) - B\left(\tfrac{1}{2}(t+1),\tfrac{3}{2}\right) = \frac{2}{t+1} - \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(t+1)\right)}{2\Gamma\left(\frac{1}{2}(t+4)\right)}$$

so the calculation of S_1, S_2, S_3 reduces to the evaluation of derivatives of the Γ-function at arguments $\frac{1}{2}$ and 2.

Correct solutions were received from: S. Bhadra, S. Dolan, M. G. Elliott, M. Hennings, D. Pinchon, H. Ricardo (parts (a),(b)) and the proposer Seán Stewart.

106.H (Isaac Sofair)

A bag contains p white marbles and q black marbles. Marbles are randomly drawn from the bag one at a time without replacement until n white marbles are obtained, where $n \leqslant p$. Prove that the mean number of marbles that need to be drawn is $n\left(1 + \frac{q}{n+1}\right)$.

First, my sincere apologies to readers and especially the proposer, Isaac Sofair, for the typo in the final formula which should read $n\left(1 + \frac{q}{p+1}\right)$. This was entirely my fault but, fortunately, was spotted and corrected by solvers.

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Solutions divided evenly between those that calculated the probability distribution involved and those that argued directly with expectations. From the latter, Mark Hennings gave the following careful induction proof. Let $E_{n,p,q}$ $(p \geqslant n \geqslant 0,\ q \geqslant 0)$ be the expected number of marbles that have to be drawn to obtain n white marbles when there are initially p white marbles and q black marbles in the bag. Clearly $E_{0,p,q} = 0$ and $E_{n,p,0} = n$. Moreover, conditioning on the type of marble first drawn, we see that

$$E_{n,p,q} = \frac{p}{p+q}\left(1 + E_{n-1,p-1,q}\right) + \frac{q}{p+q}\left(1 + E_{n,p,q-1}\right)$$

$$= 1 + \frac{p}{p+q}E_{n-1,p-1,q} + \frac{q}{p+q}E_{n,p,q-1} \quad (*)$$

for all $p \geqslant n \geqslant 1$, $q \geqslant 1$.

Let the inductive statement, P_N, be that $E_{n,p,q} = n\left(1 + \dfrac{q}{p+1}\right)$ for any $p \geqslant n \geqslant 0$, $q \geqslant 0$ such that $n + q = N$.

P_0 states that $E_{0,p,0} = 0$, which is true.

P_1 states that $E_{0,p,1} = 0$ for $p \geqslant 0$ and $E_{1,p,0} = 1$ for $p \geqslant 1$ and this is also true.

Suppose inductively that $N \geqslant 2$ and that P_{N-1} is true, and consider $p \geqslant n \geqslant 0$, $q \geqslant 0$ such that $n + q = N$.

Now $E_{0,p,N} = 0 = 0\left(1 + \dfrac{N}{p+1}\right)$ and $E_{N,p,0} = N = N\left(1 + \dfrac{0}{p+1}\right)$. But if both p and q are positive, then $(*)$ and P_{N-1} gives

$$\begin{aligned} E_{n,p,q} &= 1 + \frac{p}{p+q}E_{n-1,p-1,q} + \frac{q}{p+q}E_{n,p,q-1} \\ &= 1 + \frac{p}{p+q}\cdot(n-1)\left(1 + \frac{q}{p}\right) + \frac{q}{p+q}\cdot n\left(1 + \frac{q-1}{p+1}\right) \\ &= 1 + (n-1) + \frac{nq}{p+1} = n\left(1 + \frac{q}{p+1}\right), \text{ as required for } P_N. \end{aligned}$$

Jacob Siehler neatly calculated the expected value via the probability distribution as follows.

Suppose that all the marbles are drawn one at a time, making a sequence of $p + q$ marbles. The probability that the nth white marble is at position t is

$$\frac{\binom{t-1}{n-1}\binom{p+q-t}{p-n}}{\binom{p+q}{p}} \text{ for } n \leqslant t \leqslant n + q \text{ (and 0 otherwise)}$$

since there have to be $n - 1$ white marbles in the first $t - 1$ drawn and $p - n$ white marbles in the last $p + q - t$ drawn.

The required expectation, E, is thus

$$E = \sum_{t=n}^{n+q} t \frac{\binom{t-1}{n-1}\binom{p+q-t}{p-n}}{\binom{p+q}{p}} = \frac{n}{\binom{p+q}{p}} \sum_{t=n}^{n+q} \binom{t}{n}\binom{p+q-t}{p-n}.$$

But, echoing the argument above, $\binom{t}{n}\binom{p+q-t}{p-n}$ is the number of sequences of $p + 1$ white and q black marbles in which the $(n + 1)$th white marble is in the $(t + 1)$th position. Then $\sum_{t=n}^{n+q} \binom{t}{n}\binom{p+q-t}{p-n} = \binom{p+q+1}{q}$, the total number of such sequences.

Finally then, $E = \frac{n}{\binom{p+q}{q}} \cdot \binom{p+q+1}{q} = \frac{n(p+q+1)}{p+1} = n\left(1 + \frac{q}{p+1}\right)$.

Samuele Riccarelli identified the probability model involved in **106.H** as the inverse hypergeometric distribution, [1].

Reference

1. N. L. Johnson, A. W. Kemp and S. Kotz, *Univariate discrete distributions* (3rd edn.) John Wiley (2005).

Correct solutions were received from: S. Bhadra, N. Curwen, S. Dolan, M. G. Elliott, M. Hennings, K-W Lau, J. A. Mundie, D. Pinchon, S. Riccarelli, J. Siehler and the proposer Isaac Sofair.

N.J.L

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Appeal from the Editor

The *Gazette* relies on volunteers who are prepared to referee articles and thus maintain the quality of the journal. Currently, I am very short of referees for submissions on Euclidean Geometry. I would be very grateful for any reader who is prepared to offer their services in this specialised subject. Anyone who would like to find out more about what this entails should contact me via my e-mail address: g.leversha@btinternet.com