Combinatorics, Probability and Computing (2023), **32**, pp. 68–90
doi:10.1017/S0963548322000128

ARTICLE

On the peel number and the leaf-height of Galton–Watson trees

Luc Devroye, Marcel K. Goh* and Rosie Y. Zhao

School of Computer Science, McGill University, Montreal, QC, Canada
*Corresponding author. Email: marcel.goh@mail.mcgill.ca

(Received 29 June 2021; revised 7 April 2022; accepted 10 May 2022; first published online 9 June 2022)

Abstract

We study several parameters of a random Bienaymé–Galton–Watson tree T_n of size n defined in terms of an offspring distribution ξ with mean 1 and nonzero finite variance σ^2. Let $f(s) = \mathbb{E}\{s^\xi\}$ be the generating function of the random variable ξ. We show that the independence number is in probability asymptotic to qn, where q is the unique solution to $q = f(1-q)$. One of the many algorithms for finding the largest independent set of nodes uses a notion of repeated peeling away of all leaves and their parents. The number of rounds of peeling is shown to be in probability asymptotic to $\log n / \log(1/f'(1-q))$. Finally, we study a related parameter which we call the leaf-height. Also sometimes called the protection number, this is the maximal shortest path length between any node and a leaf in its subtree. If $p_1 = \mathbb{P}\{\xi = 1\} > 0$, then we show that the maximum leaf-height over all nodes in T_n is in probability asymptotic to $\log n / \log(1/p_1)$. If $p_1 = 0$ and κ is the first integer $i > 1$ with $\mathbb{P}\{\xi = i\} > 0$, then the leaf-height is in probability asymptotic to $\log_\kappa \log n$.

Keywords: Independence number; Bienaymé–Galton–Watson trees; protection number

2020 MSC Codes: 60J80

1. Introduction

The independence number is a fundamental graph invariant that arises often in computational complexity theory and the analysis of algorithms. In a graph $G = (V, E)$, a subset $S \subseteq V$ of vertices is said to be an *independent set* if no two elements of S are adjacent. The dual notion is that of a *vertex cover*, namely a subset $C \subseteq V$ such that every edge in G has an endpoint in C. The *independence number* $I(G)$ of G is defined to be the size of the largest independent set in G. In this paper, we concern ourselves with the case $G = T$, a random tree in the Bienaymé–Galton–Watson model. In recent years, analysis of the independence number of trees has been carried out for various other random models. C. Banderier, M. Kuba, and A. Panholzer studied various families of simply -generated trees [4], and a recent paper of M. Fuchs, C. Holmgren, D. Mitsche, and R. Neininger considers random binary search trees as well as random recursive trees [13].

Because every tree T is bipartite, the independence number $I(T)$ is always at least $|T|/2$ (we take the larger element of the bipartition). Recall that a vertex set S is a *vertex cover* of T if every edge of T intersects a vertex in S. Letting $V(T)$ denote the size of a minimum-cardinality vertex cover, we have the formula $n = V(T) + I(T)$. In a tree, there always exists a maximum-cardinality independent set that includes all of the leaves, and the following algorithm, which will be the starting point of our discussion, uses this fact to find an independent set of maximum size. Note that this is only one of many possible algorithms that accomplishes this task.

© The Author(s), 2022. Published by Cambridge University Press.

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

Algorithm 1 *Independent set* Given a directed tree T, this algorithm computes a maximum-cardinality independent set A of vertices.

I1. [Initialise.] Set $A \leftarrow \emptyset$.

I2. [Compute leaves and parents.] Let $L(T)$ be the set of leaves of T, that is, the set of vertices with out-degree 0. Let $P(T)$ be the set of parents of nodes in $L(T)$.

I3. [Update.] Set $A \leftarrow A \cup L(T)$ and $T \leftarrow T \setminus L(T) \setminus P(T)$. (At this stage, T may now be a forest.)

I4. [Loop?] If $T = \emptyset$, halt and output A; otherwise, return to step I2.

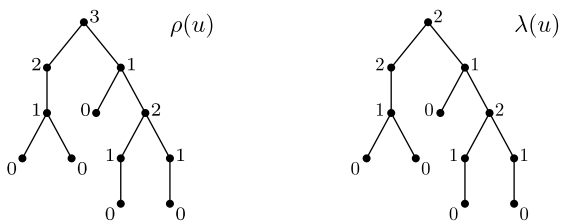

Figure 1. Peel numbers and leaf-heights of nodes in a unary-binary tree.

Algorithm I repeatedly peels away leaves and their parents to arrive at what we shall call the *layered independent set*. We refer to $L(T)$ as layer 0, to $P(T)$ as layer 1, to $L(T \setminus L(T) \setminus P(T))$ as layer 2, and so on. In this manner, each node u gets assigned a *peel number* $\rho(u)$, the layer number of the set to which it belongs. The peel number $\rho(T)$ of a tree T is the peel number of the root of T. We also let $m(T)$ denote the maximum of the peel numbers of vertices in T; this quantity is twice the number of loops that Algorithm I undergoes before termination, rounded up. Note that all the peel numbers can be computed by postorder traversal of the tree in time $O(|T|)$, and then the layered independent set is simply the collection of all nodes with even peel number.

A quantity related to the peel number is the *leaf-height* $\lambda(u)$ of a node $u \in T$. It is the length of the path to the nearest leaf in the (fringe) subtree rooted at u. The leaf-height $\lambda(T)$ of a tree T is the maximal leaf-height of any node in T. The fact that $\rho(u) = k$ implies that there is a leaf at depth k from the root, so $\lambda(u) \le \rho(u)$ for all nodes u in a tree. It is also easily seen that for any tree T, $\lambda(T) \le \rho(T) \le m(T)$. A small example is given in Figure 1; note that for nodes with few children or small subtrees, the two quantities are quite similar. One corollary of our main results is that under certain conditions, this phenomenon persists as n gets large, that is, the peel number and leaf-height have the same order of asymptotic growth.

The leaf-height goes by the name *protection number* in the literature and has enjoyed some recent attention. With this usage, a node whose minimal distance from any leaf is k is called *k-protected* and a 2-protected node is often simply said to be *protected*. In this paper, we say that a node has leaf-height k, which we believe is more illustrative than saying it is k-protected. The number of nodes with leaf-height ≥ 2 was examined by G.-S. Cheon and L. W. Shapiro for planted plane trees, Motzkin trees, full binary trees, Catalan trees, and ternary trees [8]; by T. Mansour [23] for k-ary trees; by R. R. X. Du and H. Prodinger for digital search trees [10]; by H. M. Mahmoud and M. D. Ward for binary search trees [21] and for random recursive trees [22]; and by L. Devroye and S. Janson, who considered simply generated trees and also unified some earlier results regarding binary search trees and random recursive trees. Nodes with leaf-height > 2 were studied in binary search trees by M. Bóna [5] and in planted plane trees by K. Copenhaver [9]. In the setting of simply generated trees and Pólya trees, the leaf-height of the root as well as the leaf-height of a node chosen uniformly at random was studied in [14].

The main results of this paper characterise the asymptotic behaviour of the independence number $I_n = I(T_n)$, the maximum peel number $M_n = m(T_n)$, and the maximum leaf-height $L_n = \lambda(T_n)$ for a Bienaymé–Galton–Watson tree T_n, which is conditioned on having n nodes. We also

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

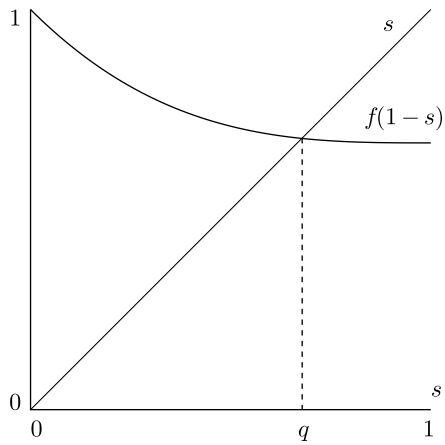

Figure 2. The parameter q satisfying $q = f(1-q)$.

include distributional properties of closely related statistics, such as the peel number and leaf-height of the root of an unconditional Bienaymé–Galton–Watson tree, as well as the leaf-height L_n' of the root and the leaf-height L_n'' of a node chosen uniformly at random in a conditional Bienaymé–Galton–Watson tree.

The Bienaymé–Galton–Watson model. For a nonnegative integer-valued random variable ξ, a *Bienaymé–Galton–Watson tree* is a random tree in which every node has i children independently with probability $p_i = \mathbb{P}\{\xi = i\}$. The random variable ξ is called the *offspring distribution* of the tree; we only consider distributions with mean $\mathbb{E}\{\xi\} = 1$ and variance $\mathbb{V}\{\xi\} = \sigma^2 \in (0, \infty)$ (standard references include [3] and [20]). Let T_n denote the tree T, conditioned on having n nodes. Note that many important simply generated families of trees can be characterised by a conditional Bienaymé–Galton–Watson tree with a certain distribution [15]. Strictly speaking, T is a graph (V, E), but we will abuse notation and write $v \in T$ to indicate that v is in the vertex set of T.

2. The independence number

We begin by studying unconditional Bienaymé–Galton–Watson trees. Recall that the generating function $f(s)$ of an offspring distribution ξ is the infinite series $\mathbb{E}\{s^\xi\}$, which converges absolutely when $0 \le s \le 1$. We can thus differentiate to obtain $f'(s) = \mathbb{E}\{\xi s^{\xi-1}\}$. A quantity that will play a key role in our story is q, the unique solution in $(0, 1)$ of $q = f(1-q)$. This is illustrated in Figure 2.

Lemma 1. *Let ξ be an offspring distribution with $0 < \mathbb{E}\{\xi\} \le 1$ and let $f(s) = \mathbb{E}\{s^\xi\}$. The probability that the root of a Bienaymé–Galton–Watson tree T with this distribution belongs to the layered independent set is q, which belongs to the interval* $(1/2, 1)$.

Proof. Note that q is the probability that all the children of the root are not in the layered independent set. By the recursive definition of a Bienaymé–Galton–Watson tree, we have

$$q = \sum_{i \ge 0} p_i (1-q)^i = f(1-q), \tag{1}$$

and the Banach fixed-point theorem guarantees the uniqueness of the solution to $s = f(1-s)$ in the compact interval $[0, 1]$. Of course, q cannot be 1 since $\mathbb{P}\{\xi = 0\} \ne 0$. The fact that $f(s) > s$ for all $s \in (0, 1)$ implies that $q = f(1-q) > 1-q$, hence $q > 1/2$. □

Lemma 1 is essentially known (see, e.g., Banderier, Kuba, and Panholzer [4]).

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

Example. There is a well-known connection between certain families of trees and conditioned Bienaymé–Galton–Watson trees. In each of the following cases, sampling a conditional Bienaymé–Galton–Watson tree T_n with the given distribution is equivalent to uniformly sampling a tree of size n from the respective tree family.

i) In *Flajolet's t-ary tree*, every node is either a leaf or has t children. This corresponds to the distribution with $p_0 = 1 - 1/t$ and $p_t = 1/t$, so we can compute q numerically by finding the unique solution to the equation

$$q = 1 - \frac{1}{t} + \frac{(1-q)^t}{t}. \tag{2}$$

in the interval $(1/2, 1)$. In the case $t = 2$ of full binary trees, we find that $q = 2 - \sqrt{2} \approx 0.585786$, and since the $(1-q)^t/t$ term is very small for larger values of t, q is approximately $1 - 1/t$ for large t.

ii) To obtain a random rooted *Cayley tree*, we set $p_i = (i!e)^{-1}$ for all $i \geq 0$. Since $f(s) = e^{s-1}$, we have $qe^q = 1$, which we can invert in terms of the *Lambert W function*. Concretely, we have

$$q = W(1) = \left(\int_{-\infty}^{\infty} \frac{dt}{(e^t - t)^2 + \pi^2} \right)^{-1} - 1 \approx 0.567143, \tag{3}$$

which is also known as the *omega constant*.

iii) *Planted plane trees* correspond to the distribution $p_i = 1/2^{i+1}$ for $i \geq 0$. In this case, $f(s) = 1/(2-s)$, yielding the equation $q^2 + q - 1 = 0$, whose solution in the correct range is $q = 1/\varphi \approx 0.618034$. (The golden ratio $\varphi = 1.618034$ is the more famous solution to this quadratic equation).

iv) *Motzkin trees*, also known as unary-binary trees, are trees in which every non-leaf node has either one tree or two children. This corresponds to the distribution $p_0 = p_1 = p_2 = 1/3$ and $p_i = 0$ for all $i \geq 3$. So we have $q = (1 + (1-q) + (1-q)^2)/3$ and we have $q = 3 - \sqrt{6} \approx 0.550510$.

v) A *binomial tree* of order d can be thought of as a tree in which every node has d "slots" for its children, some of which may be filled. Thus, a node can have r children in $\binom{d}{r}$ different ways, for $0 \leq r \leq d$. This corresponds, fittingly, to a binomial offspring distribution, where

$$p_i = \binom{d}{i} \left(\frac{1}{d}\right)^i \left(1 - \frac{1}{d}\right)^{d-i}, \tag{4}$$

for $0 \leq i \leq d$, and $p_i = 0$ otherwise. For this distribution, we have $f(s) = (s/n + 1 - 1/n)^n$, meaning that

$$q = \left(1 - \frac{1}{d} + \frac{1-q}{d}\right)^d = \left(1 - \frac{q}{d}\right)^d. \tag{5}$$

For large d, this tends to the omega constant. An important case is $d = 2$, which produces a random *Catalan tree*; it can be readily computed that $q = 4 - 2\sqrt{3} \approx 0.535898$ for these trees.

The following theorem shows the link between q and the size of the largest independent set in a conditioned Bienaymé–Galton–Watson tree.

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

Theorem 2. *Let ξ be an offspring distribution with $\mathbb{E}\{\xi\} = 1$ and let $f(s) = \mathbb{E}\{s^{\xi}\}$. The independence number $I_n = I(T_n)$ of a Bienaymé–Galton–Watson tree, conditioned on having n nodes, satisfies*

$$\frac{I_n}{n} \to q$$

in probability as $n \to \infty$, where q is the unique solution in $(1/2, 1)$ of the equation $q = f(1-q)$.

Proof. For a vertex u, we let Γ_u denote the set of children of u and let

$$g(u) = \begin{cases} 1, \text{ if the peel number of } u \text{ is even;} \\ 0, \text{ otherwise.} \end{cases} \tag{6}$$

Note that the recursive function

$$G(u) = g(u) + \sum_{v \in \Gamma_u} G(v) \tag{7}$$

is exactly the independence number of the subtree rooted at u. Since g is bounded, we can apply a result of S. Janson ([16], Theorem 1.3) to find that for a conditional Bienaymé–Galton–Watson tree T_n with root u,

$$\frac{I_n}{n} = \frac{G(u)}{n} \to \mathbb{E}\left\{g(u)\right\} = q \tag{8}$$

in probability as $n \to \infty$. □

Note that examples (ii), (iii), and the Catalan case agree with explicit computations given in [4]. For simply generated trees, that paper, which uses singularity analysis, derives the constant q in a different manner, proves the stronger statement $\mathbb{E}\left\{I(T_n)\right\} = qn + O(1)$, and also gives a formula for the variance in terms of the degree-weight generating function. In particular, they show there exists a constant ν depending on the family of trees such that the variance is $\nu n + O(1)$.

3. Minimum-size *s*-path vertex covers

This section represents a brief digression and will not be related to our remaining results, though it discusses the natural generalisation of Algorithm I and is related to the open problem we give at the end of the paper. As mentioned in the introduction, the size $V(T)$ of the minimal vertex cover of a T with n nodes has size $n - I(T)$, where $I(T)$ is the independence number. In particular, Algorithm I outputs a minimum-cardinality vertex cover alongside the maximum-cardinality independent set; it is the set of all nodes with odd peel number. We now tackle a more general notion of vertex covers. For an integer $s \geq 2$, an *s-path vertex cover* of a rooted tree T is a subset C of vertices such that any path of length $s-1$ in the tree contains a vertex in C. Thus, the common-or-garden vertex cover corresponds to $s = 2$. (The off-by-one quirk in the definition goes away if we measure a path not by its length, but instead by its *order*, that is, the number of vertices it contains.) Note that only directed paths are considered, so two children of the same node are *not* connected by a path of length 2.

One might be tempted to generalise our earlier observations by claiming that the set of nodes with peel number congruent to $s-1$ modulo s is a minimal s-path vertex cover. This is not true! Consider a tree in which the root has two children, and one of the children has itself one child. Then no node has peel number equal to 2, but of course, the minimal 3-path vertex cover consists of the root. Towards a correct generalisation, consider the fact that if in every loop of Algorithm I, we removed all subtrees of height exactly 1, then the roots of these removed subtrees are precisely the vertices with odd peel number. Thus, we arrive at an algorithm for computing a minimal s-path vertex cover.

Note that if the original tree had height less than $s-1$, the algorithm outputs the empty set, which is a valid cover, since there are no paths of length $s-1$ in the tree. The fact that

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

Algorithm P *Compute s-path vertex cover* Given $s \geq 2$, and a rooted tree T, this algorithm computes a minimal s-path vertex cover C.

P1. [Initialise.] Set $C \leftarrow \emptyset$.

P2. [Done?] If there are no subtrees with height exactly $s-1$, we output C and terminate.

P3. [Prune a subtree.] Let v be a node in T such that the subtree T_v rooted at v has height exactly $s-1$. We set $C \leftarrow C \cup \{v\}$ and set $T \leftarrow T \setminus T_v$. Return to step P2.

this algorithm actually does output a minimum-size vertex cover is proved in [6], and it is also remarked that the algorithm can be made to run in $O(|T|)$ time.

Let $V_s(T)$ denote the size of the minimum s-path vertex cover of a Bienaymé–Galton–Watson tree T. To determine this random quantity, we will have to determine the probability that a node is added to the set C in Algorithm P. We will say that a vertex $v \in T$ is "marked" if Algorithm P adds it to the cover C. The following lemma gives necessary and sufficient conditions for the root of a tree to be marked.

Lemma 3. *The root u of a tree is marked if and only if there exists a path of length $s-1$ from the root that contains no marked vertices (other than the root).*

Proof. Suppose that the root u is marked. This means that in the final iteration of Algorithm P, after all other marked nodes have been removed, the tree has height $s-1$. This means that some unmarked node v is at depth $s-1$, and no node is marked on the path to this node. (This happens when v is a leaf or all children of v are marked, since if a child of v is unmarked, then we have an unmarked path of length s in the tree and the algorithm would have to mark some node on this path before marking the root.) Conversely, if such a path exists, then Algorithm P will be in this state in the final iteration of the loop and will therefore mark the root. □

This observation can be used to derive a functional equation for the probability that a node in an unconditional Bienaymé–Galton–Watson tree is marked, as the following lemma shows.

Lemma 4. *Let T be a Bienaymé–Galton–Watson tree with offspring distribution ξ satisfying $\mathbb{E}\{\xi\} \leq 1$. Let $f(z) = \mathbb{E}\{\xi^z\}$ be the generating function of the distribution and let*

$$g(z,q) = 1 - f\left(q + (1-q)z\right) \tag{9}$$

The probability q_s that the root of the tree is in the minimum s-path vertex cover produced by Algorithm P satisfies

$$q_s = g(g(\cdots(g(0,q_s)\cdots,q_s),q_s),q_s), \tag{10}$$

where the function g is iterated $s-1$ times.

Proof. For $1 \leq j < s$, let E_j be the event that in an unconditional Bienaymé–Galton–Watson tree, there is a path of length j from the root that contains no marked vertices (except possibly the root). Thus $q_s = \mathbb{P}\{E_{s-1}\}$. Restating things slightly, E_j is the probability that there exists an unmarked child v of the root in whose subtree E_{j-1} is true. If the degree of the root is i, then the probability that all the children of the root fail to have this property is $\left(q_s + (1-q_s)\mathbb{P}\{E_{j-1}\}\right)^i$, so for $1 < j < s$,

$$\mathbb{P}\{E_j\} = \sum_{i \geq 0} p_i \left(1 - (q_s + (1-q_s)\mathbb{P}\{E_{j-1}\})^i\right) = 1 - f\left(q_s + (1-q_s)\mathbb{P}\{E_{j-1}\}\right). \tag{11}$$

Note that $\mathbb{P}\{E_1\}$ is simply the probability $1 - f(q_s) = g(0,q_s)$ that one of the children of the root is unmarked, so unravelling the above equation proves the lemma. □

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

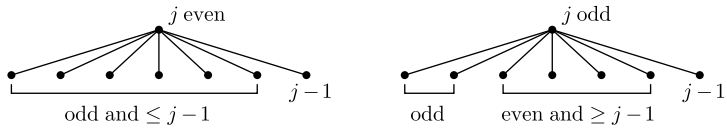

Figure 3. Children of nodes with even and odd peel numbers.

Note that when $s=2$, $q_s=1-q$, where q is the solution to $z=f(1-z)$ we studied earlier. By a recursive computation analogous to the one we performed for the independence number, we find that if $V_s(T_n)$ denotes the minimum size of an s-path vertex cover of the conditional Bienaymé–Galton–Watson tree T_n, then as $n\to\infty$,

$$\frac{V_s(T_n)}{n}\to q_s, \tag{12}$$

in probability. The function g given by Lemma 5 is rather unwieldy, so we cannot hope to find neat closed forms for the limit of $V_s(T_n)$ like we did for I_n in many special cases. However, we can, in principle, use g to numerically approximate the s-path vertex cover number for arbitrary distributions satisfying $\mathbb{E}\{\xi\}\leq 1$.

4. Distribution of the peel number

Let r_i denote the probability that the root of an unconditional Bienaymé–Galton–Watson tree has peel number i. In this section, we shall compute the distribution $(r_i)_{i\geq 0}$. It will also be convenient to set $r_i=0$ when i is negative. We will establish the notation

$$r_i^+=\sum_{j\geq i} r_j \qquad \text{and} \qquad r_i^-=\sum_{j=0}^{i} r_j\,. \tag{13}$$

There will be some asymmetry for odd and even i, so let us write $r_i^{+\text{odd}}$ for the subsum of r_i^+ consisting of odd terms and $r_i^{+\text{even}}$ for the subsum of r_i^+ consisting of even terms. Defining $r_i^{-\text{odd}}$ and $r_i^{-\text{even}}$ similarly, we have, of course, $r_i^{+\text{odd}}+r_i^{+\text{even}}=r_i^+$ and $r_i^{-\text{odd}}+r_i^{-\text{even}}=r_i^-$. The situation is depicted in Figure 3.

Clearly, $r_0=p_0$. For even indices j, all children must have an odd peel number at most $j-1$ and at least one must have peel number $j-1$. Thus, if ξ is the number of children at the root, then for $i\geq 1$,

$$r_{2i}=\mathbb{E}\left\{\left(r_{2i-1}^{-\text{odd}}\right)^{\xi}-\left(r_{2i-3}^{-\text{odd}}\right)^{\xi}\right\}=f\left(r_{2i-1}^{-\text{odd}}\right)-f\left(r_{2i-3}^{-\text{odd}}\right). \tag{14}$$

For odd indices j, all the children of the root with even peel number must have peel number at least $j-1$, and at least one must have peel number $j-1$. Since

$$\sum_{i\geq 0} r_{2i-1}=1-q \qquad \text{and} \qquad \sum_{i\geq 0} r_{2i}=q, \tag{15}$$

we find that for $i\geq 1$,

$$r_{2i-1}=\mathbb{E}\left\{\left(1-q+r_{2i-2}^{+\text{even}}\right)^{\xi}-\left(1-q+r_{2i}^{+\text{even}}\right)^{\xi}\right\}. \tag{16}$$

The following lemma describes r_i for large i.

Lemma 5. *Let r_i be the probability that the root of an unconditional Bienaymé–Galton–Watson tree with offspring distribution $\xi\sim(p_i)_{i\geq 0}$ has peel number equal to i. As $i\to\infty$, we have*

$$r_i=f'(1-q)^{i+o(i)}. \tag{17}$$

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

Proof. In the even case, we have

$$\begin{aligned} r_{2i} &= f\left(r_{2i-1}^{-\text{odd}}\right) - f\left(r_{2i-3}^{-\text{odd}}\right) \\ &\sim r_{2i-1} \sum_{j\geq 0} jp_j \left(r_{2i-3}^{-\text{odd}}\right)^{j-1} \\ &= r_{2i-1} f'\left(r_{2i-3}^{-\text{odd}}\right), \end{aligned} \tag{18}$$

which, since $r_{2i-3}^{-\text{odd}} \to 1-q$, is asymptotic to $r_{2i-1}f'(1-q)$. Similarly, we have

$$r_{2i-1} = f\left(r_{2i-2}^{+\text{even}} + 1 - q\right) - f\left(r_{2i}^{+\text{even}} + 1 - q\right) \sim r_{2i-2} \sum_{j\geq 0} jp_j (1-q)^{j-1}, \tag{19}$$

which is also asymptotic to $r_{2i-2}f'(1-q)$. □

If N_i is the number of nodes in the ith layer for our algorithm, then Aldous's theorem [1] implies that for every fixed i,

$$\frac{N_i}{n} \to r_i \tag{20}$$

in probability. The number of nodes in the layers decreases at the indicated rate, namely $f'(1-q)$. As $q \in (1/2, 1)$, we have

$$p_1 = f'(0) < f'(1-q) < f'\left(\frac{1}{2}\right) \leq \mathbb{E}\left\{\frac{1}{2^\xi}\right\}. \tag{21}$$

The next section will need the event that the maximum peel number in an unconditional tree occurs at the root. We have the following lemma.

Lemma 6. *Let T be an unconditional Bienaymé–Galton–Watson tree with offspring distribution ξ. Let R be the peel number of the root of such a tree and let M be the maximum peel number of any node in the tree. Let q be the solution to $q = f(1-q)$, where f is the reproduction generating function of this distribution. Then*

$$\tau_i := \mathbb{P}\{R = M = i\} = f'(1-q)^{i+o(i)} \tag{22}$$

as $i \to \infty$.

Proof. The fact that $\tau_i \leq \mathbb{P}\{R = i\} = r_i = f'(1-q)^{i+o(i)}$ means that we only have to worry about finding a lower bound. To that end, consider the ξ children of the root (each the root of unconditional Bienaymé–Galton–Watson trees), with peel numbers $R_1, \ldots, R_\xi$ and maximum peel numbers $M_1, \ldots, M_\xi$. We consider the odd and even cases separately.

When i is odd, the event that $R = M = i$ is implied by the event that there exists some $1 \leq j \leq \xi$ with $R_j = M_j = i-1$ and for all $k \neq j$, we have R_j odd and $M_j \leq i$. Therefore, when i is odd, we have, by the inclusion–exclusion inequality,

$$\tau_i \geq \mathbb{E}\left\{\xi \cdot \tau_{i-1}\mathbb{P}\{R \text{ odd},\ M \leq i\}^{\xi-1}\right\} - \mathbb{E}\left\{\binom{\xi}{2} \cdot {\tau_{i-1}}^2 \mathbb{P}\{R \text{ odd},\ M \leq i\}^{\xi-2}\right\}. \tag{23}$$

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

Note that $\mathbb{P}\{R \text{ odd},\ M > i\} = o(1)$ as $i \to \infty$, and so

$$\begin{aligned}\tau_i &\geq \mathbb{E}\left\{\xi \cdot \tau_{i-1}\left(1-q-o(1)\right)^{\xi-1}\right\} - \mathbb{E}\left\{\binom{\xi}{2} \cdot {\tau_{i-1}}^2\left(1-q-o(1)\right)^{\xi-2}\right\} \\ &= \tau_{i-1} f'\left(1-q-o(1)\right) - \frac{{\tau_{i-1}}^2}{2} f''(1-q) \\ &\geq \tau_{i-1} f'\left(1-q-o(1)\right) - \frac{{\tau_{i-1}}^2 \sigma^2}{2}. \end{aligned} \tag{24}$$

When i is even, the event that $R = M = i$ is implied by the event that there exists some $1 \leq j \leq \xi$ with $R_j = M_j = i-1$ and for all $k \neq j$, we have R_j odd, $R_j \leq i-2$, and $M_j \leq i$. With another application of the inclusion–exclusion inequality and by a similar argument as in the odd case, we have

$$\begin{aligned}\tau_i &\geq \mathbb{E}\left\{\xi \cdot \tau_{i-1} \mathbb{P}\left\{R \leq i-2,\ M \leq i,\ R \text{ odd}\right\}^{\xi-1}\right\} \\ &\quad - \mathbb{E}\left\{\binom{\xi}{2} {\tau_{i-1}}^2 \mathbb{P}\left\{R \leq i-2,\ M \leq i,\ R \text{ odd}\right\}^{\xi-2}\right\} \\ &\geq \tau_{i-1} f'\left(1-q-o(1)\right) - \frac{{\tau_{i-1}}^2 \sigma^2}{2}. \end{aligned} \tag{25}$$

In both the odd and even cases, we see that $\tau_i \geq f'(1-q)^{i+o(i)}$, completing the proof. □

Using this result, we can give the following property of the distribution of the maximum peel number.

Lemma 7. *The maximum peel number M in an unconditional Bienaymé–Galton–Watson tree satisfies*

$$\mathbb{P}\{M \geq i\} = f'(1-q)^{i/2+o(i)}, \tag{26}$$

as $i \to \infty$, where f is the reproduction generating function.

Proof. As before, let $R_1, \ldots, R_\xi$ denote the peel numbers of children of the root and let $M_1, \ldots, M_\xi$ denote the maximum peel numbers in their respective subtrees. Let $\mu_i = \mathbb{P}\{M = i\}$, $\mu_i^- = \mathbb{P}\{M \leq i\}$, and $\mu_i^+ = \mathbb{P}\{M \geq i\}$. The event that $M \geq i$ is implied by the event

$$\left(\max_{1 \leq j \leq \xi} M_j \geq i\right) \text{ or } \left(\max_{1 \leq j \leq \xi} M_j < i \text{ and there is some } 1 \leq j \leq \xi \text{ with } R_j = M_j = i-1\right). \tag{27}$$

Thus, letting E_j be the event that $R_j = M_j = i-1$, we have

$$\mathbb{P}\{M \geq i\} \geq \mathbb{P}\left\{\max_{1 \leq j \leq \xi} M_j \geq i\right\} + \mathbb{P}\left\{\max_{1 \leq j \leq \xi} M_j < i,\ \bigcup_{j=1}^{\xi} E_j\right\}. \tag{28}$$

Note first that

$$\mathbb{P}\left\{\max_{1 \leq j \leq \xi} M_j \geq i\right\} = 1 - \mathbb{E}\left\{(1-\mu_i^+)^\xi\right\} = 1 - f(1-\mu_i^+). \tag{29}$$

By taking the Taylor series expansion of f around 1, we have

$$f(1-s) = f(1) - sf'(1) + \frac{s^2}{2} f''(\theta) \tag{30}$$

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

for some $1-s \le \theta \le 1$, so that

$$f(1-s) = 1 - s + \frac{s^2}{2} f''(\theta) \le 1 - s + \frac{s^2}{2}\sigma^2 \tag{31}$$

and

$$\mathbb{P}\left\{\max_{1\le j\le \xi} M_j \ge i\right\} \ge \mu_i^+ - \frac{{\mu_i^+}^2\sigma^2}{2}. \tag{32}$$

Next, by the union bound, we have

$$\begin{aligned}\mathbb{P}\left\{\max_{1\le j\le \xi} M_j < i, \bigcup_{j=1}^{\xi} E_j\right\} &= \mathbb{P}\left\{\bigcup_{j=1}^{\xi} E_j\right\} - \mathbb{P}\left\{\bigcup_{j=1}^{\xi} E_j, \max_{1\le j\le \xi} M_j \ge i\right\} \\ &= 1 - \mathbb{E}\left\{(1-\tau_{i-1})^{\xi}\right\} - \mathbb{E}\left\{\xi(\xi-1)\tau_{i-1}\mu_i^+\right\} \\ &= 1 - f(1-\tau_{i-1}) - \sigma^2\tau_{i-1}\mu_i^+ \\ &\ge \tau_{i-1} - \frac{{\tau_{i-1}}^2\sigma^2}{2} - \sigma^2\tau_{i-1}\mu_i^+. \end{aligned} \tag{33}$$

Collecting these bounds back into (28), we have

$$\mu_i^+ \ge \mu_i^+ + \tau_{i-1} - \frac{{\mu_i^+}^2\sigma^2}{2} - \frac{{\tau_{i-1}}^2\sigma^2}{2} - \sigma^2\tau_{i-1}\mu_i^+ \tag{34}$$

and therefore

$$\frac{\sigma^2}{2}{\mu_i^+}^2 \ge \tau_{i-1}(1-\sigma^2\mu_i^+) - \frac{\tau_{i-1}^2\sigma^2}{2}. \tag{35}$$

Let $\phi(x)$ be a decreasing function that is $o(1)$ as $x \to \infty$. We combine (35) with Lemma 7 to conclude that

$$\frac{\sigma^2}{2}{\mu_i^+}^2 \ge \tau_{i-1}\,(1-\phi(i)) = f'(1-q)^{i+o(i)}. \tag{36}$$

To bound μ_i^+ from above, we observe that since the event that $M \ge i$ is a subset of the event (27), we have

$$\begin{aligned}\mu_i^+ &\le 1 - f(1-\mu_i^+) + \mathbb{E}\left\{1-(1-\tau_{i-1})^{\xi}\right\} \\ &\le \mu_i^+ - \frac{{\mu_i^+}^2}{2}\left(\sigma^2 + o(1)\right) + \tau_{i-1} - \frac{{\tau_{i-1}}^2}{2}\left(\sigma^2 + o(1)\right), \end{aligned} \tag{37}$$

and therefore

$$\frac{\sigma^2 + o(1)}{2}{\mu_i^+}^2 \le \tau_{i-1} = f'(1-q)^{i+o(i)}, \tag{38}$$

which is what we need. □

5. Asymptotics of the peel number

We are now ready to prove an asymptotic result for the peel number of T_n; that is, the maximum peel number over all nodes in T_n. This is the number of rounds of peeling required by Algorithm I to calculate a maximum-cardinality independent set. Our proof uses Kesten's tree T_∞, whose construction we shall briefly recall here (see [18]). Fix an offspring distribution ξ with $\mathbb{E}\{\xi\} = 1$. Starting from the root, we attach ζ children, where $\mathbb{P}\{\zeta = i\} = ip_i$ for $i \ge 0$. Now select a child

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

uniformly at random and mark it. We repeat the process at the marked child, while all other children become the roots of independent unconditional Bienaymé–Galton–Watson trees with the ordinary offspring distribution ξ. For a tree t, we let $\tau(t,k)$ denote t, truncated to include only the first k levels. Letting TV denote total variation distance, it is well known (see [17] and [25]) that if $k = o(\sqrt{n})$, then

$$\lim_{n\to\infty} \mathrm{TV}\left(\tau(T_\infty, k), \tau(T_n, k)\right) = 0. \tag{39}$$

Theorem 8. *Let M_n be the maximum peel number in T_n, a conditional Bienaymé–Galton–Watson tree on n nodes with offspring distribution ξ. Then*

$$\frac{M_n}{\log n} \to \frac{1}{\log\left(1/f'(1-q)\right)} \tag{40}$$

in probability, where f is the generating function of ξ.

Proof. For any tree t, let $h(t)$ denote its height and $m(t)$ its maximum peel number. For the lower bound, we employ Kesten's limit tree T_∞. Let S_k denote the set of nodes of T_∞ that are children of nodes on the spine of $\tau(T_\infty, k)$ (i.e. nodes that are marked in the construction of T_∞). Let

$$\alpha_n = \left\lfloor \frac{\sqrt{n}}{\log^2 n} \right\rfloor \qquad \text{and} \qquad \beta_n = \left\lfloor \frac{\sqrt{n}}{\log n} \right\rfloor.$$

By the same result of [17] and [25] that we used before, we can find a coupling of $\tau(T_n, \beta_n)$ and $\tau(T_\infty, \beta_n)$ such that

$$\mathbb{P}\left\{\tau(T_n, \beta_n) \neq \tau(T_\infty, \beta_n)\right\} = o(1). \tag{41}$$

For every node u in T_∞, let T_u be the subtree of T_∞ rooted at u. Let $M_n = m(T_n)$. Letting E_{ux} denote the event that $m(T_u) \le x$, we have

$$\begin{aligned}\mathbb{P}\{M_n \le x\} \le{}& \mathbb{P}\left\{\tau(T_n, \beta_n) \neq \tau(T_\infty, \beta_n)\right\} + \mathbb{P}\left\{\bigcap_{u\in S_{\alpha_n}} E_{ux}\right\} \\ &+ \mathbb{P}\left\{\max_{u\in S_{\alpha_n}} h(T_u) \ge \beta_n - \alpha_n\right\}.\end{aligned} \tag{42}$$

We already pointed out that $\mathbb{P}\left\{\tau(T_n, \beta_n) \neq \tau(T_\infty, \beta_n)\right\} = o(1)$; we bound the other two terms by

$$\begin{aligned}&\mathbb{P}\left\{\bigcap_{u\in S_{\alpha_n}} E_{ux}\right\} + \mathbb{P}\left\{\max_{u\in S_{\alpha_n}} h(T_u) \ge \beta_n - \alpha_n\right\} \\ &\qquad \le \mathbb{P}\left\{|S_{\alpha_n}| \le \frac{\sigma^2\alpha_n}{2}\right\} + \mathbb{P}\left\{|S_{\alpha_n}| \ge \frac{\sigma^2 3\alpha_n}{2}\right\} \\ &\qquad\quad + \mathbb{P}\left\{m(T) \le x\right\}^{\sigma^2\alpha_n/2} + \frac{2\sigma^2}{2}\alpha_n \mathbb{P}\left\{h(T) \ge \beta_n - \alpha_n\right\},\end{aligned} \tag{43}$$

where T is an unconditional Bienaymé–Galton–Watson tree. Now, $S_{\alpha_n}/(\sigma^2\alpha_n) \to 1$ in probability by the law of large numbers, as the expected number of children of any node on the spine of T_∞ is $\sigma^2 + 1$. So, the first two terms of (43) tend to zero. Next, we see that

$$\mathbb{P}\left\{m(T) \le x\right\}^{\sigma^2\alpha_n/2} = \left(1 - \mathbb{P}\left\{m(T) > x\right\}\right)^{\sigma^2\alpha_n/2} \le \exp\left(-f'(1-q)^{x/2+o(x)}\frac{\sigma^2\alpha_n}{2}\right), \tag{44}$$

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

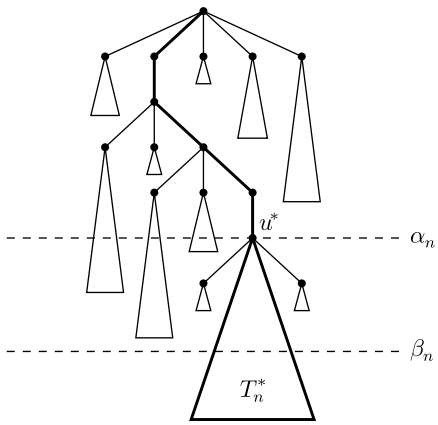

Figure 4. The proof uses Kesten's infinite tree T_∞ for both bounds.

which tends to zero if $x = (1-\epsilon)\log n/\log\left(1/f'(1-q)\right)$. For the final term, we have, by Kolmogorov's theorem (see, e.g. [20] or [2])

$$\frac{3\sigma^2\alpha_n}{2}\mathbb{P}\left\{h(T) > \beta_n - \alpha_n\right\} \sim \frac{3\sigma^2}{2}\cdot\frac{2\alpha_n}{\sigma^2(\beta_n-\alpha_n)} \sim \frac{3\alpha_n}{\beta_n} \sim \frac{3}{\log n}, \tag{45}$$

which goes to zero. We have shown that

$$\mathbb{P}\left\{M_n < (1-\epsilon)\frac{\log n}{\log\left(1/f'(1-q)\right)}\right\} \to 0 \tag{46}$$

for all $\epsilon > 0$.

For the upper bound we will again work with T_∞, truncated to level β_n, but also require some further auxiliary definitions. Let u^* denote the unique node on the spine of T_∞ at distance α_n from the root of T_n and let T_n^* be its subtree in T_n (the setup is illustrated in Figure 4). Let S be the set of children of nodes on the spine at distance $\le \alpha_n$ from the root. We then define

$$M_n' = \max_{u\in S}\rho(u) \qquad \text{and} \qquad M_n'' = \max_{u\in S} m(T_u).$$

Next, we let S^* denote the set of nodes u on the spine with the property that all of u's non-spine children have an odd peel number. In particular, let Y_n be the maximal number of *consecutive* nodes on the spine that are in S^*. Lastly, we let Y_n^* denote the number of consecutive nodes on the spine, starting at the parent of u^*, whose non-spine children all have an odd peel number. Assuming that $\tau(T_n,\beta_n) = \tau(T_\infty,\beta_n)$, we have the inequality

$$m(T_n) \le \max\left(m(T_n^*), \rho(u^*) + Y_n^*, M_n' + Y_n, M_n''\right) \le \max\left(m(T_n^*) + Y_n^*, 2M_n', 2Y_n, M_n''\right). \tag{47}$$

To explain this, we note that nodes in S^* have a peel number that is at most one more than the peel numbers of their children on the spine. Nodes on the spine that are not in S^* have a peel number that is at most one more than the maximum peel number of any of their non-spine children (and this is bounded from above by M_n').

Let $\epsilon > 0$ be given and let $x = (1+\epsilon)\log n/\log\left(1/f'(1-q)\right)$. We have

$$\begin{aligned}\mathbb{P}\{m(T_n) \ge x\} &\le \mathbb{P}\{\tau(T_n,\beta_n) \ne \tau(T_\infty,\beta_n)\} + \mathbb{P}\left\{\max_{u\in S} h(T_u) \ge \beta_n - \alpha_n\right\} + \mathbb{P}\left\{Y_n^* \ge \sqrt{\log n}\right\} \\ &\quad + \mathbb{P}\{Y_n \ge x/2\} + \mathbb{P}\left\{M_n' \ge x/2\right\} + \mathbb{P}\left\{M_n'' \ge x\right\} + \mathbb{P}\left\{m(T_n^*) \ge x - \sqrt{\log n}\right\}.\end{aligned} \tag{48}$$

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

As noted in our proof of the lower bound, the first two terms are $o(1)$, so we have reduced our task to showing that the latter five terms are also $o(1)$.

Let ζ be the offspring distribution of nodes on the spine (recall that $\mathbb{P}\{\zeta = i\} = ip_i$). For a node on the spine, the probability that it is in S^* is

$$\mathbb{E}\left\{(1-q)^{\zeta-1}\right\} = \sum_{i\geq 0} ip_i(1-q)^{i-1} = f'(1-q). \tag{49}$$

Thus, Y_n^* is a geometric random variable with parameter $1 - f'(1-q)$, and hence

$$\mathbb{P}\left\{Y_n^* \geq \sqrt{\log n}\right\} = o(1). \tag{50}$$

Also, Y_n is bounded from above in distribution by the maximum of α_n independent Geo $\left(f'(1-q)\right)$ random variables, so that

$$\mathbb{P}\{Y_n \geq x/2\} \leq \alpha_n f'(1-q)^{x/2} = o(1). \tag{51}$$

Next,

$$\mathbb{P}\{M_n' \geq x/2\} \leq \mathbb{E}\left\{|S|\right\}\mathbb{P}\{R \geq x/2\} = \sigma^2\alpha_n f'(1-q)^{x/2+o(x)} = o(1) \tag{52}$$

and

$$\mathbb{P}\{M_n'' \geq x\} \leq \mathbb{E}\left\{|S|\right\}\mathbb{P}\{M \geq x\} = \sigma^2\alpha_n f'(1-q)^{x/2+o(x)} = o(1). \tag{53}$$

This leaves us with the final term of (48). Observe that $|T_n^*| = n - \alpha_n - \sum_{u\in S}|T_u|$, which is at most $n - \max_{u\in S}|T_u|$. Thus,

$$\begin{aligned}\mathbb{P}\left\{|T_n^*| \geq n - \frac{n}{\log^5 n}\right\} &\leq \mathbb{P}\left\{\max_{u\in S}|T_u| \leq \frac{n}{\log^5 n}\right\} \\ &= \mathbb{E}\left\{\mathbb{P}\left\{|T| \leq \frac{n}{\log^5 n}\right\}^{|S|}\right\} \\ &\leq \mathbb{P}\left\{|S| \leq \frac{\sigma^2\alpha_n}{2}\right\} + \left(1 - \mathbb{P}\left\{|T| > \frac{n}{\log^5 n}\right\}\right)^{\sigma^2\alpha_n/2}. \end{aligned} \tag{54}$$

Since, as noted earlier, $|S|/(\sigma^2\alpha_n) \to 1$ in probability and since $\mathbb{P}\left\{|T| \geq n\right\} = \Theta(1/\sqrt{n})$, we have

$$\begin{aligned}\mathbb{P}\left\{|T_n^*| \geq n - \frac{n}{\log^5 n}\right\} &\leq o(1) + \exp\left(-\Theta\left(\frac{\log^{5/2} n}{\sqrt{n}}\right)\frac{\sigma^2}{2}\alpha_n\right) \\ &\leq o(1) + \exp\left(-\Theta\left(\sqrt{\log n}\right)\right), \end{aligned} \tag{55}$$

which is $o(1)$. So

$$\begin{aligned}\mathbb{P}\left\{m(T_n^*) \geq x - \sqrt{\log n}\right\} &\leq \max_{1\leq k\leq n-n/\log^5 n}\mathbb{P}\left\{m(T_n^*) \geq x - \sqrt{\log n}\,\middle|\,|T_n^*| = k\right\} \\ &+ \mathbb{P}\left\{|T_n^*| \geq n - \frac{n}{\log^5 n}\right\}. \end{aligned} \tag{56}$$

Noting that given $|T_n^*| = k$, T_n^* is again a Bienaymé–Galton–Watson tree and letting F_k be the event that there exists a node $v \in T_n$ with $|T_v| \leq k$ and $m(T_v) \geq x - \sqrt{\log n}$, we see that

$$\mathbb{P}\left\{m(T_n^*) \geq x - \sqrt{\log n}\,\middle|\,|T_n^*| = k\right\} \leq \mathbb{P}\left\{F_{n-n/\log^5 n}\right\}. \tag{57}$$

Now define $t(v)$ to be the subtree of v in the shifted preorder degree sequence $\xi_v, \xi_{v+1}, \ldots, \xi_n, \xi_1, \ldots \xi_{v-1}$. Let $\rho(v)$ be the peel number of the root v of $t(v)$ and let G_{vx} denote

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

the event that $\rho(v) \geq x - \sqrt{\log n}$ and $|t(v)| \leq n/\log^5 n$. We have

$$\mathbb{P}\{M_n \geq x\} \leq \mathbb{P}\Big\{\bigcup_{v\in T_n} G_{vx}\Big\} + o(1). \tag{58}$$

Note that $\max_{v\in T_n;|t(v)|\leq n/\log^5 n}\rho(v)$, is invariant under the cyclic shift of the preorder degree sequence. This rotational invariance, by Dwass' device [11], shows that

$$\mathbb{P}\left\{\bigcup_{v\in T_n} G_{vx}\right\} = \frac{\mathbb{P}\left\{\bigcup_{v\in T_n} G_{vx},\ \sum_{1\leq i\leq n}(\xi_i - 1) = -1\right\}}{\mathbb{P}\left\{\sum_{1\leq i\leq n}(\xi_i - 1) = -1\right\}}, \tag{59}$$

where on the right-hand side, all probabilities are with respect to an i.i.d. sequence $\xi_1, \ldots, \xi_n$. We bound (59) from above by

$$\mathbb{P}\left\{\bigcup_{v\in T_n} G_{vx}\right\} \leq n \cdot \frac{\mathbb{P}\left\{\rho(1) \geq x - \sqrt{\log n},\ |t(1)| \leq n/\log^5 n,\ \sum_{i=1}^n(\xi_i - 1) = -1\right\}}{\mathbb{P}\left\{\sum_{i=1}^n(\xi_i - 1) = -1\right\}}. \tag{60}$$

By conditioning on the size of $t(1)$, we obtain the further bound

$$\mathbb{P}\left\{\bigcup_{v\in T_n} G_{vx}\right\} \leq n \cdot \mathbb{P}\left\{\rho(1) \geq x - \sqrt{\log n}\right\} \cdot \frac{\sup_{n/\log^5 n\leq k\leq n}\mathbb{P}\left\{\sum_{i=1}^k(\xi_i - 1) = 0\right\}}{\mathbb{P}\left\{\sum_{i=1}^n(\xi_i - 1) = -1\right\}}.$$

By Kolchin's estimate [19], the fraction is $\Theta(1)$, therefore,

$$\mathbb{P}\left\{\bigcup_{v\in T_n} G_{vx}\right\} \leq nf'(1-q)^{x+o(x)},$$

which goes to 0 if $x = (1+\epsilon)\log n/\log\big(1 - f'(1-q)\big)$. □

If, instead of removing leaves and parents at each step, we only remove leaves, then it is clear that the number of rounds needed to delete all nodes is simply the height of the tree. The height of random binary trees was studied by P. Flajolet and A. Odlyzko, who showed that in this case, $H_n/\sqrt{n}$ converges in law to a theta distribution [12]. Earlier, it was shown by N. G. de Bruijn, D. E. Knuth, and S. O. Rice that the expected height of a random planted plane tree is $\sqrt{\pi n} + O(1)$. It is interesting that deleting only leaves from T_n at each step requires $\Theta\big(\sqrt{n}\big)$ rounds of deletion, but deleting leaves and their parents causes the number of rounds to decrease to $\Theta(\log n)$.

Examples. We apply Theorem 8 to calculate explicit asymptotics of the maximum peel number for the various families of trees mentioned earlier.

i.) *Flajolet's t-ary trees:* We have $f'(1-q) = 1-q$ and thus $M_n/\log n \to 1/\log(1/(1-q))$ in probability. As t gets large, q approaches $1 - 1/t$, so that the limit of $M_n/\log n$ is approximately $1/\log t$ for large t. For the case of full binary trees when $t = 2$, recall that $q = 2 - \sqrt{2}$ and thus $M_n/\log n \to -1/\log(\sqrt{2} - 1)$ in probability.

ii.) *Cayley trees:* In this case, $f'(1-q) = e^{-q}$ and hence $M_n/\log n \to 1/q$ in probability; we know from earlier that $q = W(1)$, so $1/q \approx 1.763223$.

iii.) *Planted plane trees:* We calculate $f'(1-q) = 1/(q+1)^2$. Recalling that $q = 1/\varphi$ where $\varphi = (\sqrt{5} - 1)/2$ is the golden ratio, we have in probability $M_n/\log n \to 1/\varphi^2 \approx 0.381966$.

iv.) *Motzkin trees:* The derivative $f'(1-q) = (3-2q)/3$ and substituting $q = 3 - \sqrt{6}$ we get $M_n/\log n \to 1/(\log 3 - \log(2\sqrt{6} - 3)) \approx 2.186769$.

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

v.) *Binomial trees:* In this case, we have $f'(1-q) = (1-q/d)^{d-1}$. As $d \to \infty$, it is clear to see that $M_n/\log n \to 1/W(1)$ in probability, matching the earlier calculation for Cayley trees above. For the special case $d = 2$ of Catalan trees, we have $f'(1-q) = 1 - q/2$ and thus $M_n/\log n \to -1/\log(\sqrt{3}-1)$. This constant is greater than that we obtain for full binary trees above, which is consistent with intuitive reasoning about the maximal peel numbers of these trees.

6. Distribution of the leaf-height

We now repeat the treatment given in Section 4, but this time for the distribution $(\ell_i)_{i\geq 0}$, where for each $i \geq 0$, ℓ_i denotes the probability that the root of an unconditional Bienaymé–Galton–Watson tree has leaf-height equal to i. Observe that ℓ_0 is exactly the probability p_0 that the root has no children and in general, for a node u with children Γ_u the leaf-height $\lambda(u)$ is

$$\lambda(u) = \begin{cases} 0, & \text{if } u \text{ is a leaf;} \\ \min_{v\in\Gamma_u} \lambda(v) + 1, & \text{otherwise.} \end{cases} \tag{61}$$

We define ℓ_i^+ and ℓ_i^- analogously to r_i^+ and r_i^-:

$$\ell_i^+ = \sum_{j\geq i} \ell_j \qquad \text{and} \qquad \ell_i^- = \sum_{j=0}^{i} \ell_j\,; \tag{62}$$

since $(\ell_i)_{i\geq 0}$ defines a distribution, $\ell_{i+1}^+ + \ell_i^- = 1$ for every $i \geq 0$. Letting E_i be the event that all the children of the root have leaf-height at least i, we have, for $i \geq 1$, $\ell_i = \mathbb{P}\{E_{i-1}\} - \mathbb{P}\{E_i\}$. We can then compute

$$\ell_1 = 1 - \mathbb{E}\left\{(1-\ell_0)^\xi\right\} = 1 - f(1-\ell_0) = 1 - f(1-p_0). \tag{63}$$

and, in general, for $i \geq 1$,

$$\begin{aligned} \ell_{i+1} &= \mathbb{P}\{E_i\} - \mathbb{P}\{E_{i+1}\} \\ &= \mathbb{E}\left\{(\ell_i^+)^\xi\right\} - \mathbb{E}\left\{(\ell_{i+1}^+)^\xi\right\} \\ &= f(\ell_i^+) - f(\ell_{i+1}^+) \\ &= f(1-\ell_{i-1}^-) - f(1-\ell_i^-). \end{aligned} \tag{64}$$

By convexity of f, we see that ℓ_i is nonincreasing, and this formula provides a fast method to compute the ℓ_i recursively. The following lemma describes the behaviour of ℓ_i as i gets large.

Lemma 9. *Let T be a Bienaymé–Galton–Watson tree with offspring distribution $\xi \sim (p_i)_{i\geq 0}$. If $p_1 \neq 0$, then $\ell_i = (p_1 + o(1))^i$. Otherwise if $p_1 = 0$ and $\kappa = \min\{i > 1 : p_i \neq 0\}$, then*

$$\log \ell_i = \Theta(\kappa^i) \tag{65}$$

as $i \to \infty$.

Proof. The recursive formula above is our starting point. Expanding f as a power series, for $i \geq 1$ we have, by our choice of κ,

$$\begin{aligned} \ell_{i+1} &= \sum_{j=0}^{\infty} p_j \left((\ell_i^+)^j - (\ell_{i+1}^+)^j\right) \\ &= 0 + p_1(\ell_i^+ - \ell_{i+1}^+) + \sum_{j\geq\kappa} p_j \left((\ell_i^+)^j - (\ell_{i+1}^+)^j\right) \end{aligned}$$

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

$$
\begin{aligned}
&= p_1\ell_i + p_\kappa \ell_i \left((\ell_i^+)^{\kappa-1} + (\ell_i^+)^{\kappa-2}(\ell_{i+1}^+)^1 + \cdots + (\ell_{i+1}^+)^{\kappa-1}\right) + \sum_{j>\kappa} p_j \left((\ell_i^+)^j - (\ell_{i+1}^+)^j\right). \\
&\le p_1\ell_i + \kappa p_\kappa \ell_i (\ell_i^+)^{\kappa-1} + \sum_{j>\kappa} j p_j \ell_i (\ell_i^+)^{j-1} \\
&\le p_1\ell_i + \ell_i(\ell_1^+)^{\kappa-1}\Big(\sum_{j\ge\kappa} j p_j\Big) \\
&= p_1\ell_i + \ell_i(1-p_0)^{\kappa-1}(1-p_1). \qquad (66)
\end{aligned}
$$

Letting $\alpha = p_1 + (1-p_1)(1-p_0)^{\kappa-1} < 1$, we have $\ell_{i+1} \le \ell_i\alpha$. Hence $\ell_{i+1} \le \ell_1\alpha^i$ and therefore $\ell_i \to 0$ as $i \to \infty$.

Let $\epsilon > 0$ and pick n_ϵ large enough such that $\ell_i^+ \le \epsilon$ for all $i \ge n_\epsilon$. When $p_1 \ne 0$, we have $\ell_i p_1 \le \ell_{i+1} \le \ell_i(p_1 + \epsilon^{\kappa-1})$, so we immediately conclude that $\ell_i = \big(p_1 + o(1)\big)^i$. If $p_1 = 0$, then

$$
\begin{aligned}
\ell_{i+1} &\le \kappa p_\kappa \ell_i (\ell_i^+)^{\kappa-1} + \sum_{j>\kappa} j p_j \ell_i (\ell_i^+)^{\kappa-1}(\ell_i^+)^{j-\kappa} \\
&\le \kappa p_\kappa \ell_i (\ell_i^+)^{\kappa-1}\Big(1 + \sum_{j>\kappa} j p_j \epsilon^{j-\kappa}\Big) \\
&\le \kappa p_\kappa \ell_i (\ell_i^+)^{\kappa-1}\left(1 + \frac{\epsilon}{1-\epsilon}\right) \\
&\le \kappa p_\kappa \ell_i (\ell_i)^{\kappa-1}\Big(\sum_{j=0}^{\infty} \alpha^j\Big)^{\kappa-1}\left(\frac{1}{1-\epsilon}\right) \\
&= \frac{\kappa p_\kappa}{(1-\alpha)^{\kappa-1}(1-\epsilon)} \ell_i^{\kappa}. \qquad (67)
\end{aligned}
$$

From this and the fact that $\ell_i \to 0$, we see that for some positive constants c_1, $c_2 < 1$, and c_3,

$$\ell_i \le c_1 c_2^{\kappa^{i-c_3}} \qquad (68)$$

for all $i \ge c_3$. We also have

$$\ell_{i+1} \ge \kappa p_\kappa \ell_i (\ell_i^+)^{\kappa-1} \ge \kappa p_\kappa \ell_i^k, \qquad (69)$$

so that for some positive constants c_1', $c_2' < 1$, and c_3',

$$\ell_i \ge c_1' c_2'^{\kappa^{i-c_3'}} \qquad (70)$$

for all $i \ge c_3'$. This proves that $\log \ell_i = \Theta(\kappa^i)$. We finish the proof by noting that ℓ_i^+ can be bounded in a similar manner. □

7. Asymptotics of the leaf-height

In this section, we will describe the asymptotic behaviour of the leaf-height of a tree T_n (recall that this is the maximum of $\lambda(v)$, taken over all the nodes $v \in T_n$). The result depends on whether p_1 is zero or nonzero, and we have split this into two lemmas, since both of the proofs are rather involved.

Lemma 10. *Let L_n be the leaf-height of T_n, a conditional Bienaymé–Galton–Watson tree on n nodes with offspring distribution $\xi \sim (p_i)_{i\ge0}$. If $p_1 \ne 0$, then*

$$\frac{L_n}{\log n} \to \frac{1}{\log(1/p_1)} \qquad (71)$$

in probability.

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

Proof. Let Y_n be the length of the longest string, oriented away from the root, of nodes of degree one in T_n. Clearly $L_n \geq Y_n$, so we will first show that for $\epsilon > 0$,

$$\mathbb{P}\left\{Y_n < (1-\epsilon)\log n/\log(1/p_1)\right\} \to 0.$$

Now, $\mathbb{P}\{Y_n < x\}$ is the probability that a string of 1s appears in the preorder degree sequence of the tree $(\xi_1, \xi_2, \ldots, \xi_n)$, given that the sequence is of length n and that the sequence does, in fact, define a tree; as we have used previously, this latter probability is $\Theta(n^{-3/2})$, from Dwass [11]. So letting $Y_n(\xi_1, \xi_2, \ldots, \xi_n)$ be the length of the longest subsequence of 1s in the preorder degree sequence, we have

$$\mathbb{P}\{Y_n < x\} = \Theta(n^{-3/2})\mathbb{P}\left\{Y_n(\xi_1, \xi_2, \ldots, \xi_n) < x\right\}. \tag{72}$$

We divide the sequence into n/x subsequences of length x each and let E_i be the event that the ith subsequence *does not* consist only of 1s. Then

$$\mathbb{P}\left\{Y_n(\xi_1, \xi_2, \ldots, \xi_n) < x\right\} = \mathbb{P}\left\{\bigcup_{i=1}^{n/2} E_i\right\} = \mathbb{P}\{E_i\}^{n/2}. \tag{73}$$

Since $\mathbb{P}\{E_i\} = 1 - {p_1}^x$ for all i,

$$\mathbb{P}\left\{Y_n(\xi_1, \xi_2, \ldots, \xi_n) < x\right\} = (1-{p_1}^x)^{n/x} \leq \exp\left(-\frac{n{p_1}^x}{x}\right). \tag{74}$$

When $x = (1-\epsilon)\log n/\log(1/p_1)$, this is equal to $\exp(-n^\epsilon/x)$, so

$$\mathbb{P}\left\{L_n < (1-\epsilon)\log n/\log(1/p_1)\right\} \leq \mathbb{P}\left\{Y_n < (1-\epsilon)\log n/\log(1/p_1)\right\} \leq \Theta(n^{3/2})e^{-\Theta(n^\epsilon/\log n)}, \tag{75}$$

which goes to 0 as $n \to \infty$.

To tackle the upper bound, it will be helpful for us to reorder the degrees into level (also called breadth-first) ordering and to consider the following random variable. Arrange the level-order degree sequence $\xi_1, \xi_2, \ldots, \xi_n$ in a cycle, and let Z_n be the longest string of consecutive nonzero numbers in this cyclic ordering. Clearly the probability that no sub-cycle of length x of this ordering consists only of zeroes is at most $n(1-p_0)^x$. So letting A be the event that $(\xi_1, \xi_2, \ldots, \xi_n)$ defines a tree, we can crudely bound $\mathbb{P}\{Z_n \geq x\}$ by

$$\mathbb{P}\{Z_n \geq x\} = \frac{\mathbb{P}\{Z_n \geq x,\ A\}}{\mathbb{P}\{A\}} \leq \Theta(n^{3/2})n(1-p_0)^x = \Theta(n^{5/2})(1-p_0)^x, \tag{76}$$

which goes to zero if $c > (5/2)/\log\left(1/(1-p_0)\right)$ and x is set to $c\log n$. By symmetry, if Z_n is the longest string of nonzeros in the *preorder* listing, then the same result holds, that is,

$$\mathbb{P}\left\{Z_n \geq \frac{3\log n}{\log\left(1/(1-p_0)\right)}\right\} \to 0. \tag{77}$$

Note that $L_n \leq Z_n$. Now for $1 \leq \Delta \leq n$ let $L_n(\xi_1, \xi_2, \ldots, \xi_\Delta)$ be the smallest leaf depth if we start constructing a tree using degrees $\xi_1, \xi_2, \ldots, \xi_\Delta$, in preorder. Two situations can occur: either $\xi_1, \xi_2, \ldots, \xi_\Delta$ defines at least one tree in a possible forest or $\xi_1, \xi_2, \ldots, \xi_\Delta$ defines an incomplete tree. In the former case, $L_n(\xi_1, \xi_2, \ldots, \xi_\Delta)$ is the leaf-height of the first completed tree; in the latter, set $L_n(\xi_1, \xi_2, \ldots, \xi_\Delta) = 0$. Note that if $Z_n \leq \Delta$, then $L_n(\xi_1, \xi_2, \ldots, \xi_\Delta) \leq \Delta$. For a sequence $\xi_1, \xi_2, \ldots, \xi_n$ of degrees, we define

$$L_{ni} = L_n(\xi_i, \xi_{i+1} \ldots, \xi_{i+\Delta-i}), \tag{78}$$

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

where addition in the indices is taken modulo n. If $Z_n \le \Delta$, note that $L_n \le \max_{1\le i\le n} L_{ni}$. So we have

$$\begin{aligned}\mathbb{P}\{L_n > x\} &\le \mathbb{P}\{L_n > x,\ Z_n \le \Delta\} + \mathbb{P}\{Z_n > \Delta\}\\ &\le \mathbb{P}\left\{\max_{1\le i\le n} L_{ni} > x,\ Z_n \le \Delta\right\} + \mathbb{P}\{Z_n > \Delta\}\\ &\le \mathbb{P}\left\{\max_{1\le i\le n} L_{ni} > x\right\} + \mathbb{P}\{Z_n > \Delta\}. \end{aligned} \tag{79}$$

The second term is $o(1)$ if we pick $c > (5/2)/\log\big(1/(1-p_0)\big)$ as before and set $\Delta = c\log n$. In the first term, the maximum is invariant under rotations of $(\xi_1, \xi_2, \ldots, \xi_n)$, so we use may use a version of the cycle lemma [11], obtaining

$$\begin{aligned}\mathbb{P}\left\{\max_{1\le i\le n} L_{ni} > x\right\} &= \frac{\mathbb{P}\left\{\max_{1\le i\le n} L_{ni} > x,\ \sum_{i=1}^n \xi_i = n-1\right\}}{\mathbb{P}\left\{\sum_{i=1}^n \xi_i = n-1\right\}}\\ &\le \frac{\mathbb{P}\left\{\max_{1\le i\le n} L_{ni} > x,\ \sum_{i=\Delta+1}^n \xi_i = n-1-\sum_{i=1}^{\Delta}\xi_i\right\}}{\Theta(n^{-1/2})}\\ &\le O(n^{3/2}) \sup_{\ell} \mathbb{P}\left\{L_{n1} > x,\ \sum_{i=\Delta+1}^n \xi_i = \ell\right\}\\ &= O(n^{3/2}) \cdot \mathbb{P}\{L_{n1} > x\} \cdot \sup_{\ell} \mathbb{P}\left\{\sum_{i=\Delta+1}^n \xi_i = \ell\right\}. \end{aligned} \tag{80}$$

Rogozin's inequality [24] tells us that

$$\sup_{\ell} \mathbb{P}\left\{\sum_{i=\Delta+1}^n \xi_i = \ell\right\} \le \frac{\gamma}{\sqrt{1-\Pi}} \cdot \frac{1}{\sqrt{n-\Delta}}, \tag{81}$$

where γ is a universal constant and $\Pi = \sup_j p_j$. So if $L(T)$ is the leaf-height of the root of an unconditional Bienaymé–Galton–Watson tree, then

$$\mathbb{P}\{L_n > x\} \le O(n)\mathbb{P}\{L_{n1} > x\} \le O(n)\mathbb{P}\{L(T) > x\} \le O(n)\ell_x^+ \le O(n)\left(p_1 + o(1)\right)^x, \tag{82}$$

which goes to 0 when $x = (1+\epsilon)\log n/\log(1/p_1)$. □

The next lemma handles the other case, in which p_1 is zero.

Lemma 11. *Let L_n be the leaf-height of T_n, a conditional Bienaymé–Galton–Watson tree on n nodes with offspring distribution $\xi \sim (p_i)_{i\ge 0}$. Let $\kappa = \min\{i > 1 : p_i \ne 0\}$. If $p_1 = 0$, then*

$$\frac{L_n}{\log\log n} \to \frac{1}{\log \kappa} \tag{83}$$

in probability.

Proof. Let $\epsilon > 0$, A be the event that $(\xi_1, \xi_2, \ldots, \xi_n)$ forms a tree and let $L_n(\xi_1, \xi_2, \ldots, \xi_n)$ be as in the previous lemma. If $L(T)$ is the leaf-height of the root of an unconditional Bienaymé–Galton–Watson tree T, we have, by the cycle lemma,

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

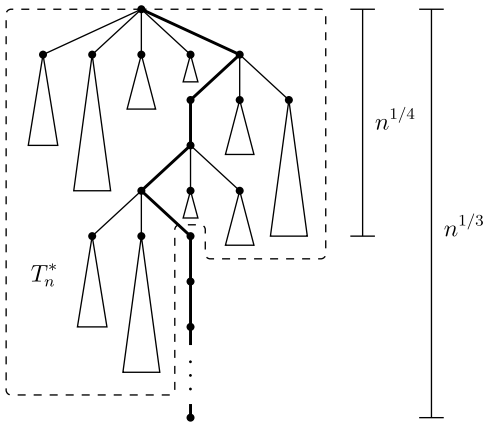

Figure 5. The construction in the lower bound. Note that none of the unconditional trees in T_n^* reach level $n^{1/3}$.

$$\begin{aligned}\mathbb{P}\{L_n \geq x\} &= \frac{\mathbb{P}\left\{L_n(\xi_1, \xi_2, \ldots, \xi_n) \geq x,\ A\right\}}{\mathbb{P}\{A\}} \\ &\leq \Theta(n^{3/2})\mathbb{P}\left\{L_n(\xi_1, \xi_2, \ldots, \xi_n) \geq x\right\} \\ &\leq \Theta(n^{3/2})n\mathbb{P}\{L(T) \geq x\} \\ &\leq \Theta(n^{5/2})c_1 c_2^{\kappa^{x-c_3}},\end{aligned} \tag{84}$$

for some positive constants c_1, $c_2 < 1$, and c_3. When $x = (1+\epsilon)\log_\kappa \log n$, this is $\Theta(n^{5/2})c_2^{\Theta((\log n)^{1+\epsilon})}$, which goes to 0 as $n \to \infty$.

Let T_∞ be Kesten's infinite tree. Our proof of the lower bound, which the reader may find easier to follow by consulting the illustration in Figure 5, uses the fact that

$$\mathbb{P}\left\{\tau(T_\infty, n^{1/3}) \neq \tau(T_n, n^{1/3})\right\} \to 0 \tag{85}$$

as $n \to \infty$. Let U be the set of all unconditional Bienaymé–Galton–Watson trees T rooted less than $n^{1/4}$ of the way down the spine. Let $h(T)$ denote the height of an unconditional Bienaymé–Galton–Watson tree; the probability that one of these trees has height greater than $n^{1/3}/2$ is bounded above by

$$\mathbb{E}\left\{|U|\right\}\mathbb{P}\left\{h(T) > n^{1/3}\right\} \leq \sigma^2 n^{1/4}\left(\frac{2+o(1)}{\sigma^2 n^{1/3}/2}\right) \sim \frac{4}{n^{1/12}}, \tag{86}$$

which goes to 0. Here we used the fact that $\mathbb{E}\{\zeta\} = \sigma^2 + 1$ and applied Kolmogorov's result (see [20] and [2]) about the height of a Bienaymé–Galton–Watson tree. If all the heights above are at most $n^{1/3}/2$, then T_n^*, the tree obtained by taking the spine up to level $n^{1/4}$ and all hanging unconditional trees up to that point, is a subtree of $\tau(T_\infty, n^{1/3})$, since $n^{1/4} + n^{1/3}/2 < n^{1/3}$.

For every tree $t \in U$, let E_t be the event that the leaf-height of the root of t is less than x and let E_T be the same event for an unconditional Bienaymé–Galton–Watson tree (since each t in U is

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

such a tree, there is no real moral distinction between these events). We have

$$\begin{aligned}\mathbb{P}\{L_n < x\} &\le \mathbb{P}\left\{\tau(T_\infty) \neq (T_n, n^{1/3})\right\} + \mathbb{P}\left\{T_n^* \neq (T_n, n^{1/3})\right\} + \mathbb{P}\left\{\bigcap_{t\in U} E_t\right\} \\ &\le o(1) + \mathbb{E}\left\{\mathbb{P}\{E_T\}^{|U|}\right\} \\ &\le o(1) + \mathbb{E}\left\{\left(1 - c_1' \cdot c_2'^{\kappa^{x-c_3'}}\right)^{|U|}\right\} \\ &\le o(1) + \mathbb{E}\left\{\left(1 - c_1' \cdot c_2'^{\kappa^{x-c_3'}}\right)^{n^{1/4}\sigma^2/2}\right\} + \mathbb{P}\left\{|U| < \frac{n^{1/4}\sigma^2}{2}\right\}, \end{aligned} \tag{87}$$

for some c_1', c_2', c_3' positive, $c_2' < 1$. Take $x = (1-\epsilon)\log_\kappa \log n$. Letting $\zeta_1, \zeta_2, \ldots, \zeta_n$ be independent and distributed as ζ, we find that

$$\mathbb{P}\{L_n < x\} \le o(1) + \exp\left(-\Theta(n^{1/4} e^{-\Theta((\log n)^{1-\epsilon})})\right) + \mathbb{P}\left\{\sum_{i=1}^{n^{1/4}} (\zeta_i - 1) < \frac{n^{1/4}\sigma^2}{2}\right\}. \tag{88}$$

Since $\mathbb{E}\{\zeta - 1\} = \sigma^2$, by the law of large numbers, this entire expression is $o(1)$. □

It is important to note that the node with maximum leaf-height in a tree usually is not the root. We have the following result for the distribution of the leaf-height of the root of a conditional Bienaymé–Galton–Watson tree T_n.

Lemma 12. *Let L_n' denote the leaf-height of the root of T_n, a conditional Bienaymé–Galton–Watson tree of size n, and offspring distribution ξ. Let f be the generating function of this distribution. Then the probability distribution of L_n' is given by*

$$\lim_{n\to\infty} \mathbb{P}\{L_n' = i\} = \prod_{j=0}^{i-1} f'(\ell_j^+). \tag{89}$$

Proof. For a node v on the spine of Kesten's infinite tree T_∞, let H^* be the leaf-height of the spine child of v, and $H(1), H(2), \ldots, H(\zeta - 1)$ be the leaf-heights of the $\zeta - 1$ independent unconditional Bienaymé–Galton–Watson trees spawned by v. Then the leaf-height of a node v on the spine is

$$1 + \min_{w\in\Gamma_v} \lambda(w) = 1 + \min\left(H(1), H(2), \ldots, H(\zeta - 1)\right). \tag{90}$$

This defines a Markov chain on the positive integers that proceeds up the spine. The state H^* (which is just a positive integer indicating the leaf-height of the node on the spine) is taken to the state

$$1 + \min\big(H^*, H(1), H(2), \ldots, H(\zeta - 1)\big) \tag{91}$$

in one step of the Markov chain; here all H_i have distribution $(\ell_i)_{i\ge 0}$. Let H^{**} be the limit stationary random variable of this Markov chain. That the limit exists and that the chain is positive recurrent follows from the fact that at each step, there is a positive probability that the next state is 1. This happens when $\zeta > 1$ and one of $H(i)$ is 0. In fact, H^{**} is the unique solution of the distributional identity

$$H^{**} \stackrel{\mathcal{L}}{=} 1 + \min\left(H^{**}, H(1), H(2), \ldots, H(\zeta - 1)\right). \tag{92}$$

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

Broutin, Devroye, and Fraiman showed that under a coalescence condition (satisfied here), the limit of the root value of T_n tends in distribution to the stationary random variable for Kesten's spinal Markov chain [7]. Thus $L'_n \to H^{**}$ in distribution.

We can describe the distribution of H^{**} more explicitly. For convenience, let $\ell_i^{**} = \mathbb{P}\{H^{**} = i\}$. For $i \geq 1$,

$$\ell_i^{**} = \mathbb{P}\{H_j \geq i+1 \text{ for all } 1 \leq j \leq \zeta - 1\}\mathbb{P}\{H^{**} \geq i-1\}. \tag{93}$$

This means that

$$\ell_i^{**} = \ell_{i-1}^{**}\mathbb{E}\left\{(\ell_{i-1}^{+})^{\zeta-1}\right\} = \ell_{i-1}^{**}\sum_{j\geq 1} jp_j(\ell_{i-1}^{+})^{j-1}, \tag{94}$$

and we can rewrite this in terms of the generating function $f(s)$ of ξ as

$$\ell_i^{**} = \ell_{i-1}^{**}f'(\ell_{i-1}^{+}) = \ell_0^{**}\prod_{j=0}^{i-1} f'(\ell_j^{+}) = \prod_{j=0}^{i-1} f'(\ell_j^{+}), \tag{95}$$

proving the lemma. □

Lastly, we can obtain a random variable by taking the leaf-height of a node chosen uniformly at random from T_n. Its distribution is asymptotically the same as the leaf-height of the root of an unconditional Bienaymé–Galton–Watson tree.

Lemma 13. *Let L''_n be a random variable obtained by taking the leaf-height of a node chosen uniformly at random from a conditional Bienaymé–Galton–Watson tree T_n. We have*

$$\lim_{n\to\infty} \mathbb{P}\{L''_n = i\} = \ell_i \tag{96}$$

for all $i \geq 0$.

Proof. By Aldous's theorem [1], if T_n^* denotes the subtree of T_n rooted at a uniformly selected random node, then for all trees t,

$$\lim_{n\to\infty} \mathbb{P}\{T_n^* = t\} = \mathbb{P}\{T = t\}, \tag{97}$$

where T is the unconditional Bienaymé–Galton–Watson tree. The result is immediate. □

Examples. We now apply Lemmas 10 and 11 to compute the maximum leaf-height (asymptotically in probability) for common families of trees. These results, along with the independence numbers and maximum peel numbers we computed earlier, are collected in Table 1. In the table, W denotes the Lambert function and $\varphi = (\sqrt{5}-1)/2$ is the golden ratio.

i) *Flajolet's t-ary trees:* For $t \geq 2$, we have $p_1 = 0$ here and $\kappa = t$, so we have $L_n/\log\log n \to 1/\log t$ in probability, by Lemma 13.

ii) *Cayley trees:* In this case, $p_1 = 1/e$, so Lemma 12 gives us $L_n/\log n \to 1$ in probability.

iii) *Planted plane trees:* For these trees, $p_1 = 1/4$, so we have $L_n/\log n \to 1/\log 4$ in probability, by Lemma 12.

iv) *Motzkin trees:* This family has $p_1 = 1/3$ and $L_n/\log_n \to 1/\log 3$ in probability.

v) *Binomial trees:* For a parameter $d \geq 2$, we have $p_1 = (1-1/d)^{d-1}$, which means that $Ł_n/\log n \to 1/((1-d)\log(1-1/d))$ in probability. As $d \to \infty$, the denominator approaches 1, which gives the same leaf-height as the case of Cayley trees. In the special case when $d = 2$, we have the Catalan trees, for which $p_1 = 1/2$ and $L_n/\log n \to 1/\log 2$ in probability.

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

Table 1. Asymptotic values of parameters for certain families of trees

Family	I_n	M_n	L_n
Full binary	$(2-\sqrt{2})n$	$\frac{\log n}{\log(1/(\sqrt{2}-1))}$	$\log_2 \log n$
(Uni{0,2})			
Flajolet t-ary	$\left(1-\frac{1+o_{t\to\infty}(1)}{t}\right)n$	$\sim_{t\to\infty} \log_t n$	$\log_t \log n$
$(p_0 = 1-1/t; p_t = 1/t)$			
Cayley	$W(1)n$	$\log n/W(1)$	$\log n$
(Poi(1))			
Planted plane	$\frac{n}{\varphi}$	$\frac{\log n}{\varphi^2}$	$\log_4 n$
(Geo(1/2))			
Motzkin	$(3-\sqrt{6})n$	$\frac{\log n}{\log 3-\log(2\sqrt{6}-3)}$	$\log_3 n$
(Uni{0,1,2})			
Catalan	$(4-2\sqrt{3})n$	$\frac{\log n}{\log(1/(\sqrt{3}-1))}$	$\log_2 n$
(Bin(2,1/2))			
Binomial	$\sim_{d\to\infty} W(1)n$	$\sim_{d\to\infty} \frac{\log n}{W(1)}$	$\frac{\log n}{(d-1)\log(1/(1-1/d))}$
(Bin(d,1/d))			

8. Further directions

The definition of the peel number and our characterisation of its asymptotic growth fully describes the running time of Algorithm I, mentioned in the introduction, which computes the layered independent set. It would be interesting to consider the runtime of the more general Algorithm P, described in Section 3. To this end, we define *higher-order peel numbers* as follows. Algorithm P generates an $(r+1)$-path vertex cover by repeatedly deleting subtrees with height exactly r (and marking their roots). If a node u is deleted in the mth iteration of the loop and is at depth i of the subtree that is deleted, then its *peel number of order r* (or *rth order peel number*) is $mk-i$. Note that the loop counter m should start at 1 and we have $0 \le i \le r$. By this definition, the peel number we studied in this paper is simply the first-order peel number. To determine the runtime of Algorithm P, one should in principle be able to approach the higher order peel numbers in the same way we approached the case $r=1$ in Sections 4 and 5. However, even in this case one had to handle the even and odd cases separately, and we anticipate that the analysis of higher-order peel numbers will require careful reasoning with respect to congruence modulo $r+1$.

Acknowledgements

All three authors are supported by the Natural Sciences and Engineering Research Council of Canada (NSERC). We also thank Anna Brandenberger for wonderful discussions and for her valuable technical feedback. Last but not least, we owe a debt of gratitude to the anonymous referees for their careful reading of our paper and for their detailed and insightful comments.

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press

References

[1] Aldous, D. (1991) Asymptotic fringe distributions for general families of random trees. *Ann. Appl. Probab.* **1** 228–266.
[2] Alsmeyer, G. (2008) Galton–Watson Processes. Course notes at the University of Münster.
[3] Athreya, K. and Ney, P. (1972) *Branching Processes*. Springer Verlag.
[4] Banderier, C., Kuba, M. and Panholzer, A. (2009) Analysis of three graph parameters for random trees. *Random Struct. Algorithms* **35** 42–69.
[5] Bóna, M. (2014) *k*-protected vertices in binary search trees. *Adv. Appl. Math.* **53** 1–11.
[6] Brešar, B., Jakovac, M., Katrenič, J., Semanišin, G. and Taranenko, A. (2013) On the vertex *k*-path cover. *Discrete Appl. Math.* **161** 1943–1949.
[7] Broutin, N., Devroye, L. and Fraiman, N. (2020) Recursive functions on conditional Galton–Watson trees. *Random Struct. Algorithms* **57** 304–316.
[8] Cheon, G.-S. and Shapiro, L. W. (2008) Protected points in ordered trees. *Appl. Math. Lett.* **21** 516–520.
[9] Copenhaver, K. (2017) *k*-protected vertices in unlabeled rooted plane trees. *Graph Combinator.* **33** 347–355.
[10] Du, R. R. X. and Prodinger, H. (2012) On protected nodes in digital search trees. *Appl. Math. Lett.* **25** 1025–1028.
[11] Dwass, M. (1969) The total progeny in a branching process. *J. Appl. Probab.* **6** 682–686.
[12] Flajolet, P. and Odlyzko, A. (1982) The average height of binary trees and other simple trees. *J. Comput. Syst. Sci.* **25** 171–213.
[13] Fuchs, M., Holmgren, C., Mitsche, D. and Neininger, R. (2021) A note on the independence number, domination number and related parameters of random binary search trees and random recursive trees. *Discrete Appl. Math.* **292**(1) 64–71.
[14] Gittenberger, B., Gołębiewski, Z., Larcher, I. and Sulkowska, M.. "Protection numbers in simply generated trees and Pólya trees,". arXiv preprint 1904. 03519, 2019.
[15] Janson, S. (2012) Simply generated trees, conditioned Galton–Watson trees, random allocations and condensation. *Probab. Surv.* **9**(none) 103–252.
[16] Janson, S. (2016) Asymptotic normality of fringe subtrees and additive functionals in conditioned Galton–Watson trees. *Random Struct. Algorithms* **48**(1) 57–101.
[17] Kersting, G.. "On the height profile of a conditioned Galton–Watson tree,". arXiv preprint 1101. 3656, 1998.
[18] Kesten, Harry (1986) Subdiffusive behavior of a random walk on a random cluster. *Ann. l'Inst. Henri Poincaré* **22** 425–487.
[19] Kolchin, V. F. (1986) *Random Mappings*. Optimisation Software Inc.
[20] Lyons, R. and Peres, Y. (2017) *Probability on Trees and Networks*. Cambridge University Press.
[21] Mahmoud, H. M. and Ward, M. D. (2012) Asymptotic distribution of two-protected nodes in random binary search trees. *Appl. Math. Lett.* **25**(12) 2218–2222.
[22] Mahmoud, H. M. and Ward, M. D. (2015) Asymptotic properties of protected nodes in random recursive trees. *J. Appl. Probab.* **52** 290–297.
[23] Mansour, T. (2011) Protected points in *k*-ary trees. *Appl. Math. Lett.* **24** 478–480.
[24] Rogozin, B. A. (1961) On an estimate of the concentration function. *Theor. Probab. Appl.* **6** 94–97.
[25] Stufler, B. (2019) Local limits of large Galton–Watson trees rerooted at a random vertex. *Ann. l'Inst. Henri Poincaré Probab. Stat.* **55** 155–183.

Cite this article: Devroye L, Goh MK, and Zhao RY (2023). On the peel number and the leaf-height of Galton–Watson trees. *Combinatorics, Probability and Computing* **32**, 68–90. https://doi.org/10.1017/S0963548322000128

https://doi.org/10.1017/S0963548322000128 Published online by Cambridge University Press