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Isometric Group Actions on Hilbert Spaces: Structure of Orbits

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Abstract. Our main result is that a finitely generated nilpotent group has no isometric action on an infinite-dimensional Hilbert space with dense orbits. In contrast, we construct such an action with a finitely generated metabelian group.

1 Introduction

The study of isometric actions of groups on affine Hilbert spaces has in recent years found applications ranging from the K-theory of C^*-algebras [HiKa] to rigidity theory [Sh2] and geometric group theory [Sh3, CTV]. This renewed interest motivates the following general problem: *How can a given group act by isometries on an affine Hilbert space?*

This paper is a sequel to [CTV], but can be read independently. In [CTV], given an isometric action of a finitely generated group G on a Hilbert space $\alpha\colon G \to \mathrm{Isom}(\mathcal{H})$, we focused on the growth of the function $g \mapsto \alpha(g)(0)$. Here the emphasis is on the structure of orbits.

We will mainly focus on actions of nilpotent groups. Let us begin by a simple example: every isometric action of $\mathbf{Z}$ on a Euclidean space is the direct sum of an action with a fixed point and an action by translations. This actually remains true for general nilpotent groups. The situation becomes more subtle when we study actions on infinite-dimensional Hilbert spaces. However, something remains from the finite-dimensional case.

We say that a convex subset of a Hilbert space is *locally bounded* if its intersection with any finite-dimensional subspace is bounded. The main result of the paper is the following theorem, proved in Section 4.

Theorem 1 *Let G be a nilpotent topological group. Let G act isometrically on a Hilbert space $\mathcal{H}$, with linear part π. Let $\mathcal{O}$ be an orbit under this action. Then there exist*

- *a closed subspace T of $\mathcal{H}$ (the "translation part"), contained in the subspace of invariant vectors of π,*
- *a closed, locally bounded convex subset U of the orthogonal subspace $T^{\perp}$,*

such that $\mathcal{O}$ is contained in $T \times U$.

We owe the following general question to A. Navas: which locally compact groups have an isometric action on an infinite-dimensional separable Hilbert space with

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dense orbits, *i.e.*, a minimal action? Theorem 1 allows us to provide a negative answer in the case of finitely generated nilpotent groups.

Corollary 2 (See Corollary 4.7) *A compactly generated, nilpotent-by-compact locally compact group does not admit any affine isometric action with dense orbits on an infinite-dimensional Hilbert space.*

In the course of our proof, we introduce the following new definition: a unitary or orthogonal representation π of a group is *strongly cohomological* if $H^1(G,\rho) \neq 0$ for every nonzero subrepresentation $\rho \leq \pi$. It is easy to observe that the linear part of an affine isometric action with dense orbits is strongly cohomological. The main step in the proof of Theorem 1 is the following result.

Proposition 3 (See Proposition 3.9) *Let π be an orthogonal or unitary representation of a second countable, nilpotent locally compact group G. Suppose that π is strongly cohomological. Then π is a trivial representation.*

Another case for which we answer Navas' question negatively is the following.

Theorem 4 (See Theorem 4.8) *Let G be a connected semisimple Lie group. Then G has no isometric action on a nonzero Hilbert space with dense orbits.*

It is not clear how Theorem 1 and Corollary 2 can be generalized, in view of the following example.

Proposition 5 (See Proposition 2.1) *There exists a finitely generated metabelian group admitting an affine isometric action with dense orbits on an infinite-dimensional separable Hilbert space.*

Another construction provides the following.

Proposition 6 (See Proposition 2.3) *There exists a countable group admitting an affine isometric action with dense orbits on an infinite-dimensional Hilbert space in such a way that every finitely generated subgroup has a fixed point.*

2 Existence Results

Here is a first positive result regarding Navas' question.

Proposition 2.1 *There exists an isometric action of a metabelian 3-generator group on $\ell^2_{\mathbf{R}}(\mathbf{Z})$, all of whose orbits are dense.*

Proof Observe that $\mathbf{Z}[\sqrt{2}]$ acts on $\mathbf{R}$ by translations with dense orbits. So the free abelian group of countable rank $\mathbf{Z}[\sqrt{2}]^{(\mathbf{Z})}$ acts by translations with dense orbits on $\ell^2_{\mathbf{R}}(\mathbf{Z})$. Observe now that the latter action extends to the wreath product $\mathbf{Z}[\sqrt{2}] \wr \mathbf{Z} = \mathbf{Z}[\sqrt{2}]^{(\mathbf{Z})} \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts on $\ell^2_{\mathbf{R}}(\mathbf{Z})$ by the shift. That wreath product is metabelian with three generators. ■

Corollary 2.2 *There exists an isometric action of a free group of finite rank on a Hilbert space with dense orbits.*

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Recall that an isometric action $\alpha\colon G \to \mathrm{Isom}(\mathcal{H})$ *almost has fixed points* if for every $\varepsilon > 0$ and every compact subset $K \subset G$ there exists $v \in \mathcal{H}$ such that $\sup_{g\in K} \|v - \alpha(g)v\| \leq \varepsilon$.

In the example given by Proposition 2.1, the given isometric action clearly does not almost have fixed points, *i.e.*, it defines a nonzero element in reduced 1-cohomology. The next result shows that this is not always the case.

Proposition 2.3 *There exists a countable group Γ with an affine isometric action α on an infinite-dimensional Hilbert space, such that α has dense orbits, and every finitely generated subgroup of Γ has a fixed point. In particular, the action almost has fixed points.*

Proof We first construct an uncountable group G and an affine isometric action of G having dense orbits and almost having fixed points.

In $\mathcal{H} = \ell^2_{\mathbf{R}}(\mathbf{N})$, let A_n be the affine subspace defined by the equations

$$x_0 = 1,\ x_1 = 1,\ \dots,\ x_n = 1,$$

and let G_n be the pointwise stabilizer of A_n in the isometry group of $\mathcal{H}$. Let G be the union of the G_n's. View G as a discrete group.

It is clear that G almost has fixed points in $\mathcal{H}$, since any finite subset of G has a fixed point. Let us prove that G has dense orbits.

Claim 1 For all $x, y \in \mathcal{H}$, we have $\lim_{n\to\infty} |d(x, A_n) - d(y, A_n)| = 0$.

By density, it is enough to prove Claim 1 when x, y are finitely supported in $\ell^2_{\mathbf{R}}(\mathbf{N})$. Take $x = (x_0, x_1, \dots, x_k, 0, 0, \dots)$ and choose $n > k$. Then

$$d(x, A_n)^2 = \sum_{j=0}^{k}(x_j - 1)^2 + \sum_{j=k+1}^{n} 1^2 = n + 1 - 2\sum_{j=0}^{k} x_j + \sum_{j=0}^{k} x_j^2,$$

so that $d(x, A_n) = \sqrt{n} + O(\frac{1}{\sqrt{n}})$, which proves Claim 1.

Denote by p_n the projection onto the closed convex set A_n, namely

$$p_n(x_0, x_1, \dots) = (1, 1, \dots, 1, x_{n+1}, x_{n+2}, \dots).$$

Claim 2 For all $x, y \in \mathcal{H}$, we have $\lim_{n\to\infty} \|p_n(x) - p_n(y)\| = 0$.

This is a straightforward computation.

Claim 3 G has dense orbits in $\mathcal{H}$.

Observe that two points $x, y \in \mathcal{H}$ are in the same G_n-orbit if and only if $d(x, A_n) = d(y, A_n)$ and $p_n(x) = p_n(y)$. Fix $x_0, z \in \mathcal{H}$. We want to show that

$$\lim_{n\to\infty} d(G_n x_0, z) = 0.$$

So fix $\varepsilon > 0$. By the second claim, for some n_0, $\|p_n(x_0) - p_n(z)\| \leq \varepsilon/2$ whenever $n \geq n_0$. Set $W = \{x \in \mathcal{H} : p_n(x) = p_n(z)\}$; this is the orthogonal affine subspace of A_n passing through z. Then $y_0 = x_0 + (p_n(z) - p_n(x_0)) \in W$. By the first claim, there exists $n_1 \geq n_0$ such that $|d(y_0, A_n) - d(z, A_n)| \leq \varepsilon/2$ for every $n \geq n_1$. Therefore there exists $y \in W$ such that $\|y - z\| \leq \varepsilon/2$ and $d(y, A_n) = d(y_0, A_n) = d(x_0, A_n)$. By the previous observation, there exists $g \in G_n$ such that $y = gy_0$. Then

$$d(gx_0, z) \leq d(gx_0, gy_0) + d(gy_0, z) \leq \varepsilon,$$

so that $d(G_n x_0, z) \leq \varepsilon$ for every $n \geq n_1$, proving the last claim.

Using separability of $\mathcal{H}$, it is now easy to construct a countable subgroup Γ of G also having dense orbits on $\mathcal{H}$. ■

Question 1 Does there exist an affine isometric action of a *finitely generated* group on a Hilbert space, having dense orbits and almost having fixed points?

3 Cohomology of Unitary Representations of Nilpotent Groups

Our non-existence results concerning nilpotent groups will be based on the following study of their unitary representations.

Definition 3.1 If G is a topological group and π a unitary representation, we say that π is *strongly cohomological* if every nonzero subrepresentation of π has nonzero first cohomology.

The following lemma is [Gu2, Proposition 3.1, Ch. III].

Lemma 3.2 *Let π be a unitary representation of a topological group G. Let z be a central element of G. Suppose that $1 - \pi(z)$ has a bounded inverse (equivalently, 1 does not belong to the spectrum of $\pi(z)$). Then $H^1(G, \pi) = 0$.*

Proof Let $b \in Z^1(G, \pi)$ be a 1-cocycle; we prove that b is bounded. If $g \in G$, expanding the equality $b(gz) = b(zg)$, we obtain that $(1 - \pi(z))b(g)$ is bounded by $2\|b(z)\|$, so that b is bounded by $2\|(1 - \pi(z))^{-1}\|\|b(z)\|$. ■

Lemma 3.3 *Let G be a locally compact, second countable group, and π a strongly cohomological unitary representation. Then π is trivial on the centre $Z(G)$.*

Proof Fix $z \in Z(G)$. As G is second countable, we may write $\pi = \int_{\hat{G}}^{\oplus} \rho \, d\mu(\rho)$, a disintegration of π as a direct integral of irreducible representations. Let $\chi \colon \hat{G} \to S^1 : \rho \mapsto \rho(z)$ be the continuous map given by the value of the central character of ρ on z. For $\varepsilon > 0$, set $X_\varepsilon = \{\rho \in \hat{G} : |\chi(\rho) - 1| > \varepsilon\}$ and $\pi_\varepsilon = \int_{X_\varepsilon}^{\oplus} \rho \, d\mu(\rho)$, so that π_ε is a subrepresentation of π. Since $|\rho(z) - 1|^{-1} < \varepsilon^{-1}$ for $\rho \in X_\varepsilon$, the operator

$$(\pi_\varepsilon(z) - 1)^{-1} = \int_{X_\varepsilon}^{\oplus} (\rho(z) - 1)^{-1} \, d\mu(\rho)$$

is bounded. We can now apply Lemma 3.2 to conclude that $H^1(G, \pi_\varepsilon) = 0$. By definition, this means that π_ε is the zero subrepresentation, meaning that the spectral measure μ is supported in $\hat{G} - X_\varepsilon$. As this holds for every $\varepsilon > 0$, we see that μ is supported in $\{\rho \in \hat{G} : \rho(z) = 1\}$, to the effect that $\pi(z) = 1$. ■

Proposition 3.4 *Let G be a topological group, and π a unitary representation of G. Suppose that $\overline{H^1}(G, \pi) \neq 0$. Then π has a nonzero subrepresentation that is strongly cohomological.*

Proof Suppose the contrary. Then by a standard application of Zorn's lemma, π decomposes as a direct sum $\pi = \bigoplus_{i \in I} \pi_i$, where $H^1(G, \pi_i) = 0$ for every $i \in I$, so that $\overline{H^1}(G, \pi) = 0$ by [Gu2, Proposition 2.6, Ch. III]. ■

Remark 3.5. The converse is false, even for finitely generated groups. Indeed, it is easy to check (see [Gu1]) that every nonzero unitary representation of the free group F_2 has non-vanishing H^1, so that every unitary representation of F_2 is strongly cohomological. But it turns out that F_2 has an irreducible representation π such that $\overline{H^1}(F_2, \pi) = 0$ [Ma, Lemma 5.1.5].

Corollary 3.6 *Let G be a locally compact, second countable group, and let π be a unitary representation of G without invariant vectors. Write $\pi = \pi_0 \oplus \pi_1$, where π_1 consists of the $Z(G)$-invariant vectors. Then*

(i) *π_0 does not contain any nonzero strongly cohomological subrepresentation; in particular, $\overline{H^1}(G, \pi_0) = 0$;*
(ii) *every 1-cocycle of π_1 vanishes on $Z(G)$, so that $H^1(G, \pi_1) \simeq H^1(G/Z(G), \pi_1)$.*

Proof (i) follows by combining Lemma 3.3 and Proposition 3.4. For (ii), we use the idea of proof of [Sh2, Theorem 3.1]: if $b \in Z^1(G, \pi_1)$, then for every $g \in G$, $z \in Z(G)$, $\pi_1(g)b(z) + b(g) = b(gz) = b(zg) = b(g) + b(z)$ as $\pi_1(z) = 1$. So $\pi_1(g)b(z) = b(z)$. This forces $b(z) = 0$ as π has no G-invariant vector. So b factors through $G/Z(G)$. ■

Observe that Corollary 3.6 provides a new proof of Shalom's Corollary 3.7 [Sh2]. Under the same assumptions, every cocycle in $Z^1(G, \pi)$ is almost cohomologous to a cocycle factoring through $G/Z(G)$ and taking values in a subrepresentation factoring through $G/Z(G)$. From Corollary 3.6 we also immediately deduce the following.

Corollary 3.7 *Let G be a locally compact, second countable, nilpotent group, and let π be a unitary representation of G without invariant vectors. Let (Z_i) be the ascending central series of G ($Z_0 = \{1\}$, and Z_i is the centre modulo Z_{i-1}). Let σ_i denote the subrepresentation of G on the space of Z_i-invariant vectors, and finally let π_i be the orthogonal of σ_{i+1} in σ_i, so that $\pi = \bigoplus \pi_i$.*

Then $H^1(G, \pi_i) \simeq H^1(G/Z_i, \pi_i)$ for all i, and π has no nonzero strongly cohomological subrepresentation. In particular, $\overline{H^1}(G, \pi) = 0$.

Note that the latter statement, namely $\overline{H^1}(G, \pi) = 0$, is a result of Guichardet [Gu1, Théorème 7], which can be stated as: G has property H_T, *i.e.*, every unitary representation with non-vanishing reduced 1-cohomology contains the trivial representation.

Definition 3.8 We say that a locally compact group G has property H_{CT} if every strongly cohomological unitary representation of G is trivial.

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It is a straightforward verification that this is equivalent to: every strongly cohomological *orthogonal* representation of G is trivial. This will be useful in the next paragraph since we will deal with orthogonal rather than unitary representations. The following proposition is contained in Corollary 3.7.

Proposition 3.9 *If G is a locally compact, second countable, nilpotent group, then G has property H_{CT}.*

As a corollary of Proposition 3.4, property H_{CT} implies property H_T. However the converse is not true, as shown by the following example.

Example 3.10 Let G be the full affine group of the real line. The dual $\hat{G}$ (*i.e.*, the space of unitary irreducible representations of G with the Fell-Jacobson topology) was described in [Fe]. It consists of two copies of the real line (corresponding to one-dimensional representations, *i.e.*, characters) plus one point $\{\sigma\}$ which is both open and dense. The only irreducible representation with non-vanishing reduced 1-cohomology is the trivial representation 1_G, so that G has property H_T; on the other hand, since σ weakly contains 1_G, one has $H^1(G,\sigma) \neq 0$ by [Gu1, Théorème 1]. So σ is strongly cohomological, meaning that G does not have property H_{CT}.

4 Non-Existence Results

Definition 4.1 (i) We say that a subset Y of a metric space (X, d) is *coarsely dense* if there exists $C \geq 0$ such that for every $x \in X$, $d(x, Y) \leq C$.

(ii) We say that a subset Y of a Hilbert space $\mathcal{H}$ is *enveloping* if its closed convex hull is all of $\mathcal{H}$.

Observe that every dense subset of a metric space is coarsely dense. Besides, in a Hilbert space $\mathcal{H}$, every coarsely dense subset Y is enveloping. Indeed, suppose that Y is contained in a closed, convex proper subset X of $\mathcal{H}$. Consider $v \notin X$ and let y denote its projection on X (excluding the trivial case $Y = \varnothing$). Then, for every $\lambda \geq 0$, we have $d(y + \lambda(v - y), Y) \geq d(y + \lambda(v - y), X) = \lambda$, which is unbounded, so that Y is not coarsely dense.

Example 4.2 In $\ell^2_{\mathbf{R}}(\mathbf{Z})$, let X denote the subset of elements with integer coefficients. Then X is enveloping. Indeed, its intersection with the subspace $V_n = \ell^2_{\mathbf{R}}(\{-n, \dots, n\})$ is coarsely dense, hence enveloping in V_n, and the increasing union $\bigcup V_n$ is dense in $\ell^2_{\mathbf{R}}(\mathbf{Z})$. But X is not coarsely dense. Indeed, for every $n \geq 0$, the element $\frac{1}{2}\mathbf{1}_{\{1,\dots,4n\}}$ is at distance $\sqrt{n}$ to X.

Note that X is the orbit of 0 for the natural action of the wreath product $\mathbf{Z} \wr \mathbf{Z} = \mathbf{Z}^{(\mathbf{Z})} \rtimes \mathbf{Z}$ on $\ell^2_{\mathbf{R}}(\mathbf{Z})$, where $\mathbf{Z}^{(\mathbf{Z})}$ acts by translations and the factor $\mathbf{Z}$ acts by shifting (compare to the example in the proof of Proposition 2.1).

Lemma 4.3 *Let G be a topological group and π an orthogonal representation, admitting a 1-cocycle b with enveloping orbits. Then π is strongly cohomological.*

Proof If σ is a nonzero subrepresentation of π, let b_σ be the orthogonal projection of b on $\mathcal{H}_\sigma$, so that $b_\sigma \in Z^1(G, \sigma)$. Then $b_\sigma(G)$ is enveloping in $\mathcal{H}_\sigma$, in particular b_σ

is unbounded. So b_σ defines a nonzero class in $H^1(G, \sigma)$. ■

Theorem 4.4 *Let G be a topological group with property H_{CT}. Let G act isometrically on a Hilbert space $\mathcal{H}$, with linear part π. Let $\mathcal{O}$ be an orbit under this action. Then there exist*

- *a subspace T of $\mathcal{H}$, contained in $\mathcal{H}^{\pi(G)}$,*
- *a closed, locally bounded convex subset U of $T^\perp$,*

such that $\mathcal{O}$ is contained in $T \times U$.

Proof We immediately reduce to the case when π has no invariant vectors, so that we must prove that the closed convex hull U of $\mathcal{O}$ is locally bounded.

Observe that a convex subset of a Hilbert space is locally bounded if and only if it contains no affine half-line. Thus denote by $\mathcal{D}$ the set of affine half-lines contained in U, and suppose by contradiction that $\mathcal{D} \neq \varnothing$. Denote by $\mathcal{D}_0$ the corresponding set of linear half-lines (where the linear half-line corresponding to a half-line $x + \mathbf{R}_+ v$ is simply $\mathbf{R}_+ v$). Then $\mathcal{D}_0$ is invariant under the linear action π of G. Let W be the closed subspace of $\mathcal{H}$ generated by all the half-lines in $\mathcal{D}_0$, and denote by σ the corresponding subrepresentation. By assumption, σ is nonzero.

We claim that σ is strongly cohomological, contradicting that π has no invariant vectors along with the H_{CT} assumption. Let ρ be a nonzero subrepresentation of σ. Then by the definition of W, there exists a half-line of U that projects injectively into the subspace of ρ. Thus $H^1(G, \rho) \neq 0$, proving the claim, and ending the proof. ■

Proof of Theorem 1 We can suppose that π has no invariant vectors. Suppose that the convex hull of $\alpha(G)(0)$ is not locally bounded. Then it contains a half-line $D = x + \mathbf{R}_+ v$. Let (x_n) be an unbounded sequence in D. Every x_n is a convex combination of elements of the form $\alpha(g)(0)$, where g ranges over a finite subset F_n of G. Also, since $\pi(G)$ has no invariant vector, there exists $g_0 \in G$ such that $\pi(g_0)v \neq v$. Let H be the subgroup of G generated by the countable subset $\{g_0\} \cup \bigcup_n F_n$. Then the convex hull of $\alpha(H)(0)$ contains D. By Proposition 3.9, H has property H_{CT}; it follows by Theorem 4.4 that D is parallel to the invariant vectors of $\pi(H)$, so that v is contained in the $\pi(H)$-invariant vectors, a contradiction. ■

Corollary 4.5 *Let G be a topological group with property H_{CT}. Let $\mathcal{H}$ be a Hilbert space on which G acts with enveloping (resp. coarsely dense, resp. dense) image. Then the action is by translations, defined by a continuous morphism: $u\colon G \to (\mathcal{H}, +)$ with enveloping (resp. coarsely dense, resp. dense) image.*

Corollary 4.6 *Let G be a locally compact, compactly generated group with property H_{CT}, and let $\mathcal{H}$ be a (real) Hilbert space. Then*

- *G has an isometric action on $\mathcal{H}$ with coarsely dense (respectively enveloping) orbits if and only $\mathcal{H}$ has finite dimension k, and G has a quotient isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$, with $n + m \geq k$.*
- *G has an isometric action on $\mathcal{H}$ with dense orbits if and only $\mathcal{H}$ has finite dimension k, and G has a quotient isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$, with $\max(n + m - 1, n) \geq k$.*

Proof Let α be an affine isometric action of G with enveloping orbits (this encompasses all possible assumptions). By Corollary 4.5, the action is by translations; let u be the morphism $G \to (\mathcal{H}, +)$. Its image generates $\mathcal{H}$ as a topological vector space. Let W denote the kernel of u.

Then $A = G/W$ is a locally compact, compactly generated abelian group, which embeds continuously into a Hilbert space. By standard structural results, A has a compact subgroup K such that A/K is a Lie group. Since K embeds into a Hilbert space, it is necessarily trivial, so that A is an abelian Lie group without compact subgroup. Accordingly, A is isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$ for some integers n, m; the embedding of A into $\mathcal{H}$ extends canonically to a linear mapping of $\mathbf{R}^{n+m}$ into $\mathcal{H}$. In particular $\mathcal{H}$ is finite-dimensional, of dimension $k \leq n + m$.

If the action has dense orbits, then either $m = 0$ and $n \geq k$, or $m \geq 1$ and $m \geq k-n+1$; this means that $k \leq \max(n+m-1, n)$. Conversely, if $k \leq n+m-1$, then, since $\mathbf{Z}$ has a dense embedding into the torus $\mathbf{R}^k/\mathbf{Z}^k$, $\mathbf{Z}^{k+1}$ has a dense embedding into $\mathbf{R}^k$, and this embedding can be extended to $\mathbf{R}^n \times \mathbf{Z}^m$. ■

From Proposition 3.9 and Corollary 4.6, we deduce the following.

Corollary 4.7 *A compactly generated, nilpotent-by-compact locally compact group does not admit any isometric action with enveloping,* e.g., *dense, orbits on an infinite-dimensional Hilbert space.*

Proposition 2.1 on the one hand, and Corollary 4.7 on the other, isolate the first test-case for Navas' question.

Question 2 Does there exist a polycyclic group admitting an affine isometric action with dense orbits on an infinite-dimensional Hilbert space?

Let us prove a related result for semisimple groups.

Theorem 4.8 *Let G be a connected, semisimple Lie group. Then G cannot act on a Hilbert space $\mathcal{H} \neq 0$ with coarsely dense,* e.g., *dense, orbits.*

Proof Suppose by contradiction the existence of such an action α, and let π denote its linear part. Then π is strongly cohomological. By Lemma 3.3, π is trivial on the centre of G. Thus the centre acts by translations, generating a finite-dimensional subspace V of $\mathcal{H}$. The action induces a map $p\colon G \to V \rtimes \mathrm{O}(V)$. Since G is semisimple, the kernel of p contains the sum G_{nc} of all noncompact factors of G, and thus factors through the compact group G/G_{nc}. Thus $H^1(G, V) = 0$, and since π is strongly cohomological, this implies that $V = 0$.

It follows that α is trivial on the centre of G, so that we can suppose that G has trivial centre. Then G is a direct product of simple Lie groups with trivial centre. We can write $G = H \times K$ where K denotes the sum of all simple factors S of G such that $\alpha(S)(0)$ is bounded (in other words, $H^1(S, \pi|_S) = 0$). Then the restriction of α to H also has coarsely dense orbits. Moreover, every simple factor of H acts in an unbounded way, so that, by a result of Shalom [Sh1, Theorem 3.4][1], the action of H is proper. That is, the map $i\colon H \to \mathcal{H}$ given by $i(h) = \alpha(h)(0)$ is metrically proper and

[1]Shalom only states the result for a simple group, but the proof generalizes immediately.

its image is coarsely dense. By metric properness, the subset $X = i(H) \subset \mathcal{H}$ satisfies: X is coarsely dense, and every ball in X (for the metric induced by $\mathcal{H}$) is compact.

Suppose that $\mathcal{H}$ is infinite-dimensional and let us deduce a contradiction. For some $d > 0$, we have $d(x, X) \leq d$ for every $x \in \mathcal{H}$. If $\mathcal{H}$ is infinite-dimensional, there exists, in a fixed ball of radius $7d$, infinitely many pairwise disjoint balls $B(x_n, 3d)$ of radius $3d$. Taking a point in $X \cap B(x_n, 2d)$ for every n, we obtain a closed, infinite and bounded discrete subset of X, a contradiction.

Thus $\mathcal{H}$ is finite-dimensional; since every simple factor of H is non-compact, it has no non-trivial finite-dimensional orthogonal representation, so that the action is by translations, and hence is trivial, so that finally $\mathcal{H} = \{0\}$. ∎

Remark 4.9. (i) The same argument shows that a semisimple, linear algebraic group over any local field cannot act with coarsely dense orbits on a Hilbert space.

(ii) The argument fails to work with enveloping orbits: indeed, in $\ell^2_{\mathbf{R}}(\mathbf{N})$, let X denote the set sequences (x_n) such that $x_n \in 2^n\mathbf{Z}$ for every $n \in \mathbf{N}$. Then X is coarsely dense in $\ell^2_{\mathbf{R}}(\mathbf{N})$, but, for the metric induced by $\mathcal{H}$, every ball in X is finite, hence compact. We do not know if a semisimple Lie group (e.g. $SL_2(\mathbf{R})$) can act isometrically on a nonzero Hilbert space with enveloping orbits.

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