

## LOGARITHMIC DE RHAM–WITT COMPLEXES VIA THE DÉCALAGE OPERATOR

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*Abstract* We provide a new formalism of de Rham–Witt complexes in the logarithmic setting. This construction generalises a result of Bhatt–Lurie–Mathew and agrees with those of Hyodo–Kato and Matsuue for log-smooth schemes of log-Cartier type. We then use our construction to study the monodromy action and slopes of Frobenius on log crystalline cohomology.

*Keywords:* de Rham–Witt complex, log crystalline cohomology, monodromy operator, de Rham cohomology, Nygaard filtration, slopes of Frobenius

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### 1. Introduction

Recently, Bhatt–Lurie–Mathew [2] gave a relatively elementary construction of the de Rham–Witt complex (hence the crystalline cohomology) for smooth varieties over a perfect field of characteristic  $p$ . Our article extends their construction to the logarithmic setting. This gives us a convenient framework to study various arithmetic properties arising from geometry in a slightly more general context. For example, we apply our construction to study the monodromy action and the slopes of Frobenius on log crystalline cohomology. Moreover, our construction leads to another proof of the comparison between  $A_{\text{inf}}$ /prismatic cohomology and log crystalline cohomology in the case of semistable reduction (see [20]).

To put this article into context, we first remind the reader of some relevant classical notions. For a smooth scheme  $X$  over a perfect field  $k$  of characteristic  $p$ , its de Rham–Witt complex is a sheaf  $W\Omega_{X/k}^*$  of commutative differential graded algebras (cdga) introduced by Illusie [10], which canonically represents the crystalline cohomology of  $X$  over the ring  $W(k)$  of Witt vectors. See the introduction of [10] for an account of the history of the subject. The de Rham–Witt complex is equipped with a Frobenius map  $F$  and a Verschiebung map  $V$ , required to satisfy various relations, including

$$FV = p, \quad VF = V(1) = p, \quad FdV = d, \quad xVy = V(Fx \cdot y), \quad \dots$$

<sup>1</sup>From these relations, one can further deduce that  $dF = pFd$ ,  $Vd = pdV$ , etc. Building on the work of Illusie–Raynaud [11], Hyodo–Kato introduced a logarithmic version of de Rham–Witt complexes, and uses them to construct the monodromy operator on the log crystalline cohomology for log schemes that are of semistable type over  $k$ . Slightly more recently, the theory of (log) de Rham–Witt complexes have been extended to more general bases by Langer–Zink (the ‘relative’ de Rham–Witt complexes [14]), Hesselholt–Madsen (the ‘absolute’ de Rham–Witt complexes [7, 8]), Matsue (for relative log schemes [15]) and many others. In this article, we follow the strategy of [2] and give an elementary construction of log de Rham–Witt complexes for log schemes over a perfect log field  $k$ . To distinguish our construction from the crystalline constructions (for example, in [9]), the complexes we construct will be named the saturated log de Rham–Witt complexes, again following [2].

#### 1.1. Saturated log de Rham–Witt complex

To assist the reader with conventions used in this article, we include a brief review of log geometry in the Appendix. We start with a perfect field  $\underline{k}$  of characteristic  $p$  endowed with a log structure (for the convention on log algebras, see Subsection 1.6). Let  $\underline{X}$  be a quasi-coherent log scheme (with underlying scheme  $X$ ) over  $\underline{k}$  and  $\underline{R} = (R, M)$  be a local

<sup>1</sup>The equality  $V(1) = p$  is a consequence of the fact that our schemes live in characteristic  $p$ .

chart for its log structure. The starting point of our construction is similar to that of [2], namely, to place the relation

$$dF = pFd$$

at a central place. For preparation, we first define the category  $DA^{\log}$  of log Dieudonné algebras, which consists of a commutative differential graded algebra (cdga)  $A^*$ , equipped with

- a Frobenius map  $F$ ,
- a log structure  $L$  and
- a log derivation  $\delta$

in a compatible manner. Most important, these data are required to satisfy

$$dF = pFd \quad \text{and} \quad \delta F = pF\delta.$$

There is a full subcategory

$$DA_{\text{str}}^{\log} \hookrightarrow DA^{\log}$$

of *strict* log Dieudonné algebras, which essentially consists of log Dieudonné algebras that admit the Verschiebung maps  $V$  and are complete with respect to a certain  $V$ -filtration. We will formulate the strictness condition using the log version of the décalage operator  $\eta_p$ , which on  $p$ -torsion-free cochain complexes is given by

$$\eta_p A^i := \{x \in p^i A^i \mid dx \in p^{i+1} A^{i+1}\}.$$

For each  $p$ -torsion-free object  $A^*$  in  $DA^{\log}$ , there is a morphism  $\phi_F : A^* \rightarrow \eta_p(A^*)$  of log Dieudonné algebras given by  $p^i F$  in degree  $i$ . The existence of the Verschiebung map  $V$  is closely related to  $\phi_F$  being an isomorphism.

This allows us to give the following primitive definition.<sup>2</sup>

**Definition 1.** The saturated log de Rham–Witt complex, as a functor from the category  $\text{Alg}_{\underline{k}}$  of log  $\underline{k}$ -algebras to strict log Dieudonné algebras, is the left adjoint of the functor

$$\iota : DA_{\text{str}}^{\log} \longrightarrow \text{Alg}_{\underline{k}}$$

that sends  $A^* \mapsto \underline{A}^0 / \text{im } V$ .

We denote by  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  the saturated log de Rham–Witt complex of  $\underline{R}$ .

**Theorem 2.** The left adjoint  $\mathcal{W}\omega_{\underline{-}/\underline{k}}^*$  of  $\iota$  exists. Moreover, the saturated log de Rham–Witt complexes  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  glue to a sheaf  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^*$  on the (usual) étale site  $X_{\text{ét}}$  of  $X$ .

### 1.2. The case of log-smooth schemes of log-Cartier type

Our version of the log de Rham–Witt complexes agrees with the existing ones in [9], [14] and [15] under additional smoothness assumptions.

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<sup>2</sup>Strictly speaking, we need to restrict to subcategories  $DA_{\text{str}}^{\log,p}$  of  $DA_{\text{str}}^{\log}$ , spanned by objects for which the log Frobenius is given by multiplication by  $p$ . We ignore this issue in the Introduction for the sake of exposition.

**Theorem 3.** *Let  $\underline{X}$  be a log scheme over  $\underline{k}$  that is log-smooth of log-Cartier type; then there exists isomorphisms of sheaves of complexes*

$$\mathcal{W}\omega_{\underline{X}/\underline{k}}^* \xrightarrow{\sim} W\Lambda_{\underline{X}/\underline{k}}^* \cong W^{HK}\omega_{\underline{X}/\underline{k}}^*$$

*compatible with the Frobenius operators, where  $W\Lambda_{\underline{X}/\underline{k}}^*$  is the complex constructed by Matsue [15] following the work of [14] (see Section 7), and  $W^{HK}\omega_{\underline{X}/\underline{k}}^*$  is the Hyodo–Kato complex constructed via the log crystalline site in [9]. In particular, in this case all three versions of log de Rham–Witt complexes agree and compute the log crystalline cohomology. More precisely, there exists a canonical isomorphism*

$$R\Gamma(X_{\acute{e}t}, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*) \cong R\Gamma_{\log\text{-cris}}(\underline{X}/W(\underline{k}))$$

*identifying the Frobenius operator induced by the map  $\phi_F = p^i F$  in degree  $i$  on the de Rham–Witt complex with the functorial Frobenius on log crystalline cohomology.*

**Remark 4.** In general, the saturated complex  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  differs from the crystalline construction of Hyodo–Kato. This distinction is already present in the case of ordinary schemes and has little to do with log structures. See [2, Section 6.3] for an explicit example of a cusp.

**1.3. The main construction**

As previously mentioned, one notable feature of our construction is that we start with the Frobenius and produce the Verschiebung and Restriction maps ‘along the way’. Another feature is that our formulation makes essential use of the décalage operator  $\eta_p$ . To be more precise, we turn the isomorphism

$$\phi_F : W\omega_{\underline{X}/\underline{k}}^* \cong \eta_p W\omega_{\underline{X}/\underline{k}}^*, \tag{1}$$

which is typically an output<sup>3</sup> of previous constructions of the log de Rham–Witt complex, as part of the input. As a result, our complex is characterised by a different universal property, which makes it easier to compare to cdgs equipped with Frobenius.

**Remark 5.** The isomorphism in (1) is what makes this formalism suitable for studying the log crystalline comparison of  $A_{\text{inf}}$ -cohomology in [20]. More precisely, let  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$  be the  $A_{\text{inf}}$ -cohomology of a formal scheme  $\mathfrak{X}$  over  $\mathcal{O}_C$  with semistable reduction, where  $C$  is a completed algebraic closure of  $W(k)[\frac{1}{p}]$ . Write  $\underline{\mathfrak{X}}$  for the formal log scheme with the divisorial log structure from its mod  $p$  fibre  $\mathfrak{X}_{\mathcal{O}_C/p}$ . In [20] we apply the main results of this article to prove the following theorem.

**Theorem 6** ([20]). *There is a Frobenius compatible quasi-isomorphism*

$$R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \widehat{\otimes}_{A_{\text{inf}}}^{\mathbb{L}} W(k) \cong R\Gamma_{\log\text{-cris}}(\underline{\mathfrak{X}}_k/W(k))$$

*relating the specialisation of  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$  to  $W(k)$  with the log crystalline cohomology of the special fibre of  $\mathfrak{X}$  over  $k$ .*

<sup>3</sup>The isomorphism (1) is due to Illusie for (ordinary) de Rham–Witt complexes and to Hyodo–Kato in the logarithmic setting. See Definition 2.3 and Lemma 7.5.

The construction of the saturated log de Rham–Witt complex is modeled after [2]. Assume for simplicity that  $R$  is reduced so  $W(R)$  is  $p$ -torsion free. We first consider the log algebra

$$W(\underline{R}) = (W(R), M),$$

where the log structure is obtained from the Teichmüller lift. One key claim is that the log de Rham complex  $\omega_{W(\underline{R})/W(\underline{k})}^*$  admits a canonical structure of a log Dieudonné algebra. It is then not difficult to construct a left adjoint

$$L(\iota) : \text{DA}^{\text{log}} \longrightarrow \text{DA}_{\text{str}}^{\text{log}}$$

of the inclusion functor  $\iota : \text{DA}_{\text{str}}^{\text{log}} \hookrightarrow \text{DA}^{\text{log}}$ , which forces the desired isomorphism in (1) in a functorial way. We then define the saturated log de Rham–Witt complex of  $\underline{R}/\underline{k}$  as the image of  $\omega_{W(\underline{R})/W}^*$  under this left adjoint. Finally, for a quasi-coherent log scheme  $\underline{X}$  over  $\underline{k}$ , we show that the construction  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^i$  satisfies étale descent and hence globalises to an étale sheaf  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^*$  of log Dieudonné algebras.

From the construction it is not difficult to prove the following comparison theorems, which ultimately amounts to the existence of the Cartier isomorphism (see Subsection A.3) in the logarithmic setting.

**Theorem 7.** *Suppose that  $\underline{R}$  is log-smooth over  $\underline{k}$  of log-Cartier type (or, more generally, satisfying the conditions in Remark 5.10). Then the saturated log de Rham–Witt complex  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  satisfies the following ‘de Rham comparisons’.*

- (mod  $V$ ). *There is a canonical isomorphism of cochain complexes*

$$\omega_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^*,$$

where  $\mathcal{W}_1\omega_{\underline{R}/\underline{k}}^*$  is the quotient cochain complex  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*/(V + dV)$ .

- (mod  $p$ ). *There is a canonical quasi-isomorphism*

$$\mathcal{W}\omega_{\underline{R}/\underline{k}}^*/p\mathcal{W}\omega_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^* \cong \omega_{\underline{R}/\underline{k}}^*.$$

- (with Frobenius lifts). *Suppose that there exists a log-Frobenius lift  $(\underline{A}, \varphi)$  of  $\underline{R}$  over  $W(\underline{k})$  in the sense of Subsection 5.2.1; then there is a quasi-isomorphism*

$$\iota_\varphi : \widehat{\omega}_{\underline{A}/W(\underline{k})}^* \xrightarrow{\sim} \mathcal{W}\omega_{\underline{R}/\underline{k}}^*.$$

As notation suggests, the map  $\iota_\varphi$  depends on the choice of  $\varphi$  (compare with the canonical isomorphism in Proposition 5.4).

**Remark 8.** These comparisons typically fail for general log rings  $\underline{R}$  without any additional assumptions. For example, the  $\mathbb{F}_p$ -algebra  $\mathcal{W}_1\omega_{\underline{R}/\underline{k}}^0$  is always reduced by Remark 2.8, but  $\omega_{\underline{R}/\underline{k}}^0 = R$  could be arbitrary.

### 1.4. Monodromy and Frobenius on log crystalline cohomology

The formalism of saturated log de Rham–Witt complexes provides a convenient framework to reconstruct the monodromy operator of Hyodo–Kato on log crystalline cohomology, in fact in a slightly more general setup.

**Theorem 9.** *Let  $\underline{X}/\underline{k}$  be a log scheme of generalised semistable type (see Section 8) over the standard log point  $\underline{k} = (k, \mathbb{N})$ . Let  $k^\circ = (k, 0)$  be the trivial log point. Denote the saturated log de Rham–Witt complex of  $\underline{X}$  over the trivial log point by  $\mathcal{W}\tilde{\omega}_{\underline{X}/\underline{k}}^* := \mathcal{W}\omega_{\underline{X}/k^\circ}^*$ . Then there is a short exact sequence of cochain complexes*

$$0 \rightarrow \mathcal{W}\omega_{\underline{X}/\underline{k}}^*[-1] \longrightarrow \mathcal{W}\tilde{\omega}_{\underline{X}/\underline{k}}^* \longrightarrow \mathcal{W}\omega_{\underline{X}/\underline{k}}^* \rightarrow 0$$

over  $X_{\acute{e}t}$ . The connecting homomorphism on cohomology

$$N : H_{\log\text{-cris}}^*(\underline{X}/W(\underline{k})) = \mathbb{H}^*(X_{\acute{e}t}, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*) \longrightarrow H_{\log\text{-cris}}^*(\underline{X}/W(\underline{k}))$$

satisfies  $N\varphi = p\varphi N$ , where  $\varphi$  is the functorial Frobenius on the log crystalline cohomology. This agrees with the monodromy operator constructed in [9] when  $\underline{X}$  is of semistable type.

Finally, we end the article with a discussion on slopes of the Frobenius operator acting on the saturated de Rham–Witt cohomology  $\mathbb{H}^*(X_{\acute{e}t}, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*)$ . Following [19], we construct and study a certain ‘Nygaard filtration of level  $n$ ’ on  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^*$ . Our framework makes it particularly simple to relate the graded pieces of the Nygaard filtration to the conjugate filtration on  $\mathcal{W}_n\omega_{\underline{X}/\underline{k}}^*$  (‘la filtration canonique’). This in turn allows us to prove the following ‘generalised Katz conjecture’ in the logarithmic setup.

Suppose that  $\underline{X}$  is a proper log scheme over  $\underline{k}$  that is log-smooth of log-Cartier type (or, more generally, satisfying a technical condition that guarantees the Cartier isomorphism. This condition implies that all relevant cohomology groups have finite length over  $W(k)$ . See Subsection 9.4.) Fix an integer  $d \geq 0$ . For each  $m \geq 1$ , consider the  $F^m$ -crystal

$$(\mathbb{H}^d(X_{\acute{e}t}, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*)_{\text{tf}}, \varphi^m)$$

given by the torsion-free quotient of the saturated log de Rham–Witt cohomology and the  $m$ th power of Frobenius. Let  $\text{New}(\varphi)$  denote the Newton polygon of the  $F$ -crystal with  $m = 1$ . Let  $\frac{1}{n}\text{Hdg}(\varphi^n)$  denote the polygon obtained from the Hodge polygon of  $\varphi^n$  by scaling by  $\frac{1}{n}$  in the vertical direction (see Subsection 9.5 for a precise definition). Finally, let  $\text{Hdg}(\underline{X}, n)$  denote the ‘geometric Hodge polygon’ of level  $n$  associated to the ‘Hodge numbers’:

$$\text{Hdg}(\underline{X}, n) := \text{Hdg}\left(\left\{\frac{\mathfrak{h}^0}{n}, \frac{\mathfrak{h}^1}{n}, \frac{\mathfrak{h}^2}{n}, \dots\right\}\right)$$

where

$$\mathfrak{h}^j(n) := \text{length}_{W(k)} H^{d-j}(X, \mathcal{W}_n\omega_{\underline{X}/\underline{k}}^j).$$

Finally, we fix the following notation for the relative position of polygons: for two polygons  $P, Q$ , we write  $P \succcurlyeq Q$  if  $P$  lies on or above  $Q$ .

**Theorem 10** (Generalised Katz conjecture). *For  $\underline{X}$  as above, the Newton polygon of  $\varphi$  always lies on or above the level  $n$  geometric Hodge polygons for any  $n$ . More precisely, we have the following relations:*

$$\text{New}(\varphi) \succcurlyeq \frac{1}{n}\text{Hdg}(\varphi^n) \succcurlyeq \text{Hdg}(\underline{X}, n)$$

on the relative position for the polygons mentioned above.

**1.5. Outline of the article**

In Section 2, we summarise relevant results from [2] on Dieudonné complexes and study the structure of the  $V$ -filtration on saturated Dieudonné complexes. In Section 3, we define log Dieudonné algebras and its subcategories consisting of  $p$ -compatible strict objects. The saturated log de Rham–Witt complex lives in the latter subcategory. We also compute some explicit examples of saturated and strict log Dieudonné algebras in this section. Section 4 consists of the technical core of the construction of  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$ . In Section 5 we compare  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  to the (completed) de Rham complex of a log-smooth ‘log-Frobenius lifting’ and prove the comparison results in Theorem 7. In Section 6, we extend the construction to the étale site  $X_{\text{ét}}$  of  $X$ . In Section 7, we show that our construction agrees with the existing ones due to Hyodo–Kato and Matsue in the case when the log scheme is sufficiently smooth. We then construct the monodromy operator on the log crystalline cohomology for semistable log schemes in Section 8. Finally, in Section 9 we study the level  $n$  Nygaard filtration on the saturated log de Rham–Witt complex and prove the generalised Katz conjecture in this setup (Theorem 10). Appendix A is included as a brief review of the necessary background on log schemes.

**1.6. Conventions**

We fix a prime  $p$  once and for all. We say that a cochain complex  $M^*$  is  $p$ -torsion free if each  $M^i$  is  $p$ -torsion free. By a cdga we mean a commutative differential graded algebra  $(A^* = \bigoplus_{i \geq 0} A^i, d)$ . In particular, the differential operator  $d$  increases grading by  $+1$  and satisfies  $d(xy) = (dx)y + (-1)^k x(dy)$  for  $x \in A^k$ ; commutativity requires that  $xy = (-1)^{kl}yx$  for  $x \in A^k, y \in A^l$  and that  $x^2 = 0$  for all  $x \in A^{2j+1}$ . The last condition is redundant except in characteristic 2.

For log schemes we refer to Appendix A, where we mostly follow [13] except that we denote a log scheme  $(X, M_X)$  by  $\underline{X}$ . In addition, by a log algebra  $(R, L)$  we mean an algebra  $R$  together with a monoid morphism  $L \rightarrow R$ ; this is the same data as giving a pre-log scheme  $\underline{Y} = (\text{Spec } R, L)$  with constant pre-log structure  $\beta : L_Y \rightarrow \mathcal{O}_Y$ . We often denote by  $\mathcal{L}^a$  the associated log structure of  $L$  on  $Y = \text{Spec } R$ , and denote its global section by  $L^{\text{sh}} = \Gamma(Y, \mathcal{L}^a)$ .

Throughout the article,  $k$  is a perfect field of characteristic  $p$ . We denote by  $\underline{W}$  a log algebra  $(W(k), N)$  where  $N$  could be arbitrary and reserve the notation  $W(\underline{k})$  for the log algebra  $(W(k), [\alpha] : N \rightarrow W(k))$ , where the log structure comes from the Teichmüller lift of a log point  $\underline{k} = (k, N)$ . The reader is welcome to take  $\underline{W} = W(\underline{k})$  for convenience.

**2. Saturated Dieudonné algebras**

The goal of this section is twofold: first we summarise and extend relevant results of Bhatt–Lurie–Mathew [2] and then we study the structure of the  $V$ -filtration of saturated Dieudonné complexes, which is essentially a reinterpretation of the relevant results of [10, I.3.1–I.3.10] using the language of saturated Dieudonné complexes.

**2.1. Dieudonné algebras, saturation and  $V$ -completion**

First recall the décalage operator.<sup>4</sup>

**Definition 2.1.** Let  $R$  be a ring and  $\mu \in R$  a nonzero divisor. Let  $(M^*, d)$  be a cochain complex of  $\mu$ -torsion-free  $R$ -modules; then  $(\eta_\mu M)^* \subset M^*[\frac{1}{\mu}]$  is defined to be the subcomplex given by

$$(\eta_\mu M)^i = \{x \in \mu^i M^i : dx \in \mu^{i+1} M^{i+1}\}.$$

$\eta_\mu$  kills  $\mu$ -torsion in the cohomology. More precisely, one has

$$H^i(\eta_\mu M^*) \cong H^i(M^*)/H^i(M^*)[\mu].$$

**2.1.1. Dieudonné complexes and Dieudonné algebras.**

**Definition 2.2.** A Dieudonné algebra is a triple  $(A^*, d, F)$  where

- $(A^* = \bigoplus_{i \geq 0} A^i, d)$  is a cdga concentrated in nonnegative degrees
- $F : A^* \rightarrow A^*$  is a graded algebra map satisfying the conditions  $dF(x) = pF(dx)$  for all  $x \in A^*$ , and  $F(x) \equiv x^p \pmod p$  for all  $x \in A^0$ .

More generally, a Dieudonné complex is a triple  $(M^*, d, F)$  consisting of a cochain complex  $(M^*, d)$  together with a Frobenius map  $F : M^* \rightarrow M^*$  satisfying  $dF = pFd$ .

**Definition 2.3.** Let  $M^* = (M^*, d, F)$  be a  $p$ -torsion-free Dieudonné complex; then  $F$  determines a map of cochain complexes  $\phi_F : M^* \rightarrow \eta_p M^*$  by sending  $x \mapsto p^n F(x)$  for  $x \in M^n$ . A Dieudonné complex  $M^*$  is saturated if it is  $p$ -torsion free and  $\phi_F$  is an isomorphism.

**Remark 2.4.** From the definition above, it follows by applying iterations of  $\eta_p$  (see Subsection 2.1.3) that, for a saturated Dieudonné complex, we have

$$d^{-1}(p^n M^{i+1}) = F^n(M^i)$$

for all  $n \geq 1$ . As remarked in the Introduction, this is one of the essential features of de Rham–Witt complexes constructed in [10] (and its logarithmic variant in [9]).

**Notation 2.5.** A Dieudonné algebra is saturated if its underlying Dieudonné complex is saturated. The category of Dieudonné algebras (respectively the full subcategory spanned by saturated algebras) is denoted by DA (respectively  $DA_{\text{sat}}$ ). Morphisms between two Dieudonné algebras are morphisms between cdgas compatible with the Frobenius maps.

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<sup>4</sup>The décalage operator was first introduced in [1]. It was later used in [11] for the crystalline construction of the de Rham–Witt complex  $W\Omega_{X/k}^*$  (and similarly in [9] to construct the log de Rham–Witt complexes from the log crystalline site.) Recently it appeared in the definition  $A_{\text{inf}}$ -cohomology in [3].



**2.1.2. Verschiebung.** Consider a saturated Dieudonné complex  $M^*$ . For each  $i \in \mathbb{Z}$ , the composed map

$$\phi_F : M^i \xrightarrow{F} \{x \in M^i : dx \in pM^{i+1}\} \xrightarrow{\times p^i} (\eta_p M)^i$$

is an isomorphism; hence,  $F$  is injective and  $F(M^*)$  contains  $pM^*$ . Therefore, for each  $x \in M^i$ , there is a unique element  $Vx \in M^i$  such that  $F(Vx) = px$ . This defines the **Verschiebung** map

$$V : M^i \longrightarrow M^i.$$

It is straightforward to check that

$$FV = VF = p, \quad FdV = d, \quad Vd = p dV, \quad xVy = V(Fx \cdot y).$$

The saturated condition imposes strong restrictions on differential, Frobenius and Verschiebung.

**Lemma 2.6.** *Suppose that  $M^*$  is saturated; then for any  $r \geq 1$ , we have*

- (1)  $d^{-1}(V^r M^{i+1}) = F^r(M^i) = d^{-1}(p^r M^{i+1})$ .
- (2) *If  $Fx \in V^r M^i$ , then  $x \in V^r M^i$ .*

**Proof.** For (1) it suffices to show that the inclusion

$$d^{-1}(p^r M^{i+1}) \subset d^{-1}(V^r M^{i+1}) \tag{2}$$

is an equality. To this end, suppose that  $x \in M^i$  is an element satisfying  $dx = V^r y$  for some  $y \in M^{i+1}$ ; then we have

$$d(F^r x) = p^r F^r dx = p^{2r} y.$$

Thus,  $F^r x = F^{2r} z$  for some  $z \in M^i$  by the saturated assumption. Because  $F$  is injective, we know that  $x = F^r z$  and hence  $dx \in p^r M^{i+1}$ . For (2), suppose that  $Fx = V^r y$  for some  $y \in M^i$ ; then

$$dy = F^r dV^r y = pF^{r+1} dx,$$

so  $y = Fz$  for some  $z \in M^i$ ; thus,  $Fx = FV^r z$  and it follows that  $x = V^r z$ . □

**2.1.3. Saturation.** Suppose that  $A^* \in \text{AD}$  is a  $p$ -torsion-free Dieudonné algebra; then  $(\eta_p A)^*$  with its inherited differential and Frobenius structure is again a Dieudonné algebra. The only nontautological point to check is that for any  $x \in (\eta_p A)^0$ ,  $F(x) = x^p + py$  for some  $y \in (\eta_p A)^0$  (not just in  $A^0$ ). The map  $\phi_F : A^* \rightarrow (\eta_p A)^*$  is a morphism of Dieudonné algebras.

The inclusion functor  $\text{AD}_{\text{sat}} \hookrightarrow \text{AD}$  admits a left adjoint,  $A^* \mapsto A_{\text{sat}}^*$ , which is called the **saturation** of  $A^*$  in [2] and is described as follows. Suppose that  $A^*$  is  $p$ -torsion free (replacing  $A^* \in \text{AD}$  by its  $p$ -torsion-free quotient if necessary); we then define  $A_{\text{sat}}^*$  to be the direct limit of

$$A^* \xrightarrow{\phi_F} (\eta_p A)^* \xrightarrow{\eta_p(\phi_F)} (\eta_p \eta_p A)^* \xrightarrow{\eta_p^2(\phi_F)} (\eta_p^3 A)^* \rightarrow \dots$$

By the discussion above, the saturation  $A_{\text{sat}}^*$  inherits the structure of a Dieudonné algebra, and the natural map  $A^* \rightarrow A_{\text{sat}}^*$  is a morphism of Dieudonné algebras.

**2.1.4. Strict Dieudonné algebras and  $V$ -completion.** Let  $M^*$  be a saturated Dieudonné complex. The  $V$ -filtration<sup>5</sup> of  $M^*$  is given by

$$\text{Fil}_V^r M^i = V^r M^i + dV^r M^{i-1}. \tag{3}$$

For each  $r \geq 1$ , we form the quotient

$$W_r(M^*) := M^*/(V^r M^* + dV^r M^*).$$

If  $A^*$  is a saturated Dieudonné algebra, then each  $W_r(A^*)$  is a cdga because  $\text{Fil}_V^r = V^r A^* + dV^r A^*$  is a differential graded ideal. The  $V$ -completion of  $M^*$  is defined to be the limit

$$W(M^*) := \varprojlim W_r M^*$$

along the natural projection maps  $R : W_r M^* \rightarrow W_{r-1} M^*$ . It is easy to check that the Frobenius and Verschiebung maps  $F$  and  $V$  on  $M^*$  induce maps

- $F : W_r(M^*) \rightarrow W_{r-1}(M^*)$  and
- $V : W_r(M^*) \rightarrow W_{r+1}(M^*)$

on the quotients.

**Definition 2.7.** A saturated Dieudonné algebra (or, more generally, a saturated Dieudonné complex)  $A^*$  is strict if it is complete with respect to the  $V$ -filtration; in other words, if the canonical map  $A^* \rightarrow W(A^*)$  is an isomorphism. The full subcategory of  $\text{DA}_{\text{sat}}$  spanned by strict algebras is denoted by  $\text{DA}_{\text{str}}$ .

**Remark 2.8.** Let  $A^*$  be a saturated Dieudonné algebra.

- $A^0/V A^0$  is a reduced  $\mathbb{F}_p$ -algebra by Lemma 3.6.1 of [2].
- The  $V$ -completion  $W(A^*)$  becomes a Dieudonné algebra with  $F$  the inverse limit of  $F : W_r(A^*) \rightarrow W_{r-1}(A^*)$ . It is in fact still saturated. One needs to check that (i)  $W(A^*)$  is  $p$ -torsion free; (ii) if  $x \in A^i := (W(A^*))^i$  is an element with  $dx \in pW^{i+1}$ , then  $x \in \text{im}(F)$ ; and (iii) the inverse limit  $F$  on  $W(A^*)$  satisfies  $F(x) \equiv x^p \pmod{p}$ . The details are left to the reader (see [2, Section 2.6 & 3.5]).

**Remark 2.9.** The  $V$ -completion  $W(A^*)$  for  $A^* \in \text{DA}_{\text{sat}}$  is strict. More precisely, for each  $r \geq 1$ , the canonical map  $W_r(A^*) \rightarrow W_r(W(A^*))$  is an isomorphism (by [2] Proposition 2.7.5). The completion functor  $A^* \mapsto W(A^*)$  provides a left adjoint of the inclusion  $\text{DA}_{\text{str}} \hookrightarrow \text{DA}_{\text{sat}}$ .

**Remark 2.10.** For a strict Dieudonné algebra  $A^*$ , each  $A^i$  is  $p$ -complete. Moreover,  $A^0$  is the ring of Witt vectors of  $W_1(A)^0 = A^0/V A^0$ . In other words, there is a canonical

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<sup>5</sup>This is sometimes called the canonical filtration in the context of de Rham–Witt complexes, though we prefer to reserve the name canonical for the conjugate filtration (‘la filtration canonique’).

isomorphism  $\mu : A^0 \rightarrow W(A^0/VA^0)$  such that the composition  $A^0 \rightarrow W(A^0/VA^0) \xrightarrow{\text{can}} A^0/VA^0$  is the projection map. The Frobenius  $F$  on  $A^0$  corresponds to the Witt vector Frobenius under  $\mu$ .

### 2.2. Saturated Dieudonné complexes

Let  $A^*$  be a  $p$ -torsion-free Dieudonné algebra (or, more generally, a Dieudonné complex). Consider the cochain complex  $(H^*(A^*/pA^*), \beta)$  where  $\beta$  is the Bockstein differential induced from

$$0 \rightarrow A^*/p \xrightarrow{p} A^*/p^2 \rightarrow A^*/p \rightarrow 0.$$

We have the following commutative diagram, where each arrow is a map of cochain complexes:

$$\begin{array}{ccc} (A^*/p, d) & \xrightarrow{F} & (H^*(A^*/p), \beta) \\ & \searrow \phi_F & \nearrow \gamma \\ & (\eta_p(A^*)/p, d) & \end{array}$$

Here  $\gamma$  is defined by sending  $x \in (\eta_p A)^i$  to  $x/p^i$ . Suppose that  $A^*$  is in addition saturated; then the composition  $F$  in the diagram factors through the quotient cochain complex  $W_1(A^*)$ , as in the diagram below:

$$\begin{array}{ccc} & (W_1(A^*), d) & \\ q \nearrow & & \searrow F_1 \\ (A^*/p, d) & \xrightarrow{F} & (H^*(A^*/p), \beta) \end{array}$$

**Proposition 2.11.** *Let  $A^*$  be a  $p$ -torsion-free Dieudonné algebra or complex; then*

- (1)  $\gamma : \eta_p(A^*)/p \rightarrow H^*(A^*/p)$  is always a quasi-isomorphism of cochain complexes.
- (2) Suppose that  $A^*$  is saturated; then  $F_1 : W_1(A^*) \rightarrow H^*(A^*/p)$  is an isomorphism. More generally, the iterated Frobenius map  $F^r$  induces an isomorphism of cochain complexes

$$F_r : (W_r(A^*), d) \xrightarrow{\sim} H^*((A^*/p^r), \beta).$$

- (3) Again suppose that  $A^*$  is saturated; then the natural projection

$$q : A^*/p^r \rightarrow W_r(A^*)$$

is a quasi-isomorphism of cochain complexes.

- (4) Let  $f : A^* \rightarrow B^*$  be a morphism of saturated Dieudonné algebras; then  $f : A^*/p \rightarrow B^*/p$  is a quasi-isomorphism if and only if  $f : W_1(A^*) \rightarrow W_1(B^*)$  is an isomorphism, if and only if  $f : W(A^*) \rightarrow W(B^*)$  is an isomorphism.

**Proof.** For (1), the map  $\gamma$  is surjective and one can show that its kernel is acyclic essentially by unwinding definitions (see [2, Proposition 2.4.5]; also compare to [5, Proposition 1.3.4] and [3, Proposition 6.12]). For (2), the surjectivity follows from

Remark 2.4; the injectivity is clear (see [2, Proposition 2.7.1]). (3) Follows from the two commutative triangles above (and their generalisations) and (4) follows from (3). See also [2, Remark 2.7.3] for a direct argument for (3).  $\square$

We record a corollary for our discussion later on Nygaard filtrations.

**Corollary 2.12.** *Let  $A^*$  be a saturated Dieudonné algebra or complex; then the iterated Frobenius  $F^r$  induces an isomorphism*

$$F^r : W_r(A^i) \xrightarrow{\sim} H^i(W_r(A^*)).$$

**Proof.** We have the following diagram:

$$\begin{array}{ccc}
 W_r(A^i) & \xrightarrow{F^r} & H^i(W_r(A^*)) \\
 \uparrow q & \searrow F_r & \nearrow \\
 A^i/p^r A^i & \xrightarrow{F^r} & H^i(A^*/p^r)
 \end{array}$$

The right-most arrow on cohomology (induced by  $q$ ) is an isomorphism by Proposition 2.11 part (3), so the top arrow is an isomorphism by Proposition 2.11 part (2).  $\square$

**Remark 2.13.** In fact, the map  $F^r$  above can be upgraded to an isomorphism

$$F^r : W_r(A^*) \xrightarrow{\sim} H^*(W_r(A^*))$$

of cochain complexes, where the target is equipped with the ‘Bockstein differential’ induced from the cofibre sequence  $W_r A^* \xrightarrow{P^r} W_{2r} A^* \rightarrow W_r A^*$ .

**2.3. Dieudonné complexes of Cartier type**

**Definition 2.14.** We say that a Dieudonné algebra (or, more generally, a Dieudonné complex)  $\Omega^*$  is of Cartier type (or satisfies the Cartier criterion) if

$$F : (\Omega^*/p, d) \rightarrow (H^*(\Omega^*/p), \beta)$$

is an isomorphism of cochain complexes.

In the case of the above definition, we denote the map  $F^{-1}$  by  $C$ , because it is evidently related to the Cartier isomorphism (see Subsection A.3.2).

**Remark 2.15.** Dieudonné complexes of Cartier type are generally not saturated. In fact, if  $\Omega^*$  is a  $p$ -complete,  $p$ -torsion-free saturated Dieudonné complex of Cartier type, then from the discussion in Subsection 2.2, the projection  $q : \Omega^*/p \rightarrow W_1 \Omega^*$  is an isomorphism. This implies that  $V\Omega^* + dV\Omega^* \subset p\Omega^*$ ; thus, for each  $i$  the Frobenius  $F : \Omega^i \rightarrow \Omega^i$  is surjective (and hence an isomorphism). Therefore, for any  $x \in \Omega^i$ ,  $dx = dF^n(F^{-n}(x)) \in p^n \Omega^{i+1}$  for arbitrarily large  $n$ ; thus, by the  $p$ -completeness assumption,  $dx = 0$ . In other words, under our assumption,  $\Omega^* = \oplus \Omega^i[-1]$  with differential  $d = 0$ . In other words, there exists no saturated Dieudonné complex of Cartier type with nontrivial differentials.

Next we consider the saturation of a Dieudonné complex of Cartier type.

**Corollary 2.16.** *Let  $\Omega^*$  be a  $p$ -torsion-free Dieudonné algebra or complex of Cartier type; then  $\phi_F : \Omega^*/p \rightarrow \eta_p \Omega^*/p$  is a quasi-isomorphism. Moreover, we have a quasi-isomorphism*

$$\Omega^*/p \longrightarrow \Omega_{\text{sat}}^*/p$$

**Corollary 2.17.** *Let  $\Omega^*$  be a  $p$ -complete and  $p$ -torsion free Dieudonné algebra or complex of Cartier type, then the canonical map  $\Omega^* \rightarrow W(\Omega_{\text{sat}}^*)$  is a quasi-isomorphism.*

**Proof.** It suffices to show that  $\Omega^*/p \rightarrow \Omega_{\text{sat}}^*/p \rightarrow W(\Omega_{\text{sat}}^*)/p$  is a quasi-isomorphism because each  $A^i$  is  $p$ -adically complete and  $p$ -torsion free. The first map is a quasi-isomorphism by Corollary 2.16 (1), and the second map is a quasi-isomorphism by Remark 2.9 and Corollary 2.16 (3). □

### 2.4. Structure of the $V$ -filtration

In this subsection we let  $M^*$  be a saturated Dieudonné complex and analyze the structure of its  $V$ -filtration  $\text{Fil}_V^*$  defined in (3). We begin by considering the following maps:

$$\beta : \text{Fil}_V^n(M^i) \rightarrow M^{i-1}/(V, F^n) \tag{4}$$

$$\beta' : \text{Fil}_V^n(M^i) \rightarrow M^i/(V, F^n d) \tag{5}$$

defined by

$$\beta(V^n x + dV^n y) = y, \quad \beta'(V^n x + dV^n y) = x.$$

First we check that these maps are well defined. Suppose that  $V^n x + dV^n y = V^n x' + dV^n y'$ ; then

$$d(y - y') = F^n dV^n(y - y') = F^n(V^n x' - V^n x) = p^n(x' - x).$$

By Remark 2.4 we know that  $y - y' = F^n z$  for some  $z \in M^{i-1}$ , so

$$\beta(V^n x + dV^n y) = \beta(V^n x' + dV^n y') \in M^{i-1}/(V, F^n).$$

On the other hand, because  $y - y' = F^n z$ , we further know that  $p^n(x' - x) = dF^n z = p^n F^n dz$ , so  $x' - x = F^n dz$ . Thus, we also have  $\beta'(V^n x + dV^n y) = \beta'(V^n x' + dV^n y')$ .

The following proposition echoes [10, Corollary I.3.9].

**Proposition 2.18.** *Let  $M^*$  be a saturated Dieudonné complex as above. The  $n$ th-graded piece of the  $V$ -filtration sits in the following short exact sequences:*

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & M^{i-1}/(V, F^{n+1}) & & & \\
 & & & \downarrow dV^n & \searrow & & \\
 0 \longrightarrow & M^i/(V, F^{n-1}d) & \xleftarrow{V^n} & gr_V^n M^i & \xrightarrow{\beta} & M^{i-1}/(V, F^n) & \longrightarrow 0 \\
 & & \searrow & \downarrow \beta' & & & \\
 & & & M^i/(V, F^n d) & & & \\
 & & & \downarrow & & & \\
 & & & 0 & & & 
 \end{array} \tag{6}$$

such that the composition  $\beta \circ dV^n$  (respectively  $\beta' \circ V^n$ ) is the natural projection. Note that when  $n = 0$ , we set  $F^{n-1}dM^{i-1} = 0$  in the horizontal diagram.

**Proof.** We check that the horizontal sequence is exact and leave the vertical sequence to the reader. Clearly, the map  $\beta$  is surjective and factors through

$$\beta : \text{gr}_V^n = (V^n M^i + dV^n M^{i-1}) / (V^{n+1} M^i + dV^{n+1} M^{i-1}) \longrightarrow M^{i-1} / (V, F^n).$$

First we check exactness in the middle. Suppose that  $\beta(V^n x + dV^n y) = 0$ ; then  $y = Vz + F^n w$  for some  $z, w \in M^{i-1}$ ; thus,

$$V^n x + dV^n y = V^n(x + F^n dw) + dV^{n+1} z,$$

which lies in  $\text{im}(V^n) \subset \text{gr}_V^n M^i$ . It remains to show injectivity of the first map. To this end, suppose that  $V^n x = V^{n+1} y + dV^{n+1} z$ ; then

$$dz = F^{n+1} dV^{n+1} z = F^{n+1}(V^n x - V^{n+1} y) = p^n F(x - Vy).$$

Let  $s = x - Vy$ ; then  $u$  satisfies  $p^n F s = dz$ ; thus, by Lemma 2.20 we have  $s = F^{n-1} dt$  for some  $t \in M^{i-1}$ , so  $x = Vy + F^{n-1} dt$  and we are done. □

In view of the lemma above, it is convenient to introduce the following filtrations on  $M^*$ .

**Definition 2.19.** Let  $M^*$  be saturated as above. For  $n \geq 1$ , we define subgroups of  $M^i$  by

- (1)  $B_n(M^i) := F^{n-1} dM^{i-1}$
- (2)  $Z_n(M^i) := F^n M^i$ .

From the definition we have a chain of inclusions

$$\begin{aligned} 0 \subset B_1(M^i) \subset B_2(M^i) \subset \dots \subset B_n(M^i) \subset B_{n+1}(M^i) \subset \dots \\ \subset Z_n(M^i) \subset \dots \subset Z_2 \subset Z_1(M^i) \subset M^i. \end{aligned}$$

The following simple lemma is used in the proof of the proposition above.

**Lemma 2.20.** Let  $M^*$  be saturated as above.

- (1)  $B_n(M^i) = \{x \in M^i \mid p^{n-1} x = dy\} = \{x \in M^i \mid p^n Fx = dy\}$ .
- (2)  $Z_n(M^i) = \{x \in M^i \mid dx = p^n y\} = \{x \in M^i \mid dx = V^n y\}$ .

**Proof.** (2) Is a restatement of Lemma 2.6 part (1). For (1), first note that

$$B_n(M^i) \subset \{x \in M^i \mid p^{n-1} x = dy\} \subset \{x \in M^i \mid p^n Fx = dy\}.$$

Now suppose that  $x \in M^i$  and  $p^n Fx = dy$ , so  $y = F^n z$  for some  $z \in M^{i-1}$ . This implies that  $p^n Fx = p^n F^n dz$ ; thus,  $x = F^{n-1} dz$ . □

**Notation 2.21.** Next consider the image of  $B_n$  and  $Z_n$  under the projection  $M^* \rightarrow W_1(M^*) = M^* / (V + dV)$ , which we denote by

$$B_n(W_1 M^i) = F^{n-1} dM^{i-1} / (dV, V \cap B_n), \quad \text{and} \quad Z_n(W_1 M^i) = F^n M^i / (dV, V \cap Z_n).$$

(Note that  $dV(M^{i-1}) \subset B_n(M^i) \subset Z_n(M^i)$ .)

**Lemma 2.22.** *Let  $M^*$  be saturated as above. As subgroups of  $W_1M^i$  we have*

- (1)  $B_1(W_1M^i) = B^i(W_1(M^*)) := \text{im}(d : W_1M^{i-1} \rightarrow W_1M^i)$ .
- (2)  $Z_1(W_1M^i) = Z^i(W_1(M^*)) := \text{ker}(d : W_1M^i \rightarrow W_1M^{i+1})$ .
- (3) *Moreover, the Frobenius  $F$  induces isomorphisms*

$$\begin{array}{ccccc}
 B_n(W_1M^i) & \xrightarrow{\sim} & B_{n+1}(W_1M^i)/B_1(W_1M^i) & \equiv & F^n dM^{i-1}/(\text{im } d + \text{im } V) \\
 \downarrow & & \downarrow & & \downarrow \\
 Z_n(W_1M^i) & \xrightarrow{\sim} & Z_{n+1}(W_1M^i)/B_1(W_1M^i) & \equiv & F^{n+1}M^i/(\text{im } d + \text{im } V) \\
 \downarrow & & \downarrow & & \\
 W_1M^i & \xrightarrow{\sim} & H^i(W_1(M^*)) & = & \frac{Z_1(W_1M^i)}{B_1(W_1M^i)}
 \end{array}$$

where the last isomorphism is taken from Corollary 2.12.

**Proof.** (1) Is tautological and (2) follows from Lemma 2.20. For (3), we know directly from construction that the Frobenius  $F$  on  $B_n(W_1M^i)$  and  $Z_n(W_1M^i)$  are both surjective. To check injectivity (in both cases), it suffices to show that if  $x \in M^i$  satisfies  $Fx = dy + Vz$ , then  $x \in \text{Fil}_V^1 = (V + dV)$ . To this end, we simply observe that

$$Vz = Fx - dy = Fx - FdVy = F(x - dVy).$$

By Lemma 2.6, this implies that  $x - dVy \in \text{im } V$ ; the injectivity thus follows. □

As a corollary, we have the following equivalent form of Proposition 2.18.

**Corollary 2.23.** *The graded pieces of the  $V$ -filtration of  $M^*$  are extensions of the following quotients of  $W_1(M^*)$ :*

$$0 \longrightarrow W_1M^i/B_n(W_1M^i) \xrightarrow{V^n} gr_V^n M^i \xrightarrow{\beta} W_1M^{i-1}/Z_n(W_1M^{i-1}) \longrightarrow 0 \tag{7}$$

$$0 \longrightarrow W_1M^{i-1}/Z_{n+1}(W_1M^{i-1}) \xrightarrow{dV^n} gr_V^n M^i \xrightarrow{\beta'} W_1M^i/B_{n+1}(W_1M^i) \longrightarrow 0. \tag{8}$$

**Remark 2.24.** Suppose that  $A^*$  is a saturated Dieudonné algebra; then the exact sequences (7) and (8) are exact sequences of  $W_1A^0 = A^0/V A^0$ -modules, where the  $W_1A^0$ -structure is given as follows (we only focus on the first sequence (7) and leave the second to the reader):

- The action of  $A^0$  on  $A^i$  makes  $W_1A^i$  into a  $W_1A^0$ -module because  $Vx \cdot y = V(x \cdot Fy)$ , though this is **not** the  $W_1A^0$ -structure we use. Instead, we twist by  $(n + 1)$ -iterates of Frobenius and consider  $W_1A^i$  as a  $W_1A^0$ -module via

$$F^{n+1} : W_1A^0 \longrightarrow W_1A^0.$$

In other words,  $a \cdot x := F^{n+1}(a)x$  for all  $a \in W_1A^0, x \in W_1A^i$ .

- The subgroups  $B_n(W_1A^i)$  (respectively  $Z_n(W_1A^i)$ ) are naturally  $W_1A^0$ -submodules via the  $W_1A^0$ -action described above, because

$$a \cdot F^{n-1}dx = F^{n-1}(F^2(a)dx) = F^{n-1}d(F^2(a)x) - p^2F^{n-1}((F^2da)x),$$

which lies in  $F^{n-1}dA^i$  modulo  $V$ . The quotients  $W_1A^i/B_n(W_1A^i)$  (respectively  $W_1A^i/Z_n(W_1A^i)$ ) are thus naturally  $W_1A^0$ -modules with the  $F^{n+1}$ -twisted structure.

- Finally, the middle term  $\text{gr}_V^n A^i$  is a  $W_1A^0$ -module via the following Frobenius map:

$$F : W_1A^0 = A^0/V A^0 \longrightarrow A^0/pA^0.$$

First note that  $\text{Fil}_V^n = V^n A^i + dV^n A^{i-1}$  is a  $A^0$ -submodule of  $A^i$ . This gives a natural  $A^0/pA^0$ -module structure on  $\text{gr}_V^n A^i$ , which thus carries a  $W_1A^0$  via Frobenius.

The sequences in Corollary 2.23 are short exact sequences of  $W_1A^0$ -modules with the module structure described above. In other words, we have

$$\begin{aligned} V^n(F^{n+1}(a)x) &= F(a)V^n(x) \quad \text{and} \\ \beta(F(a)dV^n y) &= \beta(dV^n(F^{n+1}(a)y) - V^n(F^{n+1}(da)y)) = F^{n+1}(a)\beta(dV^n y) \end{aligned}$$

for all  $a \in W_1A^0, y \in A^{i-1}$ .

### 3. Log Dieudonné algebras

In this section we describe an enhancement of Dieudonné algebras that incorporates log structures. This serves as a basis for the theory of saturated log de Rham–Witt complexes on local charts (of a log scheme). The definitions in this section are extensions of those in [2], taking (pre-) log structures into account. In particular, we prove that, for a strict Dieudonné algebra with a  $p$ -compatible log structure, its log structure is ‘valued in’ Teichmüller representatives (Proposition 3.13).

**Convention 3.1.** Recall from Subsection 1.6 (also see Appendix A) that a log algebra  $(R, L)$  is an algebra  $R$  together with a monoid map  $L \rightarrow R$ . Geometrically, this corresponds to a constant pre-log scheme  $\text{Spec}(R, L)$ . In fact, our construction of the saturated log de Rham–Witt complexes takes as input such a pre-log structure (in the form of a log algebra) instead of a log structure. Later we will see that (Proposition 4.12) the construction essentially only depends on the underlying log structures. In this section and the next, we do not require the log structures to satisfy additional properties such as being integral or coherent unless otherwise stated.

#### 3.1. Log Dieudonné algebras

**Definition 3.2.** Let  $\underline{W} = (W(k), N)$  be a log algebra. A log Dieudonné  $\underline{W}$ -algebra is a tuple  $(A^*, L, d, \delta, F, F_L)$ , consisting of the following data:

- $(A^* = \bigoplus_{i \geq 0} A^i, d)$  is a cdga over  $W$ ,
- $\underline{A}^0 = (A^0, L)$  is a log algebra over  $\underline{W}$ ,
- $\delta : L \rightarrow A^1$  is a map of monoids,
- $F : A^* \rightarrow A^*$  a graded algebra homomorphism, and
- $F_L : L \rightarrow L$  is a monoid homomorphism



satisfying the following requirements:

- (1)  $(A^*, d, F)$  is a Dieudonné algebra in the sense of Definition 2.2.
- (2)  $(d : A^0 \rightarrow A^1, \delta : L \rightarrow A^1)$  is a log derivation of  $\underline{A}^0/\underline{W}$ , where we further require that the composition  $L \xrightarrow{\delta} A^1 \xrightarrow{d} A^2$  is 0.
- (3)  $\delta$  satisfies  $\delta F_L = pF\delta$ .
- (4)  $F$  and  $F_L$  are compatible in the sense that  $F \circ \alpha = \alpha \circ F_L$ .

**Notation.** We often suppress notations and simply write  $A^*$  for a log Dieudonné algebra.

**Remark 3.3.** In contrast to the nonlogarithmic setting, for log Dieudonné algebras we specify a base  $\underline{W}$ . In the nonlogarithmic case,  $\Omega_{k/\mathbb{F}_p}^1 = 0$  for a perfect field  $k$  of char  $p$ , so the base is to some extent irrelevant (as one can see from the construction in Subsection 5.2, for example). On the other hand, if we have two distinct log structures  $N_1, N_2$  on  $k$ , the logarithmic differential  $\omega_{(k, N_1)/(k, N_2)}$  is typically nonzero.

**Definition 3.4.** A morphism between two log Dieudonné  $\underline{W}$ -algebras  $A^*$  and  $B^*$  is a pair  $(f, \psi)$ , where  $f : (A^*, d, F) \rightarrow (B^*, d', F')$  is a morphism between Dieudonna algebras over  $\underline{W}$  and  $\psi : L_A \rightarrow L_B$  is a monoid morphism over  $N$  that is compatible with  $F_L, \alpha$  and  $\delta$ :

- (1)  $f \circ d = d' \circ f$  and  $f \circ F = F' \circ f$  on each  $A^i$ .
- (2) The structure map  $N \rightarrow L_B$  agrees with  $N \rightarrow L_A \xrightarrow{\psi} L_B$  (respectively,  $W \rightarrow B^0$  agrees with  $W \rightarrow A^0 \xrightarrow{f|_{A^0}} B^0$ ). Moreover,  $f \circ \alpha_A = \alpha_B \circ \psi$  on  $L_A$ . This is simply saying that  $(f, \psi)|_{(A^0, L_A)}$  is a morphism of log  $\underline{W}$ -algebras.
- (3)  $\psi$  Satisfies  $\psi \circ F_{L_A} = F_{L_B} \circ \psi$  and  $f \circ \delta_A = \delta_B \circ \psi$ .

**Notation.** We denote the category of log Dieudonné algebras by  $\text{DA}_{\underline{W}}^{\text{log}}$ , and we often suppress notations to write  $\text{DA}^{\text{log}} = \text{DA}_{\underline{W}}^{\text{log}}$  when  $\underline{W}$  is understood.

**Lemma 3.5.** A morphism  $(f, \psi) : A^* \rightarrow B^*$  of log Dieudonné algebras over  $\underline{W}$  is an isomorphism if and only if both  $f$  and  $\psi$  are isomorphisms.

**Proof.** This is straightforward. □

### 3.2. $p$ -Compatible log Dieudonné algebras

**Definition 3.6.** We say that a log Dieudonné algebra  $A^*$  over  $\underline{W}$  is  $p$ -compatible if the Frobenius on the log structure  $L$  is given by  $F_L = p$ .

Let  $\text{DA}^{\text{log}, p}$  denote the full subcategory of  $\text{DA}^{\text{log}}$  spanned by  $p$ -compatible objects. In this article we are mostly interested in  $p$ -compatible log Dieudonné algebras (see Proposition 4.5).

**Remark 3.7.** For a log structure  $\mathcal{L}$  on an affine  $\mathbb{Z}_p$ -scheme  $\mathfrak{X} = \text{Spec } A$ , the associated log algebra  $(A, \Gamma(\mathfrak{X}, \mathcal{L}))$  is usually not  $p$ -compatible. A typical example that arises is

$(\mathbb{Z}_p[x, y]/xy, L = \mathbb{N}^2)$  with the pre-log structure  $(a, b) \mapsto x^a y^b$  and Frobenius  $F_L$  given by multiplication by  $p$ . On the (sections of the) associated log structure  $L^a = \mathbb{Z}_p^\times \oplus \mathbb{N}^2$ , the induced Frobenius is  $F_{L^a} = \text{id} \oplus F_L$ . This is one reason why we prefer to work with pre-log structures.

### 3.3. Saturated log Dieudonné algebras

In this subsection and the next we discuss saturation and  $V$ -completion in the logarithmic setting.

**Definition 3.8.** Let  $(A^*, L \xrightarrow{\alpha} A^0, d, \delta, F, F_L)$  be a  $p$ -torsion-free log Dieudonné algebra. We define  $\eta_p(A^*)$  to be the tuple  $(\eta_p(A^*), L \xrightarrow{\eta_p(\alpha)} \eta_p A^0, d, \eta_p(\delta), F, F_L)$  where

- $\eta_p A^i = \{x \in p^i A^i : dx \in p^{i+1} A^{i+1}\}$  as in Definition 2.1,
- the log algebra  $(\eta_p A^0, L)$  is given by  $\eta_p(\alpha) : L \xrightarrow{F_L} F_L(L) \xrightarrow{\alpha} \eta_p A^0$ ,
- the log derivation  $\eta_p(\delta)$  is given by  $\eta_p(\delta) : L \xrightarrow{F_L} F_L(L) \xrightarrow{\delta} \eta_p A^1$ .<sup>6</sup>

Both  $\eta_p(\alpha)$  and  $\eta_p(\delta)$  are well defined by the relations  $dF = pFd$  and  $\delta F = pF\delta$ . It is straightforward to verify that  $\eta_p(A^*)$  is in fact a log Dieudonné  $\underline{W}$ -algebra, with its  $\underline{W}$ -structure given by  $W \rightarrow A^0 \rightarrow (\eta_p A)^0$  and  $N \rightarrow L$ . Now let  $\text{DA}_{p\text{-tf}}^{\text{log}}$  be the full subcategory of  $\text{DA}^{\text{log}}$  spanned by  $p$ -torsion-free complexes; then  $\eta_p$  induces a logarithmic décalage operator  $\text{DA}_{p\text{-tf}}^{\text{log}} \rightarrow \text{DA}_{p\text{-tf}}^{\text{log}}$ . For a morphism  $f = (f, \psi) : A^* \rightarrow B^*$  between log Dieudonné algebras, one obtains a morphism  $\eta_p(f) : \eta_p(A^*) \rightarrow \eta_p(B^*)$  in a functorial way. The operator  $\eta_p$  clearly preserves the  $p$ -compatible objects. Moreover, for a  $p$ -torsion-free object  $A^*$  in the category  $\text{DA}^{\text{log}}$  (respectively in  $\text{DA}^{\text{log}, p}$ ), we have a natural map

$$\phi_F = (\phi_F, \text{id}) : A^* \longrightarrow \eta_p(A^*)$$

of log Dieudonné algebras, where  $\phi_F$  is given by the map  $p^i F$  on  $A^i$  and the identity on the monoid  $L$ .

**Definition 3.9.** A log Dieudonné algebra  $A^* \in \text{DA}^{\text{log}}$  is saturated if  $A^*$  is  $p$ -torsion free and  $\phi_F$  is an isomorphism; in other words, if the underlying Dieudonné algebra is saturated.

**Notation.** The full subcategory of  $\text{DA}^{\text{log}}$  spanned by saturated objects (respectively  $p$ -compatible saturated objects) is denoted by  $\text{DA}_{\text{sat}}^{\text{log}}$  (respectively  $\text{DA}_{\text{sat}}^{\text{log}, p}$ ).

**Remark 3.10.** The definition of being saturated has nothing to do with the log structure  $L$ . Moreover, by the same discussion in Subsection 2.1.2, a saturated log Dieudonné algebra admits the Verschiebung operator  $V : A^i \rightarrow A^i$  satisfying the relations listed there.

Because the log structure is irrelevant in the condition of being saturated, the construction of the saturation of a log Dieudonné algebra is essentially the same as in the

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<sup>6</sup>Equivalently,  $\eta_p(\delta)$  is given by the composition  $L \xrightarrow{\delta} A^1 \xrightarrow{pF} \eta_p A^1$ . Likewise,  $\eta_p(\alpha)$  is given by the composition  $L \xrightarrow{\alpha} A^0 \xrightarrow{F} \eta_p A^0$ .

nonlogarithmic setup. First we replace  $\underline{A}^* = (A^*, L)$  by  $(A^*/A^*[p^\infty], L)$  to reduce to the  $p$ -torsion-free case while keeping the monoid  $L$ . For a  $p$ -torsion-free log Dieudonné algebra  $A^*$ , we apply the log décalage operator  $\eta_p$  to the morphism  $\phi_F = (\phi_F, \text{id}) : A^* \rightarrow \eta_p(A^*)$  repeatedly and obtain

$$A_{\text{sat}}^* := \text{colim} \left( A^* \xrightarrow{\phi_F} \eta_p(A^*) \xrightarrow{\phi_F} \eta_p \eta_p(A^*) \rightarrow \dots \right).$$

It is a log Dieudonné algebra with monoid maps  $\alpha : L \rightarrow A^0 \rightarrow A_{\text{sat}}^0$  and  $\delta : L \rightarrow A^1 \rightarrow A_{\text{sat}}^1$ .

### 3.4. Strict log Dieudonné algebras and $V$ -completion

Let  $A^* \in \text{DA}^{\text{log}}$  be a saturated object. In Subsection 2.1 we formed the inverse limit

$$W(A^*) = \varprojlim W_r(A^*)$$

along the restriction maps  $R : W_r(A^*) \rightarrow W_{r-1}(A^*)$ , where  $W_r(A^*) = A^*/\text{Fil}^r$  is the quotient of  $A^*$  by the differential graded ideal  $\text{Fil}^r = V^r(A^*) + \text{d}V^r(A^*) \subset A^*$  (which defines the  $V$ -filtration on  $A^*$ ). The Dieudonné algebra  $W(A^*)$  inherits a log structure  $L \rightarrow A^0 \rightarrow W(A)^0$  and a log derivation  $L \rightarrow A^1 \rightarrow W(A)^1$  from  $A^*$ , which makes it a log Dieudonné algebra. The canonical map  $\rho : A^* \rightarrow W(A^*)$  of saturated Dieudonné algebras upgrades to  $(\rho, \text{id})$  as a morphism in  $\text{DA}^{\text{log}}$ , which we still denote by  $\rho$ . By Remark 2.8, we know that, if  $A^* \in \text{DA}^{\text{log}}$  is saturated (respectively saturated and  $p$ -compatible), then  $W(A^*)$  is again saturated (respectively saturated and  $p$ -compatible).

**Definition 3.11.** A saturated log Dieudonné algebra  $A^*$  is strict if  $\rho : A^* \rightarrow W(A^*)$  is an isomorphism; in other words, if its underlying Dieudonné algebra is strict.

**Notation.** The full subcategory of  $\text{DA}_{\text{sat}}^{\text{log}}$  (respectively  $\text{DA}_{\text{sat}}^{\text{log}, p}$ ) spanned by strict objects is denoted by  $\text{DA}_{\text{str}}^{\text{log}}$  (respectively  $\text{DA}_{\text{str}}^{\text{log}, p}$ ).

**Lemma 3.12.** (1) *The saturation functor  $\text{sat} : \text{DA}^{\text{log}} \rightarrow \text{DA}_{\text{sat}}^{\text{log}}$  is the left adjoint of the inclusion  $\text{DA}_{\text{sat}}^{\text{log}} \subset \text{DA}^{\text{log}}$ . The same is true for the  $p$ -compatible subcategory  $\text{DA}_{\text{sat}}^{\text{log}, p}$ .*

(2) *The completion  $W(A^*)$  of a saturated log Dieudonné algebra is strict. Moreover, the completion functor  $W : \text{DA}_{\text{sat}}^{\text{log}} \rightarrow \text{DA}_{\text{str}}^{\text{log}}$  provides a left adjoint of the inclusion  $\text{DA}_{\text{str}}^{\text{log}} \subset \text{DA}_{\text{sat}}^{\text{log}}$ , similarly for the  $p$ -compatible subcategory.*

**Proof.** Immediate from Subsection 2.1. □

The condition of being strict and  $p$ -compatible imposes some rather strict restrictions on the log structures. For the setup, let  $A^* \in \text{DA}_{\text{str}}^{\text{log}, p}$ , and recall from Section 2 that we have a canonical isomorphism  $\mu : A^0 \xrightarrow{\sim} W(A^0/VA^0)$  respecting Frobenii on both sides. The following proposition will be important for the construction of log de Rham–Witt complexes in Section 4.

**Proposition 3.13.** *Let  $A^* \in \text{DA}_{\text{str}}^{\text{log}}$  be a strict Dieudonné algebra. If  $x \in A^0$  is an element satisfying  $F(x) = x^p$ , then  $x = [\bar{x}]$ , where  $\bar{x}$  is the image of  $x$  in  $W_1A^0 = A^0/VA^0$  and  $[\bar{x}]$  is its Teichmüller lift.*

**Proof.** It suffices to show that  $x - [\bar{x}] \in \text{Fil}^s(A^0) = V^s(A^0)$  for every  $s \in \mathbb{Z}_{\geq 1}$  because  $A^0$  is complete with respect to the  $V$ -filtration. We proceed by induction. By assumption,

$x - [\bar{x}] \in \text{Fil}^1$ . We claim that if  $x - [\bar{x}] \in \text{Fil}^r$  for  $r \geq 1$ , then  $x - [\bar{x}] \in \text{Fil}^{2r}$ . Suppose that  $x - [\bar{x}] = V^r b$  for some  $b \in A^0$ ; then

$$F^r(x) - F^r([\bar{x}]) = x^{p^r} - [\bar{x}]^{p^r} = p^r \cdot b.$$

In the first equality we use the assumption on  $x$ . On the other hand,

$$\begin{aligned} x^{p^r} &= \left([\bar{x}] + V^r b\right)^{p^r} \\ &= [\bar{x}]^{p^r} + p^r [\bar{x}]^{p^r-1} \cdot V^r b + \sum_{k=2}^{p^r} \binom{p^r}{k} [\bar{x}]^{p^r-k} \cdot (V^r b)^k. \end{aligned}$$

Note that because  $xV^r y = V^r((F^r x)y)$ , we have

$$(V^r b)^2 = V^r((F^r V^r b)b) = V^r(p^r b^2) = p^r V^r(b^2).$$

Therefore,

$$p^r b = p^r [\bar{x}]^{p^r-1} \cdot V^r b + p^r V^r(b^2) \cdot \sum_{k=2}^{p^r} \binom{p^r}{k} [\bar{x}]^{p^r-k} \cdot (V^r b)^{k-2}.$$

Because  $A^0$  is  $p$ -torsion free, we conclude that  $b \in V^r(A^0)$  and this completes the proof. □

**Corollary 3.14.** *Let  $A^* \in DA_{\text{str}}$  be a strict Dieudonné algebra. Let  $\alpha : L \rightarrow A^0$  be a log structure on  $A^0$  such that  $F(\alpha(m)) = \alpha(m)^p$  for all  $m \in L$ .*

- (1) *The image of  $\alpha$  consists of Teichmüller lifts of elements of  $A^0/V A^0$ .*
- (2) *Moreover, let  $\underline{A}^0/V \underline{A}^0$  be the log algebra  $(A^0/V A^0, \bar{\alpha} : L \rightarrow A^0/V \underline{A}^0)$  and  $W(\underline{A}^0/V \underline{A}^0)$  be the log algebra obtained via the Teichmüller lift. Then  $\mu : A^0 \xrightarrow{\sim} W(A^0/V A^0)$  induces an isomorphism of log algebras*

$$\underline{\mu} = (\mu, id) : \underline{A}^0 = (A^0, L) \rightarrow W(\underline{A}^0/V \underline{A}^0).$$

*This in particular applies to the case of a strict  $p$ -compatible log Dieudonné algebra.*

**Proof.** This is immediate from Proposition 3.13. □

**3.5. Examples**

We end this subsection with some toy examples of saturated and strict log Dieudonné algebras in the form of saturation and  $V$ -completion.

**Example 3.15** (Affine spaces with partial log structures). In this example we consider the log Dieudonné algebra  $E^*$  where

$$E^0 = \mathbb{Z}_p[t_1, \dots, t_{r+s}]$$

and  $E^i$  is a free  $E^0$  module on generators

$$\eta_1 \wedge \dots \wedge \eta_i, \eta_i \in \left\{ \frac{dt_1}{t_1}, \dots, \frac{dt_r}{t_r}, dt_{r+1}, \dots, dt_{s+r} \right\}.$$

As a log Dieudonné algebra,  $E^*$  is equipped with

- the log structure  $\mathbb{N}^r \rightarrow \mathbb{Z}_p[t_1, \dots, t_{r+s}]$  that sends 1 in the  $k$ th position in  $\mathbb{N}^r$  to  $t_k$ ,
- the log Frobenius map  $F_L = p$  on  $\mathbb{N}^r$ ,
- the Frobenius map  $F$  that sends  $t_k \mapsto t_k^p$  and  $dt_k \mapsto t_k^{p-1} dt_k$  for each  $k$ .

To describe its saturation, we first note that  $E_{\text{sat}}^*$  is a subcomplex of

$$E^0[F^{-1}] \xrightarrow{d} E^1[F^{-1}][p^{-1}] \xrightarrow{d} \dots \xrightarrow{d} E^{r+s}[F^{-1}][p^{-1}]$$

where  $E^i[F^{-1}]$  denotes the direct limit of  $E^i$  under the Frobenius map  $F$ . More concretely,

- $E^0[F^{-1}] \cong \mathbb{Z}_p[t_1^{1/p^\infty}, \dots, t_{r+s}^{1/p^\infty}] = \bigoplus_{\alpha_1, \dots, \alpha_{r+s} \in \mathbb{N}[\frac{1}{p}]} \mathbb{Z}_p t_1^{\alpha_1} \dots t_{r+s}^{\alpha_{r+s}}$
- $E^1[F^{-1}][p^{-1}] \cong \left( \bigoplus_k \mathbb{Q}_p t_k^{\alpha_k} \frac{dt_k}{t_k} \right) \oplus \left( \bigoplus_l \mathbb{Q}_p t_l^{\alpha_l} dt_l \right)$ ,

where the first direct sum is taken over  $k \in \{1, \dots, r\}, \alpha_k \in \mathbb{N}[\frac{1}{p}]$  and the second direct sum is taken over  $l \in \{r+1, \dots, r+s\}, \alpha_l \in \mathbb{N}[\frac{1}{p}] \setminus \{0\}$ . (Also note that  $E^1[F^{-1}][p^{-1}]$  is isomorphic to  $\bigoplus_k \mathbb{Q}_p [t_k^{1/p^\infty}] \frac{dt_k}{t_k} \oplus \bigoplus_l \Omega^1_{\mathbb{Q}_p[t_l^{1/p^\infty}]/\mathbb{Q}_p}$ .)

Unwinding definitions, the saturation of  $E^*$  is given by the subcomplex of  $E^*[F^{-1}]$  consisting of ‘integral forms’, which are forms with integral  $p$ -adic coefficients and whose differentials also have integral coefficients. In other words, we have

$$E_{\text{sat}}^i = \{ \omega \in E^i[F^{-1}] : d\omega \in E^{i+1}[F^{-1}] \}.$$

The log structure on  $E_{\text{sat}}^*$  is clearly given by

$$\alpha : 1_k \in \mathbb{N}^r \mapsto t_k \in E_{\text{sat}}^0, \quad \delta : 1_k \in \mathbb{N}^r \mapsto \frac{dt_k}{t_k} \in E_{\text{sat}}^1$$

where  $1_k \in \mathbb{N}^r$  denotes the element  $(0, \dots, 1, \dots, 0) \in \mathbb{N}^r$  where 1 is the  $k$ th-position. Finally, to obtain a strict log Dieudonné algebra (with trivial log structure) we take the  $V$ -adic completion.

Next we discuss what the saturation looks like in the special cases where  $r = 0$  or where  $s = 0$  in the previous example.

**Special case:  $\mathbb{A}^1$  with no log structure.** Take  $r = 0$  and  $s = 1$  in the previous example, then we have  $E^* = (\mathbb{Z}_p[t] \xrightarrow{d} \mathbb{Z}_p[t]dt)$  with trivial log structure  $0 \rightarrow \mathbb{Z}_p[t]$  and with Frobenius  $t \mapsto t^p$ . In this case, the saturation (given by the subcomplex of ‘integral forms’) is

$$E_{\text{sat}}^0 = \left\{ x \in \mathbb{Z}_p[t^{1/p^\infty}] : dt \in \mathbb{Z}_p[t^{1/p^\infty}] \frac{dt}{t} \right\} = \left\{ \sum c_k t^{\alpha_k} \in \mathbb{Z}_p[t^{1/p^\infty}] : \text{ord}_p(c_k \alpha_k) \geq 0 \right\}$$

$$E_{\text{sat}}^1 = (\mathbb{Z}_p[t^{1/p^\infty}] \setminus \mathbb{Z}_p) \frac{dt}{t} := \bigoplus_{\alpha \in \mathbb{N}[\frac{1}{p}] \setminus \{0\}} \mathbb{Z}_p t^\alpha \frac{dt}{t}.$$

This is essentially the example discussed in [10, Section I.2].

**Special case: logarithmic  $\mathbb{A}^1$ .** For a more relevant version of the first example, we take  $r = 1$  and  $s = 0$ . In other words, we consider the log Dieudonné algebra  $E^* = (\mathbb{Z}_p[t] \xrightarrow{d} \mathbb{Z}_p[t] \frac{dt}{t})$  with log structure  $\mathbb{N} \xrightarrow{1 \mapsto t} \mathbb{Z}_p[t]$  and Frobenius  $F_L = p$ . Similar to the previous example, we have

$$E_{\text{sat}}^0 = \left\{ x \in \mathbb{Z}_p[t^{1/p^\infty}] : dt \in \mathbb{Z}_p[t^{1/p^\infty}] \frac{dt}{t} \right\}$$

$$E_{\text{sat}}^1 = \mathbb{Z}_p[t^{1/p^\infty}] \frac{dt}{t}.$$

In this example, the log structure of the saturation is given by

$$\alpha : \mathbb{N} \xrightarrow{1 \mapsto t} E_{\text{sat}}^0, \quad \delta : \mathbb{N} \xrightarrow{1 \mapsto dt/t} E_{\text{sat}}^1.$$

We further remark that  $E_{\text{sat}}^0$  consists of elements of the form  $\dots, t^{p^2}, t^p, t, pt^{1/p}, p^2t^{1/p^2}, \dots$  but not  $t^{1/p}$ . The  $V$ -filtration on  $E_{\text{sat}}^1$  agrees with the  $p$ -adic filtration, whereas on  $E_{\text{sat}}^0$  it is given by

- $V(E_{\text{sat}}^0)$  is generated by  $\left\{ pt^i, p^m t^{\frac{k}{p^m}} \mid i, k \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{\geq 1}, (k, p) = 1 \right\}$ .
- More generally,  $V^n(E_{\text{sat}}^0)$  is generated by  $\left\{ p^n t^i, p^n t^{\frac{j_1}{p}}, \dots, p^n t^{\frac{j_m}{p^m}}, p^m t^{\frac{k}{p^m}} \mid i, j_*, k \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{\geq n+1}, (j_*, p) = (k, p) = 1 \right\}$ .

**Example 3.16** (Normal crossing). Next we examine examples related to a simple normal crossing over  $\mathbb{F}_p$ . Let

$$A^* = (A^0 \xrightarrow{d} A^1) = (\mathbb{Z}_p[x, y]/(xy) \xrightarrow{d} \mathbb{Z}_p[x, y]/(xy) \frac{dx}{x})$$

be a log Dieudonné algebra, with

- differential  $d : x \mapsto dx, y \mapsto -y \frac{dx}{x}$ ,
- log structure  $\alpha : \mathbb{N}^2 \xrightarrow{(a,b) \mapsto x^a y^b} A^0$ , and log derivation  $\delta : \mathbb{N}^2 \xrightarrow{(a,b) \mapsto (a-b) \frac{dx}{x}} A^1$ .
- Frobenius given by  $p$  on  $\mathbb{N}^2$  and by  $F : x \mapsto x^p, y \mapsto y^p, \frac{dx}{x} \mapsto \frac{dx}{x}$  on  $A^*$ .

The log saturation of  $A^*$  is given by

$$A_{\text{sat}}^0 = \{\omega \in A^0[F^{-1}] : d\omega \in A^1[F^{-1}]\} \xrightarrow{d} A_{\text{sat}}^1 = A^1[F^{-1}]$$

where

- $A^0[F^{-1}] \cong \mathbb{Z}_p[x^{1/p^\infty}, y^{1/p^\infty}]/(x^{1/p^m} y^{1/p^m})_{m \geq 0} = \mathbb{Z}_p[x^{1/p^\infty}] \cup_{\mathbb{Z}_p} \mathbb{Z}_p[y^{1/p^\infty}]$ ,<sup>7</sup>
- $A^1[F^{-1}] \cong (\mathbb{Z}_p[x^{1/p^\infty}, y^{1/p^\infty}]/(x^{1/p^m} y^{1/p^m})_{m \geq 0}) \frac{dx}{x} = (\mathbb{Z}_p[x^{1/p^\infty}] \cup_{\mathbb{Z}_p} \mathbb{Z}_p[y^{1/p^\infty}]) \frac{dx}{x}$ .

<sup>7</sup>Namely, a union of two isomorphic copies of subalgebras whose intersection is  $\mathbb{Z}_p$ .

The  $V$ -filtration can be described similarly as the example above. Finally, we define the following useful submodule of  $A_{\text{sat}}^0$  consisting of all terms that only involve powers of  $x^{1/p^m}$ :

- Write  $A_{\text{sat}}^0(x) := \{\omega \in \mathbb{Z}_p[x^{1/p^\infty}] \subset A^0[F^{-1}] : d\omega \in A^1[F^{-1}]\}$ .

In particular,  $A_{\text{sat}}^0 = A_{\text{sat}}^0(x) \cup A_{\text{sat}}^0(y)$  and  $A_{\text{sat}}^0(x) \cap A_{\text{sat}}^0(y) = \mathbb{Z}_p$ .

**Example 3.17** (Normal crossing over a trivial log point). In this example, we consider the 3-term Dieudonné complex  $B^* = B^0 \xrightarrow{d} B^1 \xrightarrow{d} B^2$  given by

$$\mathbb{Z}_p[x, y]/(xy) \xrightarrow{d} \mathbb{Z}_p[x, y]/(xy) \left\langle \frac{dx}{x}, \frac{dy}{y} \right\rangle \xrightarrow{d} \mathbb{Z}_p[x, y]/(xy) \frac{dx}{x} \wedge \frac{dy}{y}$$

with log structure given by

$$\begin{aligned} \alpha : \mathbb{N}^2 &\xrightarrow{(a,b) \mapsto x^a y^b} B^0 = \mathbb{Z}_p[x, y]/(xy), \\ \delta : \mathbb{N}^2 &\xrightarrow{(a,b) \mapsto a \frac{dx}{x} + b \frac{dy}{y}} B^1 = \mathbb{Z}_p[x, y]/(xy) \frac{dx}{x} \oplus \mathbb{Z}_p[x, y]/(xy) \frac{dy}{y} \end{aligned}$$

and Frobenius given by  $p$  on  $\mathbb{N}^2$ , while sending  $x \mapsto x^p, y \mapsto y^p$  as above. Now we describe the saturation of  $B^*$  (we use notations from the previous example regarding  $A^*$ ).

$$\begin{aligned} B_{\text{sat}}^0 &= \{\omega \in B^0[F^{-1}] : d\omega \in B^1[F^{-1}]\} \cong A_{\text{sat}}^0 \\ B_{\text{sat}}^1 &= \{\omega \in B^1[F^{-1}] : d\omega \in B^2[F^{-1}]\} \\ &\cong \left( \mathbb{Z}_p[x^{1/p^\infty}] \cup_{\mathbb{Z}_p} A_{\text{sat}}^0(y) \right) \frac{dx}{x} \oplus \left( \mathbb{Z}_p[y^{1/p^\infty}] \cup_{\mathbb{Z}_p} A_{\text{sat}}^0(x) \right) \frac{dy}{y} \\ B_{\text{sat}}^2 &= B^2[F^{-1}] \cong \left( \mathbb{Z}_p[x^{1/p^\infty}] \cup_{\mathbb{Z}_p} \mathbb{Z}_p[y^{1/p^\infty}] \right) \frac{dx}{x} \wedge \frac{dy}{y}. \end{aligned}$$

#### 4. Saturated log de Rham–Witt complexes

The main goal of this article is to construct the saturated log de Rham–Witt complexes  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^*$  for a log scheme  $\underline{X}$  over  $\underline{k}$ , which will be a sheaf on  $X_{\text{ét}}$  valued in log Dieudonné algebras. In this section, we construct this object locally on charts (in particular, the construction in this section *a priori* takes as input a pre-log affine scheme), and compare it with de Rham complexes in the subsequent subsections. In Section 6 we glue the local construction on the étale site of  $X$ .

For this section, we fix the base log algebra  $\underline{W} = W(\underline{k})$  where  $\underline{k} = (k, N)$  as before.<sup>8</sup> In Subsection 4.1, we do not impose any condition on the log structures, but later on we gradually impose restrictions on the log algebra to prove various forms of comparison theorems.

<sup>8</sup>This is less general than the previous section, namely, we require that the log structure on  $W(k)$  comes from the Teichmüller lift from log structure on  $k$ . Examples include the trivial log structure on  $W(k)$ , or  $\mathbb{N} \xrightarrow{1 \mapsto 0} W(k)$ , but excludes  $\mathbb{N} \xrightarrow{1 \mapsto p} W(k)$ .

**4.1. Saturated log de Rham–Witt complexes for log algebras**

Let  $\text{Alg}_k^{\log}$  be the category of log algebras over  $k$ . An element  $\underline{R} = (R, M) \in \text{Alg}_k^{\log}$  can be equipped with a pair of Frobenius  $(F, F_M)$  where  $F(x) = x^p$  for all  $x \in R$  and  $F_M(m) = pm$  for all  $m \in M$ . From the previous section, we have a functor from  $p$ -compatible strict log Dieudonné algebras over  $\underline{W}$  to log algebras over  $k$ :

$$\text{DA}_{\text{str}}^{\log,p} \longrightarrow \text{Alg}_k^{\log} \quad \text{given by } A^* \mapsto \underline{A}^0/V(A^0),$$

where  $\underline{A}^0/V(A^0)$  consists of the data  $(A^0/V(A^0), \alpha : L \rightarrow A^0 \twoheadrightarrow A^0/V(A^0))$ . Now we are ready to give the definition of saturated log de Rham–Witt complexes for log algebras (Definition 1 in the Introduction).

**Definition 4.1.** The saturated log de Rham–Witt complex  $\mathcal{W}\omega_{-/W}^*$  is a functor

$$\mathcal{W}\omega_{-/W}^* : \text{Alg}_k^{\log} \longrightarrow \text{DA}_{\text{str}/W}^{\log,p}$$

that is the left adjoint of the functor  $\text{DA}_{\text{str}/W}^{\log,p} \longrightarrow \text{Alg}_k^{\log}$  sending  $A^* \mapsto (A^0/V(A^0), L)$ .

In other words, the saturated log de Rham–Witt complex of a log algebra  $\underline{R}$  over  $k$  is a strict  $p$ -compatible log Dieudonné  $W(k)$ -algebra  $\mathcal{W}\omega_{\underline{R}/k}^*$ , equipped with an isomorphism

$$e : \underline{R} \xrightarrow{\sim} \mathcal{W}_1\omega_{\underline{R}/k}^0$$

such that for any  $A^* \in \text{DA}_{\text{str}}^{\log,p}$ , we have a natural identification

$$\text{Hom}_{\text{DA}_{\text{str}}^{\log}}(\mathcal{W}\omega_{\underline{R}/k}^*, A^*) \xrightarrow{\sim} \text{Hom}_{\text{Alg}_k^{\log}}(\underline{R}, \underline{A}^0/V(A^0)).$$

The main goal of this section is to show the existence of this left adjoint  $\mathcal{W}\omega_{-/W}^*$  by construction.

**Remark 4.2.** The restriction to  $p$ -compatible objects is necessary to obtain the desired universal property. In particular, we do not have a left adjoint of the functor  $\text{DA}_{\text{str}}^{\log} \rightarrow \text{Alg}_k^{\log}$  from the larger category  $\text{DA}_{\text{str}}^{\log}$ .

**4.2. The construction**

**4.2.1. Preliminaries I.**

**Notation 4.3.** Let  $\underline{A} = (A, L_A)$  and  $\underline{B} = (B, L_B)$  be two log  $W$ -algebras equipped with Frobenius maps  $(\varphi_A, F_{L_A})$  and  $(\varphi_B, F_{L_B})$ , where  $\varphi_A$  (respectively  $\varphi_B$ ) lifts the  $p$ -power Frobenius on  $A/p$  (respectively  $B/p$ ). We denote by  $\text{Hom}_F(\underline{A}, \underline{B})$  the set of Frobenius-compatible  $W$ -morphisms between  $\underline{A}$  and  $\underline{B}$ , which are pairs  $(f, \psi)$  of morphisms making every square in the cube below commute:

$$\begin{array}{ccccc}
 & & L_A & \xrightarrow{\psi} & L_B \\
 & F_{L_A} \nearrow & \downarrow \psi & & \nearrow F_{L_B} \\
 L_A & \xrightarrow{\psi} & L_B & & \\
 \downarrow \varphi_A \nearrow & & \downarrow \varphi_B & & \\
 A & \xrightarrow{f} & B & & 
 \end{array}$$



The following setup is of particular interest. Let  $\underline{R}, \underline{R}' \in \text{Alg}_k^{\text{log}}$  be two log algebras over  $k$ . Let  $W(\underline{R}) = (W(R), [\alpha])$ , where  $[\alpha] = [\ ] \circ \alpha : M \rightarrow R \rightarrow W(R)$  is the Teichmüller lift of the log structure on  $R$ .  $W(\underline{R})$  is equipped with the Frobenius map  $(F, p)$  where  $F$  is the Witt vector Frobenius. Similarly define  $W(\underline{R}')$ . Then  $\text{Hom}_F(W(\underline{R}), W(\underline{R}'))$  consists precisely of the homomorphisms  $(f, \psi) : W(\underline{R}) \rightarrow W(\underline{R}')$  of log  $\underline{W}$ -algebras, such that  $f \circ F = F \circ f$ .

**Lemma 4.4.** *Let  $\underline{R}, \underline{R}' \in \text{Alg}_k^{\text{log}}$  be as above. Suppose that both  $R$  and  $R'$  are reduced; then*

$$\text{Hom}_F(W(\underline{R}), W(\underline{R}')) \cong \text{Hom}_{\text{Alg}_k^{\text{log}}}(\underline{R}, \underline{R}').$$

**Proof.** Let  $(\bar{f}, \psi) : \underline{R} \rightarrow \underline{R}'$  be a map over  $k$ . Because  $W(R)$  is  $p$ -torsion free (as  $R$  is reduced) and  $F : W(R) \rightarrow W(R)$  lifts the Frobenius on  $R$ , the map  $W(R) \rightarrow R \xrightarrow{\bar{f}} R'$  lifts uniquely<sup>9</sup> to a map  $f : W(R) \rightarrow W(R')$  that is compatible with Frobenius on both  $W(R)$  and  $W(R')$ . It remains to show the compatibility between  $f$  and  $\psi$ , but this is implied by the compatibility between  $f$  and taking Teichmüller lifts: namely, that  $f([x]) = [\bar{f}(x)]$  for every  $x \in R$ . To prove this claim, it suffices to show that  $f([x]) - [\bar{f}(x)] \in \text{Fil}^s = V^s W(R')$  for every  $s \in \mathbb{N}$ . This can be done by a simple induction on  $s$ , and we leave the details to the reader. □

**4.2.2. A key example of log Dieudonné algebra.** The construction of log de Rham–Witt complexes for  $\underline{R} \in \text{Alg}_k^{\text{log}}$  relies on the following proposition.

**Proposition 4.5.** *Let  $(A, \alpha : L \rightarrow A)$  be a log algebra over  $\underline{W}$ . Suppose that  $A$  is  $p$ -torsion free and that the pair  $(A, L)$  is equipped with morphisms  $\varphi : A \rightarrow A$  and  $F_L : L \rightarrow L$  such that*

- $\varphi(x) \equiv x^p \pmod p$  is a lifting of the absolute Frobenius mod  $p$ ;
- $F_L = p$  on  $L$  and is compatible with  $\varphi$ , that is,  $\varphi \circ \alpha = \alpha \circ F_L$ .

*Then there exists a unique graded ring homomorphism  $F : \omega_{\underline{A}/\underline{W}}^* \rightarrow \omega_{\underline{A}/\underline{W}}^*$  that extends  $\varphi$  on  $A$ , such that*

$$F(dx) = x^{p-1} dx + d\left(\frac{\varphi(x) - x^p}{p}\right) \quad \text{for all } x \in A.$$

*Moreover, this makes  $\underline{\omega}^* := (\omega_{\underline{A}/W(k)}^*, L, d, \delta, F, F_L)$  a  $p$ -compatible log Dieudonné  $\underline{W}$ -algebra.*

**Proof.** For the first claim we need to show the existence of  $F : \omega_{\underline{A}/W(k)}^1 \rightarrow \omega_{\underline{A}/W(k)}^1$ . We regard the second copy of  $\omega_{\underline{A}/W(k)}^1$  as an  $A$ -module where  $A$  acts by a twist of  $\varphi$ , namely,  $a \cdot \lambda = \varphi(a)\lambda$  for all  $\lambda \in \omega_{\underline{A}/W(k)}^1$ . By the universal property of  $\omega_{\underline{A}/W(k)}^1$  and Remark A.5, it suffices to supply the structure of a log derivation from  $(A, L)$  to  $\omega_{\underline{A}/W(k)}^1$  with this

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<sup>9</sup>This is the Cartier’s Dieudonné–Dwork lemma. Another way to produce this lifting is to observe that the Witt vectors provide a right adjoint to the forgetful functor from the category of  $\delta$ -rings to rings, in the sense of [12].

twisted  $A$ -module structure. For this we define

$$\tilde{\delta} : L \rightarrow \omega_{\underline{A}/W(\underline{k})}^1, \quad \text{and} \quad \tilde{d} : A \rightarrow \omega_{\underline{A}/W(\underline{k})}^1$$

respectively as follows:

- we set  $\tilde{\delta}(l) := \delta(l)$  and
- $\tilde{d}(x) := x^{p-1}dx + d\theta(x)$  where  $\theta(x) = \frac{\varphi(x) - x^p}{p}$ .

To proceed we need to show that  $\tilde{\delta}$  and  $\tilde{d}$  form a log derivation of  $\underline{A}$  over  $W(\underline{k})$ , which requires

- (1)  $\tilde{d}(\gamma x) = \gamma \cdot \tilde{d}x := F(\gamma)\tilde{d}x$  for all  $\gamma \in W(k), x \in A$ ;
- (2)  $\tilde{d}(x + y) = \tilde{d}(x) + \tilde{d}(y)$  for all  $x, y \in A$ ;
- (3)  $\tilde{d}(xy) = x \cdot \tilde{d}y + y \cdot \tilde{d}x := \varphi(x)\tilde{d}y + \varphi(y)\tilde{d}x$  for all  $x, y \in A$ ;
- (4)  $\alpha(l) \cdot \tilde{\delta}(l) := \varphi(\alpha(l))\tilde{\delta}(l) = \tilde{d}(\alpha(l))$  for all  $l \in L$ ;
- (5)  $\tilde{\delta}(\iota(n)) = 0$  for all  $n \in N$ , where  $\iota : N \rightarrow L$  is map of monoids in  $W(\underline{k}) \rightarrow \underline{A}$ .

(1) Follows from (3) because  $p\tilde{d}(\lambda) = d(F(\lambda)) = 0$ , where  $F : W(k) \rightarrow W(k)$  is the Frobenius map on Witt vectors. (2) and (3) follow from the explicit description of  $\theta(x)$ . Now we check (4). Because  $\theta(\alpha(l)) = 0$ , we have

$$\varphi(\alpha(l))\tilde{\delta}(l) = \alpha(l)^p\delta(l) = \alpha(l)^{p-1}d(\alpha(l)) = \tilde{d}(\alpha(l)).$$

(5) Is clear because  $\delta(\iota(n)) = 0$  for all  $n \in N$ . We thus get the desired map  $F : \omega_{\underline{A}/W(\underline{k})}^1 \rightarrow \omega_{\underline{A}/W(\underline{k})}^1$  that satisfies  $F(\delta(l)) = \tilde{\delta}(l) = \delta(l)$  and  $Fd(x) = \tilde{d}(x) = x^{p-1}dx + d\theta(x)$ .

To finish the proof of the proposition, we need to check that (i)  $dF = pFd$  on all of  $\omega_{\underline{A}/W(\underline{k})}^*$  and (ii)  $\delta F_L = pF\delta$  on  $L$ . For (i), note that  $\omega_{\underline{A}/W(\underline{k})}^*$  is generated over  $A$  by  $\omega_{\underline{A}/W(\underline{k})}^1$ , and it is clear that we only need to check the relation on  $x, dx$  for all  $x \in A$  and on  $\delta(l)$  for all  $l \in L$ . The fact that  $dF(x) = pFd(x)$  follows directly from the construction of  $F$  on  $\omega_{\underline{A}/W(\underline{k})}^1$ , because  $Fd = \tilde{d}$  by construction; on  $dx$  and  $\delta(l)$ , both  $dF$  and  $pFd$  evaluate to 0 (because  $d\delta = 0$  on  $L$ ). Part (ii) is automatic by construction of  $F$  again, because  $F\delta = \delta$ . □

We record a variant of the proposition above, which allows slightly more flexible Frobenius  $F_L$  on the monoid  $L$ .

**Proposition 4.6.** *Let  $(A, \alpha : L \rightarrow A)$  be a log algebra over  $\underline{W}$ . Suppose that  $A$  is  $p$ -torsion free and  $p$ -separated and that  $\omega_{\underline{A}/\underline{W}}^1$  is  $p$ -torsion free. Also suppose that the pair  $(A, L)$  is equipped with Frobenius morphisms  $\varphi : A \rightarrow A$  and  $F_L : L \rightarrow L$  such that*

- $\varphi(x) \equiv x^p \pmod{p}$  is a lifting of the absolute Frobenius mod  $p$ ;
- $F_L$  is compatible with  $\varphi$ .

*Then the same conclusions of Proposition 4.5 hold except for  $p$ -compatibility.*

**Proof.** First we need the following sublemma.

**Sublemma 4.7.** Retain the notation and assumption of the proposition. For any  $k \in \mathbb{N}$ , if  $p^k | \varphi(x)$ , then  $p^k | (x^{d-1} dx + d\theta(x))$ .

**Proof of the sublemma.** The case  $k = 0$  is tautological. We proceed by induction. Suppose that  $k \geq 1$  and that the sublemma has been verified up to  $k - 1$ . Now let  $\varphi(x) = p^k y = x^p + p\theta(x)$ ; because  $p | \varphi(x)$ , we know that  $p | x$ , so  $x = pz$  for some  $z \in A$ . Because  $A$  is  $p$ -torsion free,  $\varphi(x) = p\varphi(z) = p^k y$  implies that  $\varphi(z) = p^{k-1} y$ , so  $p^{k-1} | (z^{p-1} dz + d\theta(z))$  by induction hypothesis. Note that  $x^{p-1} dx + d\theta(x) = (pz)^{p-1} d(pz) + d\theta(pz) = p(z^{p-1} dz + d\theta(z))$ , so the lemma follows.

Now we proceed as in the proof of Proposition 4.5 but replace  $\tilde{\delta}$  by

$$\tilde{\delta}(l) := \delta(F_L(l))/p.$$

We need to show that  $p | \delta(F_L(l))$  for any  $l \in L$  for  $\tilde{\delta}$  to be well defined. For this we need the assumption that  $A$  is  $p$ -separated: write  $x = \alpha(l)$ ; then there exists some  $k \in \mathbb{N}$  such that  $p^k | \varphi(x)$  but  $p^{k+1} \nmid \varphi(x)$ . By the sublemma above we know that  $p^{k+1} | d(\varphi(x))$ . Now because

$$\varphi(x) \delta(F_L(l)) = \alpha(F_L(l)) \delta(F_L(l)) = d\alpha(F_L(l)) = d\varphi(x)$$

and  $p^{k+1} \nmid \varphi(x)$ ,  $p^{k+1} | d\varphi(x)$ , we know that  $p | \delta(F_L(l))$  (here we use the assumption that  $\omega_{A/W(k)}^1$  is  $p$ -torsion free). The rest of the proof of part (1) is the same, except that to check  $\alpha(l) \cdot \tilde{\delta}(l) = \tilde{d}(\alpha(l))$ , we need to use

$$\begin{aligned} p\tilde{d}(\alpha(l)) &= d\varphi(\alpha(l)) \\ &= \alpha(F_L(l))\delta(F_L(l)) \\ &= p\varphi(\alpha(l))\tilde{\delta}(l) = p\alpha(l) \cdot \tilde{\delta}(l). \end{aligned}$$

The rest of the proof is identical to that of Proposition 4.5. □

**4.2.3. An interlude: Frobenius on finite level.** Proposition 4.5 will be applied to  $A = W(R)$  where  $R$  is a reduced  $\mathbb{F}_p$ -algebra. For applications later, we also record the following remark, which the reader may skip in first reading.

**Remark 4.8.** The graded ring homomorphism  $F$  can be constructed on the finite-length Witt vectors; namely, there is a natural graded ring homomorphism

$$F : \omega_{W_n(\underline{R})/W_n(\underline{k})}^* \rightarrow \omega_{W_{n-1}(\underline{R})/W_{n-1}(\underline{k})}^*$$

extending the Frobenius  $F : W_n(R) \rightarrow W_{n-1}(R)$ . We prove the claim using the same strategy as Proposition 4.5, by constructing a log derivation  $(\tilde{d}, \tilde{\delta})$  of  $W_n(\underline{R})/W_n(\underline{k})$  into  $F_*\omega_{W_{n-1}(\underline{R})}^1$ , where  $\omega_{W_{n-1}(\underline{R})}^1$  is regarded as a  $W_n(R)$  structure via  $F : W_n(R) \rightarrow W_{n-1}(R)$ . We define  $\tilde{d}$  as follows: write  $x \in W_n(R)$  uniquely as  $x = [x_0] + Vx'$  with  $x' \in W_{n-1}(R)$  and  $x_0 \in R$  and then define

$$\tilde{d}x := [x_0]^{p-1} d[x_0] + dx'$$

where  $[x_0]$  now lives in  $W_{n-1}(R)$ . We then define  $\tilde{\delta}(m) := \delta(m)$ .

• Let us first check that  $\tilde{d}(x + y) = \tilde{d}(x) + \tilde{d}(y)$ , for which it suffices to show that if  $[x_0] + [y_0] = [x_0 + y_0] + Vz'$ , then

$$[x_0]^{p-1}d[x_0] + [y_0]^{p-1}d[y_0] = [x_0 + y_0]^{p-1}d[x_0 + y_0] + dz'.$$

We check this equality in  $\omega_{W(\underline{R})/W(\underline{k})}^1$ . For any element  $z = [x] + Vy \in W(R)$ , we claim that

$$[x]^{p-1}d[x] + dy = z^{p-1}dz + d\left(\frac{F(z) - z^p}{p}\right).$$

This is straightforward by expanding the terms on the right-hand side:

$$([x] + Vy)^{p-1}d([x] + Vy) + d\left(\frac{[x]^p + py - ([x] + Vy)^p}{p}\right).$$

Now the desired additivity follows from the proof of Proposition 4.5.

• Next we check that

$$\begin{aligned} \tilde{d}(xy) &= ([x_0]^p + px')([y_0]^{p-1}d[y_0] + dy') + ([y_0]^p + py')([x_0]^{p-1}d[x_0] + dx') \\ &= F(x)\tilde{d}y + F(y)\tilde{d}x = x \cdot \tilde{d}y + y \cdot \tilde{d}x. \end{aligned}$$

Because  $(\tilde{d}, \tilde{\delta})$  forms a log  $W_n(\underline{k})$ -derivation from  $W_n(\underline{R})$  into  $\omega_{W_{n-1}(\underline{R})/W_{n-1}(\underline{k})}^1$ , we have the desired map  $F : \omega_{W_n(\underline{R})/W_n(\underline{k})}^1 \rightarrow \omega_{W_{n-1}(\underline{R})/W_{n-1}(\underline{k})}^1$ . Finally, this extends to a graded ring homomorphism on  $\omega_{W_n(\underline{R})/W_n(\underline{k})}^*$ .

**4.2.4. Preliminaries II.** Now the final ingredient for the construction of the saturated log de Rham–Witt complexes.

**Lemma 4.9.** *Let  $\underline{A}$  be a log algebra over  $\underline{W} = W(\underline{k})$  satisfying the conditions in either Proposition 4.5 or 4.6 and  $\underline{\omega}^* = \underline{\omega}_{\underline{A}/\underline{W}}^*$  be the corresponding log Dieudonné  $\underline{W}$ -algebra constructed there. Let  $B^*$  be a  $p$ -torsion-free log Dieudonné  $\underline{W}$ -algebra. There is a canonical bijection between*

$$\text{Hom}_{DA^{\text{log}}}(\underline{\omega}^*, B^*) \cong \text{Hom}_F(\underline{A}, B^0),$$

where  $\text{Hom}_F$  is as defined in Notation 4.3.

**Proof.** Given a morphism  $(f, \psi) \in \text{Hom}_F(\underline{A}, B^0)$ , we need to extend it to a morphism  $f : \underline{\omega}^* \rightarrow B^*$  of log Dieudonné  $\underline{W}$ -algebras. As in the proof of Proposition 4.5, we first construct a map  $f : \omega_{\underline{A}/\underline{W}}^1 \rightarrow B^1$  that fits into the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{d} & \omega_{\underline{A}/W(\underline{k})}^1 \\ \downarrow f & & \downarrow f \\ B^0 & \xrightarrow{d_B} & B^1 \end{array}$$

Define  $d_f : A \rightarrow B^1$  by  $d_f = d_B \circ f$  and  $\delta_f : L \rightarrow B^1$  by  $\delta_f = \delta_B \circ \psi$ . One easily verifies that  $d_f$  is a derivation of  $A/W(k)$  into  $B^1$ . To show that  $(d_f, \delta_f)$  is a log derivation of  $\underline{A}/\underline{W}$ , we check that  $\delta_f(\iota(n)) = 0$  and that

$$\begin{aligned} \alpha(l) \cdot \delta_f(l) &= \alpha_B(\psi(l)) \delta_B(\psi(l)) \\ &= d_B \alpha_B(\psi(l)) = d_B f(\alpha(l)) = \delta_f(\alpha(l)). \end{aligned}$$

Therefore, we have the dotted map  $f : \omega^1_{A/W(k)} \rightarrow B^1$  in the diagram above.

Next we extend  $f$  to  $\omega^*_{A/W}$  as a differential graded morphism (because  $\omega^*_{A/W}$  is generated over  $A$  by  $\omega^1_{A/W}$ ). Note that  $f$  is compatible with  $\delta$  by construction, so it remains to show its compatibility with  $F$ . For this we proceed as the proof of Proposition 4.5; namely, we check it on  $x, dx$  for  $x \in A$  and  $\delta(l)$  for  $l \in L$  because both  $F$  and  $f$  are algebra morphisms. We know that  $Ff(x) = fF(x)$  by assumption. For  $dx$  (where  $x \in A^0$ ), we have

$$\begin{aligned} pfF(dx) &= fdF(x) = d(fF(x)) \\ &= dF(f(x)) = pF(df(x)) = pFf(dx), \end{aligned}$$

and because  $B^*$  is  $p$ -torsion free, we get  $fF(dx) = Ff(dx)$ . The proof for  $\delta(l)$  is similar. □

**4.2.5. The construction.** Now we prove our first main result (Theorem 2 in the Introduction). We continue to assume that  $\underline{k}$  has the form  $(k, N)$  where  $N \setminus \{0\} \mapsto 0 \in k$ .

**Theorem 4.10.** *Let  $\underline{R} = (R, M) \in \text{Alg}_k^{\text{log}}$  be a log algebra over  $\underline{k}$ . The saturated log de Rham–Witt complex  $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$  of  $\underline{R}/\underline{k}$  exists. Moreover, the association of  $\underline{R} \mapsto \mathcal{W}\omega^*_{\underline{R}/\underline{k}}$  is functorial.*

**Proof.** Without loss of generality, assume that  $R$  is reduced by replacing it with its reduced quotient if necessary (for an object  $B^* \in \text{DA}_{\text{str}}^{\text{log}, p}$ ,  $B^0/V(B^0)$  is reduced by Remark 2.8). Let  $W(\underline{R}) = (W(R), L = M \xrightarrow{[\alpha]} W(R))$  be the Witt vector of  $\underline{R}$ . Because  $R$  is reduced,  $W(R)$  is  $p$ -torsion free. By Proposition 4.5, there exists a unique log Dieudonné  $\underline{W}$ -algebra on the relative log de Rham complexes:

$$\omega^* = (\omega^*_{W(\underline{R})/W(\underline{k})}, L, d, \delta, F, F_L)$$

where  $L = M, F_L = p$  and  $F(dx) = x^{p-1}dx + d(\frac{\varphi(x) - x^p}{p})$ . Define

$$\mathcal{W}\omega^*_{\underline{R}/\underline{k}} = W(\omega^*)_{\text{sat}}.$$

The construction is clearly functorial in  $\underline{R}$ . We claim that  $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$  is the log de Rham–Witt complex of  $\underline{R}/\underline{k}$ . To prove the claim, consider any strict  $p$ -compatible log Dieudonné  $W(\underline{k})$ -algebra  $B^* \in \text{DA}_{\text{str}}^{\text{log}, p}$ ; then we have

$$\begin{aligned} \text{Hom}_{\text{DA}_{\text{str}}^{\text{log}, p}}(\mathcal{W}\omega^*_{\underline{R}/\underline{k}}, B^*) &= \text{Hom}_{\text{DA}^{\text{log}, p}}(\omega^*, B^*) && \text{by Lemma 3.12} \\ &= \text{Hom}_F(W(\underline{R}), B^0) && \text{by Lemma 4.9} \\ &= \text{Hom}_F(W(\underline{R}), W(B^0/V B^0)) && \text{by Corollary 3.14} \\ &= \text{Hom}_{\text{Alg}_k^{\text{log}}}(\underline{R}, B^0/V B^0) && \text{by Lemma 4.4.} \end{aligned}$$

□

**Notation.** We denote  $\mathcal{W}_n \omega_{\underline{R}/\underline{k}}^* := W_n((\omega^*)_{\text{sat}})$ .

**Remark 4.11.** For each  $n \geq 1$ , write  $\omega_n^* := \omega_{W_n(\underline{R})/W_n(\underline{k})}^*$ ; then in the construction above we may use  $\varprojlim \omega_n^*$  instead of  $\omega_{W(\underline{R})/W(\underline{k})}^*$ . In other words, consider  $\varprojlim \omega_n^*$  as a  $p$ -compatible log Dieudonné algebra (see, for example, Remark 4.8); then we may define  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  as  $W(\varprojlim \omega_n^*)_{\text{sat}}$ . It suffices to show that the canonical map  $\omega_{W(\underline{R})/W(\underline{k})}^* \rightarrow \varprojlim \omega_n^*$  induces a bijection<sup>10</sup>

$$\text{Hom}_{\text{DA}^{\log, p}}(\omega^*, B^*) \xrightarrow{\sim} \text{Hom}_{\text{DA}^{\log, p}}(\varprojlim \omega_n^*, B^*).$$

For this, we observe that any map  $\omega^* \rightarrow B^*$  of log Dieudonné algebras induces a map  $\omega_n^* \rightarrow W_n(B^*)$ , which is compatible with transition maps as well as Frobenius on both sides, hence giving rise to  $\varprojlim \omega_n^* \rightarrow B^*$  on the inverse limit. Note that we have  $\omega_n^0 = W_n(R) \rightarrow W_n B^0 = B^0/V^n(B^0)$ . Similar to the proof of Lemma 4.9, we claim that the map  $\omega_{W(\underline{R})/W(\underline{k})}^1 \rightarrow B^1$  factors through  $\omega_n^1 \rightarrow W_n B^1$ . This follows from the fact that kernel of  $\omega_{W(\underline{R})/W(\underline{k})}^1 \rightarrow \omega_n^1$  is generated by  $\text{im}(V^n)$  and  $\text{im}(dV^n)$ .

**4.3. Choice of charts of the log structure**

We may have different log structures on a  $k$ -algebra  $R$  that give rise to the same saturated affine log scheme  $(\text{Spec } R, \mathcal{M}^a)$ , which amounts to different choices of charts for  $\mathcal{M}^a$ . This choice is almost irrelevant in forming the saturated log de Rham–Witt complexes, because the underlying Dieudonné algebras will be the same. More precisely, let  $(R, \alpha : M \rightarrow R)$  be a log algebra over  $\underline{k}$  and  $\mathcal{M}^a$  be the log structure associated to the constant pre-log structure  $M$  on  $\text{Spec } R$  (see Conventions in the Introduction 1.6). Let  $M^{\text{sh}} = \Gamma(\text{Spec } R, \mathcal{M}^a)$  and denote the log algebra  $(R, M^{\text{sh}})$  by  $\underline{R}^{\text{sh}}$ . We then have the following lemma.

**Lemma 4.12.** *The induced map*

$$\mathcal{W}\omega_{\underline{R}/\underline{k}}^* \rightarrow \mathcal{W}\omega_{\underline{R}^{\text{sh}}/\underline{k}}^*$$

*of the saturated log de Rham–Witt complexes from  $\underline{R} \rightarrow \underline{R}^{\text{sh}}$  is an isomorphism on the underlying Dieudonné algebras. In particular, if  $(R, M)$  and  $(R, M')$  are two charts for the affine log scheme  $(\text{Spec } R, \mathcal{M}^a)$ , then the underlying Dieudonné algebra of their saturated log de Rham–Witt complexes is canonically isomorphic.*

**Proof.** It suffices to show that for each  $n \geq 1$ , the pre-log structures  $\beta : M \rightarrow W_n(R)$  and  $\beta^{\text{sh}} : M^{\text{sh}} \rightarrow W_n(R)$  induce isomorphic log structures over  $\text{Spec } W_n(R)$ ; thus, we have

$$\omega_{(W_n(R), M)/W_n(\underline{k})}^* \xrightarrow{\sim} \omega_{(W_n(R), M^{\text{sh}})/W_n(\underline{k})}^*.$$

The proposition then follows from Remark 4.11. Now write  $X = \text{Spec } R$  and  $Y = \text{Spec } W_n(R)$ . Let  $\mathcal{O}_X$  (respectively  $\mathcal{O}_Y$ ) be the structure sheaf of  $X_{\text{ét}}$  (respectively  $Y_{\text{ét}}$ ) and

<sup>10</sup>This is slightly different from the setup in Lemma 5.3, where  $\widehat{\omega}^*$  there denotes the  $p$ -adic completion, whereas our limit here is the ‘V-completion’.

$W_n(\mathcal{O}_X)$  be the sheaf sending  $U \mapsto W_n(\Gamma(U, \mathcal{O}_X))$ . We need to show that the following two pushouts

$$\begin{array}{ccc} \beta^{-1}(\mathcal{O}_Y^\times) & \longrightarrow & \mathcal{O}_Y^\times \\ \downarrow & & \downarrow \\ M & & M^{\text{sh}} \end{array} \qquad \begin{array}{ccc} (\beta^{\text{sh}})^{-1}(\mathcal{O}_Y^\times) & \longrightarrow & \mathcal{O}_Y^\times \\ \downarrow & & \downarrow \\ M^{\text{sh}} & & M^{\text{sh}} \end{array}$$

of étale sheaves of monoids over  $Y_{\text{ét}}$  are isomorphic. By topological invariance of étale sites, we may compute the pushout in  $X_{\text{ét}}^\sim$ . Over  $X$ , the morphism  $\beta$  corresponds to  $M \xrightarrow{\alpha} \mathcal{O}_X \xrightarrow{\gamma} W_n(\mathcal{O}_X)$ , where  $\gamma$  is given by Teichmüller liftings. Note that  $\gamma^{-1}(W_n(\mathcal{O}_X)^\times) = \mathcal{O}_X^\times$  because for an algebra  $R$  we have  $W_n(R)^\times \cap [R] = [R^\times]$ , so we have  $\beta^{-1}(W_n(\mathcal{O}_X)^\times) = \alpha^{-1}(\mathcal{O}_X^\times)$ . Thus, the first pushout can be computed via two steps: first take the pushout of  $\alpha^{-1}(\mathcal{O}_X^\times) \rightarrow M$  along  $\alpha^{-1}(\mathcal{O}_X^\times) \rightarrow \mathcal{O}_X^\times$  and then along  $\mathcal{O}_X^\times \rightarrow W_n(\mathcal{O}_X)^\times$ . The first step precisely computes  $\mathcal{M}^a$ , which is isomorphic to  $(\mathcal{M}^{\text{sh}})^a$  by Lemma A.1; therefore,  $M$  and  $M^{\text{sh}}$  indeed induce the same log structures on  $Y = \text{Spec } W_n(R)$ .  $\square$

**Remark 4.13.** The proof we give of Lemma 4.12 is somewhat indirect (for example, it involves Remark 4.11 and it seems more involved to directly show that  $M$  and  $M^{\text{sh}}$  give rise to the same log structures over  $\text{Spec } W(R)$ ). However, under the assumption that  $\underline{R}/\underline{k}$  is log-smooth of log-Cartier type, the proposition immediately follows from Corollary 2.16 and Theorem 5.8, because we have an isomorphism  $\omega_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} \omega_{\underline{R}^{\text{sh}}/\underline{k}}^*$  by Lemma A.1.

### 5. Log Frobenius liftings and de Rham comparison

In this section we discuss the saturated log de Rham–Witt complexes for log algebras that admit liftings to  $W(k)$  together with lifts of Frobenius. This is useful for at least two reasons: firstly it provides a way to explicitly compute log de Rham–Witt complexes (étale locally); secondly it allows us to compare the saturated log de Rham–Witt complex to the log de Rham complex when the log algebra  $\underline{R}$  is sufficiently log smooth.

#### 5.1. Log Frobenius liftings of log algebras

We record a (variant of a) definition from [14].

**Definition 5.1.** Let  $(f, \psi) : \text{Spec } \underline{R} = (\text{Spec } R, M) \rightarrow \text{Spec } \underline{k}$  be a morphism of affine pre-log schemes with integral pre-log structures.

- (1) A Witt lifting of  $f$  consists of a system  $(\underline{A}_n = (A_n, M_n), \delta_n : W_n(\text{Spec } \underline{R}) \rightarrow \text{Spec } \underline{A}_n)_{n \geq 1}$ , where  $(A_1, M_1) = (R, M)$ , and satisfies the following list of conditions:
  - $\text{Spec } \underline{A}_n = (\text{Spec } A_n, M_n)$  is log-smooth over  $W_n(\text{Spec } \underline{k})$ .

- For each  $n$ , the following diagram is Cartesian:

$$\begin{array}{ccc}
 \text{Spec } \underline{A}_n & \xleftarrow{R} & \text{Spec } \underline{A}_{n+1} \\
 \downarrow & \square & \downarrow \\
 W_n(\text{Spec } \underline{k}) & \xrightarrow{R} & W_{n+1}(\text{Spec } \underline{k})
 \end{array}$$

where  $R: A_{n+1} \rightarrow A_n$  is the transition map of the inverse system.

- The maps  $\delta_n$  are compatible with the system  $(\text{Spec } \underline{A}_n)$  under the transition maps.
- (2) A log Frobenius lifting of  $f$  is a projective system  $(\underline{A}_n, \delta_n, \varphi_n)_{n \geq 1}$  where  $(\underline{A}_n, \delta_n)$  is a Witt lifting of  $f$  and  $\varphi_n: \text{Spec } \underline{A}_n \rightarrow \text{Spec } \underline{A}_{n+1}$  is a morphism of pre-log schemes such that
- the maps  $\varphi_n$  are compatible with the Frobenius on  $\text{Spec } \underline{R}$  given by the  $p$ th-power Frobenius on  $R$  and multiplication by  $p$  on  $M$ .
  - $\varphi_n$  are compatible with the Frobenius of Witt vectors on both  $\text{Spec } \underline{k}$  and  $\text{Spec } \underline{R}$ .
- (3) A log Frobenius lifting  $(\underline{A}_n = (A_n, M_n), \delta_n, \varphi_n)$  of  $f$  is called  $p$ -compatible if the log-restriction map  $R: M_{n+1} \rightarrow M_n$  is identity for all  $n \geq 1$  (so, in particular,  $M_n = M$  for all  $n$ ) and the log Frobenius map  $\varphi_n: M \rightarrow M$  is multiplication by  $p$ .

The requirements on  $\varphi_n$  in the definition of log Frobenius liftings can be summarised in the following commutative diagrams:

$$\begin{array}{ccccc}
 & & \text{Spec } \underline{A}_{n+1} & \xrightarrow{R} & W_{n+1}(\text{Spec } \underline{R}) \\
 & \nearrow \varphi_n & \downarrow & & \nearrow F_n \\
 \text{Spec } \underline{A}_n & \xrightarrow{R} & W_n(\text{Spec } \underline{R}) & & \\
 \downarrow & & \downarrow & & \\
 & \nearrow F_n & W_{n+1}(\text{Spec } \underline{k}) & & \\
 & & \downarrow & & \\
 & & W_n(\text{Spec } \underline{k}) & & 
 \end{array}$$

Also note that, though we stated the definitions on log algebras following [14], it is clear that one can globalise and extend the definitions to pre-log schemes in general.

**Lemma 5.2.** *Suppose that  $\underline{R}$  is log-smooth over  $\underline{k}$ , then there exists a  $p$ -compatible log Frobenius lifting  $(\underline{A}_n, \delta_n, \varphi_n)$  of  $\underline{R}$ . More generally, let  $f: \underline{X} \rightarrow \text{Spec } \underline{k}$  be a log-smooth morphism of fine (pre-)log schemes, then a  $p$ -compatible log Frobenius liftings exist étale locally.*

**Proof.** This follows from Proposition 3.2 in [14] and Lemma 5.5 of [15]. The morphism  $(k, N) \rightarrow (R, M)$  factors through the morphism  $(k, N) \rightarrow (k \otimes_{\mathbb{Z}[N]} \mathbb{Z}[M], M) \rightarrow (R, M)$ , where the second map is étale on the underlying rings. The first arrow admits a  $p$ -compatible log Frobenius lifting given by  $(T_n, M)$  where  $T_n := W_n(k) \otimes_{\mathbb{Z}[N]} \mathbb{Z}[M]$  and  $\varphi_n$  is given by  $a \otimes b \mapsto F(a) \otimes b^p \in T_{n-1}$ . Then we need to lift the étale morphism  $T_1 \rightarrow R$  along  $\cdots T_n \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T_1$ , which exists by [14] Proposition 3.2. Note that the log



structure on each  $A_n$  is given by  $M \rightarrow T_n \rightarrow A_n$  (while  $\varphi_n|_M$  is still multiplication by  $p$ ), so in particular we have constructed a  $p$ -compatible log Frobenius lifting.  $\square$

### 5.2. Saturated log de Rham–Witt complex via log Frobenius liftings

For the rest of this subsection we make the following assumption.

#### Assumption.

$\underline{R}/\underline{k}$  is integral and admits a  $p$ -compatible log Frobenius lifting  $(\underline{A}_n, \delta_n, \varphi_n)$ . (\*)

Examples include log algebras  $\underline{R}$  that are log-smooth and integral over  $\underline{k}$  (Lemma 5.2).<sup>11</sup>

**5.2.1. A  $p$ -completed log de Rham complex.** Write  $\widehat{\underline{A}} = (\widehat{\underline{A}}, M)$  where  $\widehat{\underline{A}} = \varprojlim A_n$ , which is equipped with the log structure from the filtered inverse limit. We then form the  $p$ -adically completed log de Rham complex

$$\widehat{\omega}_{\widehat{\underline{A}}/W(\underline{k})}^* (= \widehat{\omega}^*) := \varprojlim_n \omega_{\underline{A}_n/W_n(\underline{k})}^* \cong \varprojlim_n (\omega_{\underline{A}/W(\underline{k})}^*/p^n).$$

From Subsection A.3.1,  $f$  being integral means that the map  $\mathbb{Z}[N] \rightarrow \mathbb{Z}[M]$  induced from the map of monoids is flat; therefore, for each  $n$ ,  $\text{Spec } \underline{A}_n \rightarrow \text{Spec } W(\underline{k})$  is log-smooth and integral and hence flat. Now, because the restriction map  $R : A_{n+1} \rightarrow A_n$  is surjective, the inverse limit  $\widehat{\underline{A}}$  is also flat over  $W(\underline{k})$  and hence  $p$ -torsion free. This allows us to apply Proposition 4.5 and get a log Dieudonné algebra  $\omega_{\widehat{\underline{A}}/W(\underline{k})}^*$ . The completed log de Rham complex  $\widehat{\omega}_{\widehat{\underline{A}}/W(\underline{k})}^*$  has a unique cdga structure such that the canonical map  $\omega_{\widehat{\underline{A}}/W(\underline{k})}^* \rightarrow \widehat{\omega}_{\widehat{\underline{A}}/W(\underline{k})}^*$  is a map of cdgas. Moreover,  $\widehat{\omega}_{\widehat{\underline{A}}/W(\underline{k})}^*$  inherits the Frobenius structure from  $\omega_{\widehat{\underline{A}}/W(\underline{k})}^*$ , which makes it a  $p$ -compatible log Dieudonné algebra.

**5.2.2. A key lemma.** The following lemma is a variant of Lemma 4.9. In this lemma we do not need to assume that  $\underline{A}$  comes from log Frobenius liftings.

**Lemma 5.3.** *Let  $\underline{A}$  be a log algebra over  $W(\underline{k})$  satisfying the conditions in either Proposition 4.5 or Proposition 4.6, and let  $\omega^* = \omega_{\underline{A}/W(\underline{k})}^*$  be the log Dieudonné algebra constructed there. Let  $\widehat{\omega}^*$  be the term-wise  $p$ -completion of  $\omega^*$ . Then for any  $p$ -torsion-free and  $p$ -complete log Dieudonné  $W(\underline{k})$ -algebra  $B^*$ , the canonical map*

$$\text{Hom}_{DA^{\log}}(\widehat{\omega}^*, B^*) \cong \text{Hom}_F(\underline{A}, \underline{B}^0)$$

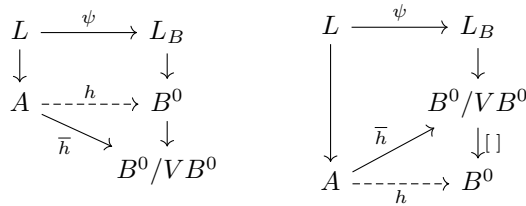
*is a bijection. Suppose that  $B^*$  is in addition strict and  $p$ -compatible; then, in fact,*

$$\text{Hom}_{DA^{\log}}(\widehat{\omega}^*, B^*) \cong \text{Hom}(\underline{A}/p, \underline{B}^0/V\underline{B}^0),$$

*where the second set denotes homomorphisms of log algebras over  $\underline{k}$ .*

<sup>11</sup>Note that our definition of log-smooth requires the monoid  $M$  to be integral and coherent, but this is different from requiring the morphism  $\text{Spec } \underline{R} \rightarrow \text{Spec } \underline{k}$  to be integral; see Appendix A.3.1.

**Proof.** The first assertion follows directly from Lemma 4.9. For the second assertion, we appeal to the Cartier–Dieudonné–Dwork lemma, which in our setup says that, because  $A$  is equipped with a lift of Frobenius  $\varphi$  satisfying  $\varphi(a) \equiv a^p$  for all  $a \in A$ , a homomorphism  $\bar{h} : A \rightarrow B^0/VB^0$  has a unique lift to a homomorphism  $h : A \rightarrow W(B^0/VB^0) \xrightarrow{\mu^{-1}} B^0$  such that  $F \circ h = h \circ \varphi$ . We still need to check that, given a morphism of log algebras  $(\bar{h}, \psi)$ , its lift  $(h, \psi) : \underline{A} \rightarrow \underline{B}^0$  is a morphism of log algebras; namely, the top square in the left diagram commutes:



This is indeed the case, because the log structure on  $\bar{B}^0$  factors through the Teichmüller lifts by Corollary 3.14, as the diagram on the right indicates. Moreover, this also shows that any map  $(h, \psi) : \underline{A} \rightarrow \underline{B}^0$  of log algebras comes from the lifting of a pair  $(\bar{h}, \psi)$ . Finally, because  $p$  is 0 in  $B^0/VB^0$ , we get  $\text{Hom}_F(\underline{A}, \underline{B}^0) = \text{Hom}(\underline{A}/p, \underline{B}^0/VB^0)$  as desired.  $\square$

**5.2.3. Comparison with log de Rham–Witt complexes.** Now let us apply the lemma above to the setup in Subsection 5.2.1, where  $\hat{A}$  comes from a log Frobenius lifting of  $\underline{R}$ . By Lemma 5.3 there is a canonical bijection

$$\text{Hom}_{\text{DA}^{\log, p}}(\hat{\omega}_{\hat{A}/W(\underline{k})}^*, \mathcal{W}\omega_{\underline{R}/\underline{k}}^*) \cong \text{Hom}(\underline{R}, \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^0).$$

Let  $e : \underline{R} \rightarrow \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^0$  be the tautological map from the definition of the log de Rham–Witt complex  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$ . Then  $(e, \text{id}) : \underline{R} \rightarrow \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^0$  gives rise to a map  $v : \hat{\omega}_{\hat{A}/W(\underline{k})}^* \rightarrow \mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  in the category  $\text{DA}^{\log, p}$  of  $p$ -compatible log Dieudonné  $\underline{W}$ -algebras.

**Proposition 5.4.** *Keep the assumption as in the beginning of Subsection 5.2. The map  $v$  induces a canonical isomorphism (again denoted by  $v$ )*

$$v : W(\hat{\omega}_{\hat{A}/W(\underline{k})}^*)_{\text{sat}} \xrightarrow{\sim} \mathcal{W}\omega_{\underline{R}/\underline{k}}^*.$$

**Proof.** By construction,  $W(\hat{\omega}_{\hat{A}/W(\underline{k})}^*)_{\text{sat}} \in \text{DA}_{\text{stt}}^{\log, p}$ , so it suffices to show that for any  $B^* \in \text{DA}_{\text{stt}}^{\log, p}$ , the map  $v : W(\hat{\omega}_{\hat{A}/W(\underline{k})}^*)_{\text{sat}} \rightarrow \mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  induces a bijection between

$$\text{Hom}_{\text{DA}_{\text{stt}}^{\log, p}}(\mathcal{W}\omega_{\underline{R}/\underline{k}}^*, B^*) \stackrel{?}{=} \text{Hom}_{\text{DA}_{\text{stt}}^{\log, p}}(W(\hat{\omega}_{\hat{A}/W(\underline{k})}^*)_{\text{sat}}, B^*).$$

This follows from the following commutative diagram

$$\begin{array}{ccc}
 \mathrm{Hom}_{\mathrm{DA}^{\mathrm{log},p}_{\mathrm{str}}}(\mathcal{W}\omega_{\underline{R}/\underline{k}}^*, B^*) & \xrightarrow{v} & \mathrm{Hom}_{\mathrm{DA}^{\mathrm{log},p}_{\mathrm{str}}}(W(\widehat{\omega}_{\widehat{A}/W(\underline{k})}^*), B^*) \\
 \parallel & & \parallel \\
 \text{Thm 4.10.} & & \mathrm{Hom}_{\mathrm{DA}^{\mathrm{log},p}}(\widehat{\omega}_{\widehat{A}/W(\underline{k})}^*, B^*) \\
 \parallel & & \parallel \\
 \mathrm{Hom}_{\mathrm{Alg}_{\underline{k}}^{\mathrm{log}}}(\underline{R}, \underline{B}^0/V\underline{B}^0) & \xlongequal{\quad} & \mathrm{Hom}_{\mathrm{Alg}_{\underline{k}}^{\mathrm{log}}}(\underline{R}, \underline{B}^0/V\underline{B}^0) \\
 & & \text{Lemma 5.3}
 \end{array}$$

□

Therefore, if  $\underline{R}/\underline{k}$  is integral and admits a  $p$ -compatible log Frobenius lifting, the saturated log de Rham–Witt complex of  $\underline{R}/\underline{k}$  can be computed by taking  $\widehat{\omega}_{\widehat{A}/W(\underline{k})}^*$  and then taking its associated  $V$ -complete saturation (cf. Subsection 3.4).

### 5.3. Aside: comparison with more general log Frobenius liftings

As an aside, we prove a more general form of Theorem 7 (3). In general, we cannot construct  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  from a non- $p$ -compatible Frobenius lifting of  $\underline{R}/\underline{k}$  as in Subsection 5.2. However, in some cases we can still compare the completed log de Rham complexes of non- $p$ -compatible Frobenius liftings with  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  in the derived category. This crucially relies on the notion of log-Cartier type (cf. Subsection A.3.2). In contrast, the Cartier criterion is not needed in Subsection 5.2.

**Assumption.**  $\underline{R}/\underline{k}$  is log-smooth of log-Cartier type (which is in particular integral), and  $(\underline{A}_n, \varphi_n, \delta_n)$  is a log Frobenius lifting such that  $\widehat{A} = \varprojlim A_n$  is  $p$ -torsion free.

Each  $\omega_{\underline{A}_n/W_n(\underline{k})}^1$  is a finitely generated locally free module over  $A_n = \widehat{A}/p^n$ , by log smoothness of  $\underline{A}_n/W_n(\underline{k})$ , so  $\widehat{\omega}_{\widehat{A}/W(\underline{k})}^1 = \varprojlim \omega_{\underline{A}_n/W_n(\underline{k})}^1$  is locally free over  $\widehat{A}$  and hence  $p$ -torsion free. Then  $\widehat{A}$  satisfies the condition in Proposition 4.6 with  $\widehat{\omega}_{\widehat{A}/W(\underline{k})}^1$  replacing  $\omega_{\underline{A}/W(\underline{k})}^1$ . The conclusion of Proposition 4.6 holds by the same proof. To summarise,  $\widehat{\omega}_{\widehat{A}/W(\underline{k})}^*$  is equipped with a Frobenius extending  $\varphi = \varprojlim \varphi_n$ , which makes it a log Dieudonné  $W(\underline{k})$ -algebra,<sup>12</sup> and there is a bijection

$$\mathrm{Hom}_{\mathrm{DA}^{\mathrm{log}}}(\widehat{\omega}_{\widehat{A}/W(\underline{k})}^*, \mathcal{W}\omega_{\underline{R}/\underline{k}}^*) \cong \mathrm{Hom}(\underline{R}, \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^0)$$

by Lemma 5.3. The tautological map  $\underline{R} \rightarrow \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^0$  again induces a map  $v : \widehat{\omega}_{\widehat{A}/W(\underline{k})}^* \rightarrow \mathcal{W}\omega_{\underline{R}/\underline{k}}^*$ .

<sup>12</sup>Though we construct the log Dieudonné algebra  $\widehat{\omega}_{\widehat{A}/W(\underline{k})}^*$  directly using universal properties of  $\widehat{\omega}_{\widehat{A}/W(\underline{k})}^1$ , Lemma 5.3 still holds by inspecting its proof (as well as the proof of Lemma 4.9).

**Remark 5.5.** Unlike the previous subsection, in general  $v$  will not induce an isomorphism from the complete saturation  $W((\widehat{\omega}_{\underline{A}/W(\underline{k})}^*)_{\text{sat}})$  of  $\widehat{\omega}_{\underline{A}/W(\underline{k})}^*$  to the saturated log de Rham–Witt complex.

**Theorem 5.6.** *Let  $\underline{R}$  and  $\widehat{A}$  be as in the setup above; then the map*

$$v : \widehat{\omega}_{\underline{A}/W(\underline{k})}^* \longrightarrow \mathcal{W}\omega_{\underline{R}/\underline{k}}^*$$

*induces a quasi-isomorphism on the underlying cochain complexes.*

**Proof.** (1). First suppose that the log Frobenius lift  $(\underline{A}_n, \varphi_n, \delta_n)$  in the Assumption above is  $p$ -compatible; then by Proposition 5.4, we have

$$v : \widehat{\omega}_{\underline{A}/W(\underline{k})}^* \rightarrow W(\widehat{\omega}_{\underline{A}/W(\underline{k})}^*)_{\text{sat}} \xrightarrow{\sim} \mathcal{W}\omega_{\underline{R}/\underline{k}}^*.$$

Because both  $\widehat{\omega}_{\underline{A}/W(\underline{k})}^*$  and  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  are  $p$ -torsion free and  $p$ -complete, it suffices to show that  $v$  is a quasi-isomorphism after mod  $p$ . In other words, we need to show that

$$\widehat{\omega}_{\underline{A}/W(\underline{k})}^*/p \rightarrow (\widehat{\omega}_{\underline{A}/W(\underline{k})}^*)_{\text{sat}}/p \rightarrow W(\widehat{\omega}_{\underline{A}/W(\underline{k})}^*)_{\text{sat}}/p \xrightarrow{\sim} \mathcal{W}\omega_{\underline{R}/\underline{k}}^*/p$$

is a quasi-isomorphism. The rest of the proof is similar to the proof of Corollary 2.17. The key point is that the underlying Dieudonné algebra  $\widehat{\omega}_{\underline{A}/W(\underline{k})}^*$  satisfies the Cartier criterion (cf. Definition 2.14) by Proposition A.10. In other words, we have

$$F : \widehat{\omega}_{\underline{A}/W(\underline{k})}^*/p \cong \omega_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} H^*(\omega_{\underline{R}/\underline{k}}^\bullet) \cong H^*(\widehat{\omega}_{\underline{A}/W(\underline{k})}^\bullet/p).$$

To finish the proof, note that both of the maps of  $(\widehat{\omega}_{\underline{A}/W(\underline{k})}^*)_{\text{sat}}/p \rightarrow W(\widehat{\omega}_{\underline{A}/W(\underline{k})}^*)_{\text{sat}}/p$  and  $\widehat{\omega}_{\underline{A}/W(\underline{k})}^*/p \rightarrow (\widehat{\omega}_{\underline{A}/W(\underline{k})}^*)_{\text{sat}}/p$  are quasi-isomorphisms by Corollary 2.16.

(2). For the general case, again we need to show that  $\bar{v} : \widehat{\omega}_{\underline{A}/W(\underline{k})}^*/p \cong \omega_{\underline{R}/\underline{k}}^* \xrightarrow{v \bmod p} \mathcal{W}\omega_{\underline{R}/\underline{k}}^*/p$  is a quasi-isomorphism. The map  $\bar{v}$  depends on the lifting  $\widehat{A}$  of  $R$  and  $\varphi$  of Frobenius, but the induced map in the derived category does not. More precisely, let  $\widehat{A}'/W$  be a  $p$ -compatible formal lift of  $\underline{R}/\underline{k}$  (which exists thanks to Lemma 5.2), and consider the following diagram:

$$\begin{array}{ccccccc} \widehat{\omega}_{\underline{A}/W(\underline{k})}^* & \longrightarrow & \omega_{\underline{R}/\underline{k}}^* & \xrightarrow{\bar{v}} & \mathcal{W}\omega_{\underline{R}/\underline{k}}^*/p & \xrightarrow{\text{pr}} & \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^* \\ & & \parallel & & & & \parallel \\ \widehat{\omega}_{\underline{A}'/W(\underline{k})}^* & \longrightarrow & \omega_{\underline{R}/\underline{k}}^* & \xrightarrow{\bar{v}'} & \mathcal{W}\omega_{\underline{R}/\underline{k}}^*/p & \xrightarrow{\text{pr}} & \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^* \end{array}$$

where  $\text{pr} : \mathcal{W}\omega_{\underline{R}/\underline{k}}^*/p \rightarrow \mathcal{W}\omega_{\underline{R}/\underline{k}}^*/\text{Fil}^1 = \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^*$  is the canonical projection. Unwinding definitions, we see that  $\text{pr} \circ \bar{v} = \text{pr} \circ \bar{v}'$  (the square commutes). From step (1), we know that  $\bar{v}'$  is a quasi-isomorphism. By Theorem 5.8,  $\text{pr} \circ \bar{v}'$  is an isomorphism; hence,  $\text{pr}$  is a quasi-isomorphism. This in turn (using Theorem 5.8 again) implies that  $\bar{v}$  is a quasi-isomorphism. □

**Remark 5.7.** (1) The proof of the general case (part (2)) uses Theorem 5.8, whose proof builds on the conclusion of the special case (part (1)) of the theorem but does not use part (2) of this proof. Therefore, there is no circular argument.

(2) In general,  $\bar{v}$  does not agree with  $\bar{v}'$  if  $R$  is nonperfect. Note that  $v$  comes from the map  $f_\varphi : \widehat{A} \rightarrow W(\widehat{A}) \rightarrow W(R)$ , where the first map is determined (on ghost coordinates) by

$$\widehat{A} \longrightarrow W(\widehat{A}) \xrightarrow{\text{gh}} \widehat{A}^{\mathbb{N}}, \quad \tilde{x} \mapsto (\tilde{x}, \varphi(\tilde{x}), \varphi^2(\tilde{x}), \dots).$$

On Witt coordinates,  $f_\varphi(\tilde{x}) = (\tilde{x}, \theta(\tilde{x}), \dots)$  where  $\theta(\tilde{x}) = (\varphi(\tilde{x}) - \tilde{x}^p)/p \pmod{p} \in R$ . The reduction mod  $p$  of  $f_\varphi$  gives  $\bar{v}|_R : R = \widehat{A}/p \rightarrow W(R)/p$ , which depends on  $\widehat{A}$  and  $\varphi$ . This is not surprising: without the liftings, we cannot produce a map from  $R = W(R)/V \rightarrow W(R)/p$  when  $R$  is nonperfect.

**5.4. Comparison with de Rham complexes in characteristic  $p$**

Let  $\underline{R}/\underline{k}$  be a log algebra and let  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  be its saturated log de Rham–Witt complex. We have a map  $(e, \text{id}) : \underline{R} \rightarrow \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^0$ , which extends uniquely to a map  $\nu : \omega_{\underline{R}/\underline{k}}^* \rightarrow \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^*$  of cdgas that is compatible with log structures and log differentials (in fact,  $\nu = \text{pr} \circ \bar{v} = \text{pr} \circ \bar{v}'$  in the proof of Theorem 5.6). Our first goal of this subsection is to prove the following proposition.

**Theorem 5.8.** *Suppose that  $\underline{R}/\underline{k}$  is log-smooth of log-Cartier type; then*

$$\nu : \omega_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^*$$

*is an isomorphism.*

**Proof.** The proof is similar to that of [2, Proposition 4.3.2]. Let  $(\widehat{A}, M)$  be a  $p$ -compatible formal lift of  $\underline{R}$  over  $\underline{W}$ , which exists by Lemma 5.2. Then we have a commutative diagram of cochain complexes

$$\begin{array}{ccc} \omega_{\underline{R}/\underline{k}}^* & \xrightarrow{C^{-1}} & H^*(\omega_{\underline{R}/\underline{k}}^*) = H^*(\widehat{\omega}_{\widehat{A}/W(k)}^*/p) \\ \downarrow \bar{v} & \searrow \nu & \downarrow H(\bar{v}) \\ & \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^* & \\ \downarrow q & \nearrow F_1 & \\ \mathcal{W}\omega_{\underline{R}/\underline{k}}^*/p & \xrightarrow{F} & H^*(\mathcal{W}\omega_{\underline{R}/\underline{k}}^*/p) \end{array}$$

where both complexes on the right are equipped with the Bockstein differentials (see Subsection 2.2). The vertical maps are both induced by  $\nu : \widehat{\omega}_{\widehat{A}/W(k)}^* \rightarrow \mathcal{W}\omega_{\underline{R}/\underline{k}}^*$ , and the bottom triangle is the triangle above Lemma 2.11 applied to the saturated log de Rham–Witt complex. From the proof of part (1) of Theorem 5.6,  $\bar{v}$  is a quasi-isomorphism, so  $H(\bar{v})$  is an isomorphism. The map  $F_1$  is an isomorphism by part (2) of Lemma 2.11. Finally, by the assumption that  $\underline{R}/\underline{k}$  is log-smooth of log-Cartier type,  $C^{-1}$  is an isomorphism. Therefore,  $\nu$  is an isomorphism. □

**Corollary 5.9.** *We continue to assume that  $\underline{R}/\underline{k}$  is log-smooth of log-Cartier type. The projection  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^* \rightarrow \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^* \xrightarrow{\nu^{-1}} \omega_{\underline{R}/\underline{k}}^*$  induces a quasi-isomorphism of cochain complexes*

$$pr : \mathcal{W}\omega_{\underline{R}/\underline{k}}^*/p\mathcal{W}\omega_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} \omega_{\underline{R}/\underline{k}}^*.$$

In other words,  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  is a deformation of  $\omega_{\underline{R}/\underline{k}}^*$  in the derived category.

**Proof.** Immediate from the commutative diagram above, because  $\bar{\nu}$  is an isomorphism. □

Our next remark is helpful for the discussion on the monodromy operator and for [20].

**Remark 5.10.** In the proof of Theorem 5.6 and Theorem 5.8, the assumption that  $\underline{R}/\underline{k}$  is log-smooth of log-Cartier type is only used to guarantee that the following two criteria hold:

- (1) There is a lift  $\underline{A}/W(\underline{k})$  of  $\underline{R}$  with a lift of Frobenius such that  $\omega_{\underline{A}/W(\underline{k})}^*$  is  $p$ -torsion free.
- (2) The Cartier isomorphism  $C^{-1} : \omega_{\underline{R}/\underline{k}}^i \xrightarrow{\sim} H^i(\omega_{\underline{R}/\underline{k}}^*)$  holds.

Thus, the conclusions of Theorem 5.6 and Theorem 5.8 continue to hold for log algebras  $\underline{R}/\underline{k}$  that meet the two criterions above.

### 6. Étale base change

In this section, we show that the saturated log de Rham–Witt complex  $\mathcal{W}\omega_{\underline{R}/\underline{k}}^*$  behaves well with respect to certain étale base change. Therefore, for any quasi-coherent log scheme  $\underline{X}$  over  $\underline{k}$ , we obtain a sheaf  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^*$  of log Dieudonné algebras on the étale site  $X_{\text{ét}}$ .

To formulate the étale base change we consider the category of log cdgas, consisting of  $\underline{A}^* = (A^*, M, d, \delta)$  where  $(A^*, d)$  is a cdga,  $M \xrightarrow{\alpha} A^0$  a log algebra and  $\delta : M \rightarrow A^1$  a monoid morphism (with the additive monoid structure on  $A^1$ ) satisfying  $d \circ \delta = 0$  and  $\alpha(m)\delta(m) = d(\alpha(m))$  for any  $m \in M$ . Morphisms between  $\underline{A}^*, \underline{B}^*$  are pairs of maps  $(f, \psi)$  where  $f : A^* \rightarrow B^*$  is a morphism of cdgas and  $\psi : M_A \rightarrow M_B$  is a morphism of monoids, compatible with  $\alpha$  and  $\delta$ .

**Definition 6.1.** • A morphism  $(f, \psi) : (A, L_A) \rightarrow (B, L_B)$  of log algebras is *naively étale* if  $f : A \rightarrow B$  is étale and  $\psi$  is an isomorphism of monoids.

- A morphism  $(f, \psi) : \underline{A}^* \rightarrow \underline{B}^*$  between log cdgas is étale if its restriction on the log algebra  $(f, \psi) : (A^0, M_A) \rightarrow (B^0, M_B)$  is naively étale and  $f$  induces an isomorphism  $A^* \otimes_{A^0} B^0 \rightarrow B^*$  of graded algebras.
- If  $\underline{A}^*, \underline{B}^* \in \text{DA}_{\text{str}}^{\text{log}}$  are strict log Dieudonné algebras, then  $(f, \psi) : \underline{A}^* \rightarrow \underline{B}^*$  is *V-adically étale* if for each  $n \geq 1$ , the induced map of log cdgas  $W_n(A^*) \rightarrow W_n(B^*)$  is étale.

If  $\underline{A}^*$  is a log cdga, we denote by  $\acute{\text{E}}\text{t}_{\underline{A}^*}$  the category of log cdgas étale over  $\underline{A}^*$ . Similarly, for  $\underline{A}^* \in \text{DA}_{\text{str}}^{\text{log}}$ , we denote by  $V\acute{\text{E}}\text{t}_{\underline{A}^*}$  the category of  $V$ -adically étale  $\underline{A}^*$ -algebras in  $\text{DA}_{\text{str}}^{\text{log}}$ .

**Proposition 6.2.** (1) The functor  $\underline{B}^* \mapsto B^0$  induces an equivalence of categories

$$\acute{E}t_{\underline{A}^*} \xrightarrow{\sim} \acute{E}t_{A^0} = \{\acute{\text{étale}} A^0\text{-algebras}\}.$$

(2) Let  $\underline{A}^* \in \text{DA}_{\text{str}}^{\text{log}}$ . The functor that sends a  $V$ -adically étale  $\underline{A}^*$ -algebra  $\underline{B}^* \in \text{DA}_{\text{str}}^{\text{log}}$  to the  $A^0/V(A^0)$ -algebra  $B^0/V(B^0)$  induces an equivalence of categories

$$\text{V}\acute{E}t_{\underline{A}^*} \xrightarrow{\sim} \acute{E}t_{A^0/V(A^0)}.$$

**Proof.** The functor  $\underline{B}^* \mapsto B^*$  forgetting the log structure on  $\underline{B}^*$  is an equivalence of categories between étale  $\underline{A}^*$ -algebras and étale  $A^*$ -algebras. Likewise,  $\underline{B}^* \mapsto B^*$  induces an equivalence between  $\text{V}\acute{E}t_{\underline{A}^*}$  and  $\text{V}\acute{E}t_{A^*}$ . Therefore, claims (1) and (2) follow from Proposition 5.3.2 and Theorem 5.3.4 in [2] respectively.  $\square$

By the same proof of Corollary 5.3.5 of [2], we arrive at the following.

**Proposition 6.3** (étale base change). Let  $\underline{R} = (R, M)$  be a log algebra over  $\underline{k}$  and  $R \rightarrow S$  be an étale morphism of  $k$ -algebras. Let  $\underline{S} = (S, M)$  be the (naively étale) log algebra over  $\underline{R}$ . For any  $n \geq 1$ , the map  $\mathcal{W}_n \omega_{\underline{R}/\underline{k}}^* \rightarrow \mathcal{W}_n \omega_{\underline{S}/\underline{k}}^*$  is étale. In other words,  $\mathcal{W} \omega_{\underline{R}/\underline{k}}^* \rightarrow \mathcal{W} \omega_{\underline{S}/\underline{k}}^*$  is  $V$ -adically étale. Moreover, we have a natural isomorphism of graded algebras

$$\mathcal{W}_n \omega_{\underline{R}/\underline{k}}^* \otimes_{W_n(R)} W_n(S) \xrightarrow{\sim} \mathcal{W}_n \omega_{\underline{S}/\underline{k}}^*.$$

**Remark 6.4.** When  $\underline{R}$  is log-smooth over  $\underline{k}$  of log-Cartier type, we may also deduce the étale base change  $\mathcal{W}_n \omega_{\underline{R}/\underline{k}}^* \otimes_{W_n(R)} W_n(S) \xrightarrow{\sim} \mathcal{W}_n \omega_{\underline{S}/\underline{k}}^*$  by comparing to the log de Rham–Witt complexes constructed by Hyodo–Kato (which is defined globally on the étale site) or Matsue (using Proposition 3.6 in [15]), once we prove Propositions 7.7 and 7.10.

Now let  $\underline{X}$  be a quasi-coherent log scheme over  $\text{Spec } \underline{k}$ . We construct the sheaf of log de Rham–Witt complexes  $\mathcal{W} \omega_{\underline{X}/\underline{k}}^*$  on  $X_{\acute{\text{ét}}}$ .

**Notation 6.5.** For a log scheme  $\underline{X} = (X, M_X)$  over  $\underline{k}$ , let  $X_{\acute{\text{ét}}, \text{aff}}$  be the site of ‘small enough’ affine étale opens where the log structure of  $\underline{X}$  admits charts. The objects of  $X_{\acute{\text{ét}}, \text{aff}}$  are étale morphisms  $U \xrightarrow{h} X$  over  $k$  where  $U = \text{Spec } R$  is affine, such that there exists a constant log structure  $L \rightarrow R$  and an isomorphism  $(\mathcal{L}_U)^a \xrightarrow{\sim} M_X|_U := h^* M_X$ . Here  $(\mathcal{L}_U)^a$  is the associated log structure of the pre-log structure  $L_U \rightarrow \mathcal{O}_U$ . We give  $X_{\acute{\text{ét}}, \text{aff}}$  the étale topology.

It is clear that  $X_{\acute{\text{ét}}, \text{aff}}$  indeed forms a site. Moreover, if the log structure  $M_X$  admits charts étale locally, then these ‘small enough’ affine opens form a basis for the étale topology on  $X$ .

**Lemma 6.6.** Further assume that  $\underline{X}$  is a quasi-coherent log scheme; then the restriction of an étale sheaf from  $X_{\acute{\text{ét}}}$  to  $X_{\acute{\text{ét}}, \text{aff}}$  induces an equivalence of topoi.

**Proof.** The canonical functor  $u : X_{\acute{\text{ét}}, \text{aff}} \rightarrow X_{\acute{\text{ét}}}$  is continuous, cocontinuous, and satisfies the property that any étale open  $Y \in X_{\acute{\text{ét}}}$  admits a cover by  $\{U_i\}$  in  $X_{\acute{\text{ét}}, \text{aff}}$ , as the log structure  $M_X$  is quasi-coherent. Thus,  $u$  induces an equivalence  $\text{Sh}(X_{\acute{\text{ét}}, \text{aff}}) \xrightarrow{\sim} \text{Sh}(X_{\acute{\text{ét}}})$  by [4, Tag 03A0].  $\square$

**Theorem 6.7.** *Let  $\underline{X} = (X, M_X)$  be a quasi-coherent log scheme over  $\underline{k}$ . There exists a unique sheaf  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^*$  on  $X_{\text{ét}}$  valued in log Dieudonné algebras such that on each étale local chart  $(U, L) = \text{Spec}(R, L)$  of  $\underline{X}$ , there is a canonical map of log Dieudonné algebras*

$$\mathcal{W}\omega_{R/\underline{k}}^* \longrightarrow \Gamma(U, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*),$$

which is an isomorphism on the underlying Dieudonné algebras and is given by  $L \rightarrow \Gamma(U, M_X)$  on the log structures.

**Proof.** It suffices to construct the sheaf of abelian groups  $\mathcal{W}_n\omega_{\underline{X}/\underline{k}}^i$  for each  $n \geq 1, i \geq 0$  and then take the inverse limit as  $n$  varies. By Lemma 6.6 it suffices to construct  $\mathcal{W}_n\omega_{\underline{X}/\underline{k}}^i$  on  $X_{\text{ét, aff}}$ . To this end we define a presheaf of abelian groups  $\mathcal{W}_n\omega_{\underline{X}/\underline{k}}^i : X_{\text{ét, aff}} \rightarrow \text{Ab}$  as follows. For each  $U = \text{Spec} R \rightarrow X$  in  $X_{\text{ét, aff}}$ , we write  $M(U) := \Gamma(U, M_X)$  and define

$$\mathcal{W}_n\omega_{\underline{X}/\underline{k}}^i(U \rightarrow X) := \mathcal{W}_n\omega_{(R, M(U))/\underline{k}}^i.$$

Note that the value of  $\mathcal{W}_n\omega_{\underline{X}/\underline{k}}^i(U \rightarrow X)$  is independent of the choice of chart on  $U$ , by Lemma 4.12. (In fact, the constant monoid  $M(U)$  gives a (functorial) choice of chart on each  $U \in X_{\text{ét, aff}}$ , see Lemma A.1.) To show that this presheaf is a sheaf on  $X_{\text{ét, aff}}$ , it suffices to check that, on each étale open neighbourhood  $U = \text{Spec} R \in X_{\text{ét, aff}}$ , the restriction  $\mathcal{W}_n\omega_{\underline{X}/\underline{k}}^i|_{U_{\text{ét, aff}}}$  defines a sheaf on  $U_{\text{ét, aff}} = X_{\text{ét, aff}}/U$ . The latter category consists precisely of all étale  $R$ -algebras. In other words, we may assume that  $X = \text{Spec} R$  is a ‘small enough’ affine log scheme that admits a chart for its log structure. Let  $R \rightarrow S$  be an étale map and write  $V = \text{Spec} S$ ; then we have morphisms

$$\mathcal{W}_n\omega_{(R, M(X))/\underline{k}}^i \xrightarrow{\gamma_1} \mathcal{W}_n\omega_{(S, M(X))/\underline{k}}^i \xrightarrow{\gamma_2} \mathcal{W}_n\omega_{(S, M(V))/\underline{k}}^i$$

by functoriality.  $\gamma_2$  is an isomorphism by Lemma A.2 and 4.12. We know that

$$\mathcal{W}_n\omega_{(S, M(V))/\underline{k}}^i \cong \mathcal{W}_n\omega_{(R, M(X))/\underline{k}}^i \otimes_{W_n(R)} W_n(S)$$

by Proposition 6.3. From the topological invariance of étale sites, we know that the association  $S \mapsto W_n(S)$  identifies the étale algebras over  $R$  with étale algebras over  $W_n(R)$  (see also Theorem 5.4.1 of [2]). Therefore, on the affine étale site  $(\text{Spec} W_n(R))_{\text{ét, affine}}$ , the presheaf  $\mathcal{W}_n\omega_{\underline{X}/\underline{k}}^i$  is in fact a sheaf associated to the  $W_n(R)$ -module  $\mathcal{W}_n\omega_{(R, M(X))/\underline{k}}^i$ , by the usual étale descent. It is straightforward to keep track of differentials, Frobenius operators and the monoid morphisms  $M_X \rightarrow \mathcal{W}\omega_{\underline{X}/\underline{k}}^0$  and  $M_X \xrightarrow{\delta} \mathcal{W}\omega_{\underline{X}/\underline{k}}^1$ ; hence, the theorem follows.  $\square$

**Corollary 6.8.** *Let  $\underline{X} = (X, M_X)$  be a quasi-coherent smooth log scheme of log Cartier type over  $\underline{k}$  and  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^*$  be the saturated log de Rham–Witt complex constructed in Theorem 6.7.*

(1) *The identification  $\mathcal{O}_X \xrightarrow{\sim} \mathcal{W}_1\omega_{\underline{X}/\underline{k}}^0$  induces a canonical isomorphism*

$$\omega_{\underline{X}/\underline{k}}^* \xrightarrow{\sim} \mathcal{W}_1\omega_{\underline{X}/\underline{k}}^*.$$

(2) *The natural projection  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^*/p \rightarrow \mathcal{W}_1\omega_{\underline{X}/\underline{k}}^*$  induces a quasi-isomorphism*

$$\mathcal{W}\omega_{\underline{X}/\underline{k}}^*/p \xrightarrow{\sim} \omega_{\underline{X}/\underline{k}}^*.$$



**Proof.** This follows from Theorem 6.7. □

Furthermore, our discussion on the structure of the  $V$ -filtration of the saturated log de Rham–Witt complexes on local charts (see Subsection 2.4) carries over to the globalisation. To state this more precisely, we introduce the following sheaves on  $X_{\text{ét}}$ :

- $\mathcal{B}_n^i := \text{im}(F^{n-1}d : \mathcal{W}_n\omega_{\underline{X}/\underline{k}}^{i-1} \rightarrow \mathcal{W}_1\omega_{\underline{X}/\underline{k}}^i).$
- $\mathcal{Z}_n := \text{im}(F^n : \mathcal{W}_{n+1}\omega_{\underline{X}/\underline{k}}^i \rightarrow \mathcal{W}_1\omega_{\underline{X}/\underline{k}}^i).$

We view  $\mathcal{W}_n\omega_{\underline{X}/\underline{k}}^{i-1}$  as an  $(F_X^2)_*\mathcal{O}_X$ -module via the Techmuller lift

$$\mathcal{O}_X \longrightarrow \mathcal{W}_n\omega_{\underline{X}/\underline{k}}^0 = \mathcal{W}_n(\mathcal{O}_X)$$

and  $\mathcal{W}_1\omega_{\underline{X}/\underline{k}}^i$  as an  $(F_X^{n+1})_*\mathcal{O}_X$ -module as in Remark 2.24. In fact, both sheaves are quasi-coherent  $\mathcal{O}_X$ -modules by the proof of Theorem 6.7. Note that the map  $F^{n-1}d : \mathcal{W}_n\omega_{\underline{X}/\underline{k}}^{i-1} \rightarrow \mathcal{W}_1\omega_{\underline{X}/\underline{k}}^i$  is a map of  $\mathcal{O}_X$ -modules; indeed, on local sections we have

$$F^{n-1}d(x \cdot y) = F^{n-1}d([x]^{p^2}y) = [a]^{p^{n+1}}F^{n-1}dy = a \cdot F^{n-1}dy.$$

In particular,  $\mathcal{B}_n^i$  is a quasi-coherent submodule of  $(F_X^{n+1})_*\mathcal{W}_1\omega_{\underline{X}/\underline{k}}^i$ . Moreover, on small enough affine open schemes  $U = \text{Spec}R \in X_{\text{ét,aff}}$ ,  $\Gamma(U, \mathcal{B}_n^i)$  is precisely  $B_n(\mathcal{W}_1\omega_{\underline{R}/\underline{k}}^i)$  defined in Notation 2.21. Likewise,  $\Gamma(U, \mathcal{Z}_n^i) = Z_n(\mathcal{W}_1\omega_{\underline{R}/\underline{k}}^i)$ .

**Corollary 6.9.** *Let  $\underline{X} = (X, M_X)$  be a quasi-coherent log scheme over  $\underline{k}$  and  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^*$  be the saturated log de Rham–Witt complex. The graded pieces of the  $V$ -adic filtration on  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^*$  sits in a short exact sequence of quasi-coherent sheaves on  $X_{\text{ét}}$  as follows:*

$$0 \rightarrow (F_X^{n+1})_*\mathcal{W}_1\omega_{\underline{X}/\underline{k}}^i/\mathcal{B}_n^i \rightarrow \text{gr}_V^n \mathcal{W}\omega_{\underline{X}/\underline{k}}^i \xrightarrow{\beta} (F_X^{n+1})_*\mathcal{W}_1\omega_{\underline{X}/\underline{k}}^{i-1}/\mathcal{Z}_n^{i-1} \rightarrow 0.$$

*In particular, if  $\underline{X}$  is moreover log smooth of log Cartier type or, more generally, if  $\mathcal{W}_1\omega_{\underline{X}/\underline{k}}^i$  is a coherent  $\mathcal{O}_X$ -module, then every term in the short exact sequence is coherent. Consequently,  $\mathcal{W}_n\omega_{\underline{X}/\underline{k}}^i$  is coherent for each  $n$ .*

**Proof.** This follows from the discussion above. Note that if  $\underline{X}$  is log smooth of log Cartier type, then  $\mathcal{W}\omega_{\underline{X}/\underline{k}}^1 \cong \omega_{\underline{X}/\underline{k}}^1$  is coherent by Corollary 6.8. □

**Remark 6.10.** Although it might be possible to develop the theory directly by working with sheaves of log Dieudonné algebras over  $X_{\text{ét}}$ , we prefer to work locally and globalise in the last step (which avoids sheafifying at each step). For many purposes it is in fact crucial for us to construct the saturated log de Rham–Witt complexes in local charts. In particular, for the application in [20], we do not know how to extract log structures from certain  $A_{\text{inf}}$ -cohomology unless restricting to sufficiently small local charts of the log scheme. We refer the interested reader to [20] for details.

### 7. Comparison theorems

The goal of the present section is to show that the saturated log de Rham–Witt complexes agree with existing constructions for sufficiently smooth log schemes. In this section, we

assume that our log schemes (for example,  $\underline{X}$  and  $\underline{Y}$ ) are log-smooth of log-Cartier type over  $\underline{k} = (k, N)$ , where the (pre-)log structure on the perfect field  $k$  satisfies  $N \setminus \{0\} \mapsto 0$ . In particular, all log structures are assumed to be fine and thus quasi-coherent.

**7.1. Comparison with the constructions of Hyodo–Kato**

In [9], Hyodo–Kato constructed a log de Rham–Witt complex  $W_n^{\text{HK}} \omega_{\underline{X}/\underline{k}}^*$  using the log crystalline site, as a generalisation of [11].

We briefly recall their construction. Denote by  $\underline{W}_n$  the log algebra  $W_n(\underline{k})$ , which is equipped with the standard PD structure. Denote by  $(\underline{X}/\underline{W}_n)_{\text{log-cris}}$  the log crystalline site of  $\underline{X}$  over  $\underline{W}_n$ , which consists of PD-thickenings equipped with a compatible log structure. More precisely, objects in  $(\underline{X}/\underline{W}_n)_{\text{log-cris}}$  are tuples of the form

$$(U, T, M_T, i, \delta),$$

where  $U$  is an étale scheme over  $X$ ,  $(T, M_T)$  is a fine log scheme over  $\underline{W}_n$ ,  $i : (U, M_{XU}) \rightarrow (T, M_T)$  is an exact closed immersion (cf. Section A) and  $\delta$  is a PD structure on (defining ideal of) the closed subscheme  $U \subset T$  that is compatible with the PD structure on  $\underline{W}_n$  (see [13]). The log crystalline site is naturally equipped with a canonical projection to the étale site

$$u_{\underline{X}/\underline{W}_n}^{\text{log}} : (\underline{X}/\underline{W}_n)_{\text{log-cris}} \rightarrow X_{\text{ét}}.$$

Hyodo–Kato then defined their (sheaf of) log de Rham–Witt complexes to be

$$W_n^{\text{HK}} \omega_{\underline{X}/\underline{k}}^i := R^i u_{\underline{X}/\underline{W}_n, *}^{\text{log}} (\mathcal{O}_{\underline{X}/\underline{W}_n}),$$

where  $\mathcal{O}_{\underline{X}/\underline{W}_n}$  is the structure sheaf on the log crystalline site. These objects are equipped with (collections of) operators  $d, F, V$  and the ‘canonical projections’  $R$ .

To explain these operators in slightly more detail, we consider the following crystalline complex  $C_n^*$ , following [9, Section 4.1 & 4.2].

**Construction 7.1** (Hyodo–Kato). Setup as above. Let  $\underline{X}^\bullet \hookrightarrow \underline{Z}^\bullet$  be an ‘embedding system’ for the log scheme  $\underline{X}$  over  $\underline{W}_n$ ; that is, a pair of simplicial log schemes  $\underline{X}^\bullet = (X^\bullet, M^\bullet)$  and  $\underline{Z}^\bullet = (Z^\bullet, N^\bullet)$  (with fine log structures), endowed with structure maps

$$\underline{X}^\bullet \rightarrow \underline{X}, \quad \underline{Z}^\bullet \rightarrow \text{Spec } \underline{W}_n$$

and closed immersions

$$\underline{X}^\bullet \hookrightarrow \underline{Z}^\bullet$$

over  $\underline{X} \rightarrow \text{Spec } \underline{W}_n$ , such that

- Each  $\underline{Z}^i = (Z^i, N^i)$  is log smooth over  $\underline{W}_n$ .
- The map  $X^\bullet \rightarrow X$  of schemes forms a hypercover in the étale site, with  $M^i = M_X|_{X^i}$ .

For such an embedding system, let  $C_n^* = C_{\underline{X}/\underline{W}_n}^*$  be the following complex on  $X_{\text{ét}}^\bullet$ :

$$C_n^* := (\mathcal{O}_D^\bullet \xrightarrow{\nabla} \mathcal{O}_D^\bullet \otimes_{\mathcal{O}_Z^\bullet} \omega_{\underline{Z}^\bullet/\underline{W}_n}^1 \xrightarrow{\nabla} \mathcal{O}_D^\bullet \otimes_{\mathcal{O}_Z^\bullet} \omega_{\underline{Z}^\bullet/\underline{W}_n}^2 \xrightarrow{\nabla} \dots).$$

Here  $D^\bullet$  is the log PD envelop of  $\underline{X}^\bullet \hookrightarrow \underline{Z}^\bullet$ . This complex is referred to as the ‘(log) crystalline complex’ of  $\underline{X}/W_n$  and computes  $Ru_{\underline{X}/W_n,*}^{\text{log}}(\mathcal{O}_{\underline{X}/W_n})$ . More precisely, let  $\theta : X_{\text{ét}}^\bullet \rightarrow X_{\text{ét}}$  be the projection from the simplicial étale site of  $X^\bullet$  to the étale site of  $X$ ; then we have

$$Ru_{\underline{X}/W_n,*}^{\text{log}}(\mathcal{O}_{\underline{X}/W_n}) \cong R\theta_* C_n^*$$

(cf. ([9, Proposition 2.20])). Now let us return to Hyodo–Kato’s construction of the operators  $d, F, V, R$  on their log de Rham–Witt complex.

- First, define the differential  $d : W_n^{\text{HK}}\omega_{\underline{X}/k}^i \rightarrow W_n^{\text{HK}}\omega_{\underline{X}/k}^{i+1}$  to be the ‘Bockstein differential’ induced from the exact sequence  $0 \rightarrow C_n^* \xrightarrow{p^n} C_{2n}^* \rightarrow C_n^* \rightarrow 0$ .
- Let  $F : W_n^{\text{HK}}\omega_{\underline{X}/k}^i \rightarrow W_{n-1}^{\text{HK}}\omega_{\underline{X}/k}^i$  be the map induced by  $C_n^* \rightarrow C_{n-1}^*$ .
- Let  $V : W_{n-1}^{\text{HK}}\omega_{\underline{X}/k}^i \rightarrow W_n^{\text{HK}}\omega_{\underline{X}/k}^i$  be the map induced by  $C_{n-1}^* \xrightarrow{p} C_n^*$ .
- Finally, we define  $R_n : W_n^{\text{HK}}\omega_{\underline{X}/k}^i \rightarrow W_{n-1}^{\text{HK}}\omega_{\underline{X}/k}^i$ , which is the most subtle piece of structure, as follows. The starting point is to consider a certain subcomplex ‘ $\eta_p C_{n-1}^* \subset C_{n-1}^*$ ’,<sup>13</sup> which is defined to be  $(\eta_p C_m)^*/p^{n-1}$  for  $m$  sufficiently large, where

$$\eta_p C_m^i := \{x \in p^i C_m^i : dx \in p^{i+1} C_m^{i+1}\}$$

as usual. For  $m > n + i + 1$ , the quotient  $(\eta_p C_m)^i/p^{n-1}$  becomes well defined, and Hyodo–Kato constructed a quasi-isomorphism ([9, Lemma 2.25])  $C_{n-1}^* \rightarrow \eta_p C_{n-1}^*$  that induces an isomorphism

$$\psi : H^i(C_{n-1}^*) \xrightarrow{\sim} H^i(\eta_p C_{n-1}^*).$$

Now, let  $\mu_n : H^i(C_n^*) \rightarrow H^i(\eta_p C_{n-1}^*)$  be the map induced by  $x \mapsto p^i x$  on the level of complexes. Finally, define

$$R_n := \mu_n \circ \psi^{-1}.$$

We then take the inverse limit along the restriction maps to form the Hyodo–Kato complex  $W^{\text{HK}}\omega_{\underline{X}/k}^* := \varprojlim_R W_n^{\text{HK}}\omega_{\underline{X}/k}^*$ .

**Theorem 7.2.** *Let  $\underline{X}$  be a log scheme over  $k$  that is log-smooth of log-Cartier type. There is a natural isomorphism of sheaves of log Dieudonné algebras*

$$\mathcal{W}\omega_{\underline{X}/k}^* \xrightarrow{\sim} W^{\text{HK}}\omega_{\underline{X}/k}^*.$$

**Remark 7.3.** In particular,  $\mathcal{W}\omega_{\underline{X}/k}^*$  computes the log crystalline cohomology for quasi-coherent log-smooth schemes of log-Cartier type.

To prove the theorem it suffices to work locally. The main point is to show that  $W^{\text{HK}}\omega_{\underline{X}/k}^*$  indeed takes values in log Dieudonné algebras. This is straightforward but quite tedious to carry out, which we prove in Lemmas 7.4 and 7.5. Now we restrict our

<sup>13</sup>The complex ‘ $\eta_p C_{n-1}^*$ ’ is denoted by  $E_{n-1}^*$  in [9]. We have chosen this notation because in modern language, their complex is essentially obtained by lifting  $C_{n-1}^*$  to characteristic 0 and then applying the décalage operator  $\eta_p$  to it.

attention to the affine case and let  $\underline{Y} = (\text{Spec } R, M)$  be an affine log scheme over  $\underline{k}$  and consider the Hyodo–Kato complex

$$W^{\text{HK}}\omega_{\underline{R}/\underline{k}}^* := \Gamma(Y, W^{\text{HK}}\omega_{\underline{Y}/\underline{k}}^*) = \varinjlim_{R_n} \Gamma(Y, W_n^{\text{HK}}\omega_{\underline{Y}/\underline{k}}^*).$$

The canonical projections  $R_n$  commute with operators  $d$ ,  $F$  and  $V$  on each finite level from the definitions; hence, we obtain a cochain complex in the inverse limit, equipped with  $F$  and  $V$ . By relations given in [9, Display 4.1.1],  $W^{\text{HK}}\omega_{\underline{R}/\underline{k}}^*$  is a Dieudonné complex.

**Lemma 7.4.** *The Frobenius  $F$  on  $W^{\text{HK}}\omega_{\underline{R}/\underline{k}}^*$  is a graded ring homomorphism.*

This is, of course, of no surprise and probably well known to experts, but we cannot track down an explicit proof, so we record the proof here.

**Proof.** For notational simplicity we write  $W^{\text{HK},*}$  (respectively  $W_n^{\text{HK},*}$ ) for the complex  $W^{\text{HK}}\omega_{\underline{R}/\underline{k}}^*$  (respectively  $W_n^{\text{HK}}\omega_{\underline{R}/\underline{k}}^* := \Gamma(Y, W_n^{\text{HK}}\omega_{\underline{Y}/\underline{k}}^*)$ ) in this proof. By [9, Proposition 4.7] and its proof, there is a canonical surjection

$$\psi : \omega_{W_n(\underline{R})/W_n(\underline{k})}^* \twoheadrightarrow W_n^{\text{HK},*}$$

with kernel  $I_n \subset \omega_{W_n(\underline{R})/W_n(\underline{k})}^*$  a graded differential ideal, generated by elements of the form  $\{\eta_{i,j,a,m}, d\eta_{i,j,a,m}\}$  for  $0 \leq j \leq i < n, a \in R, m \in M$ , where

$$\eta_{i,j,a,m} := V^i([a])dV^j([\alpha(m)]) - V^i([\alpha(m)^{p^{i-j}}])d\log(m).$$

To further ease notation, we write  $\omega_{W_n(\underline{R})}^* := \omega_{W_n(\underline{R})/W_n(\underline{k})}^*$ . The map  $\psi$  is compatible with the canonical projection maps. By (the corrected version in [18, Section 7] of) [9, Diagram 4.9.1], we have the following isomorphism of cochain complexes:

$$s = \psi \circ t : (W_n^{\text{HK},*})'' \xrightarrow{t} \omega_{W_n(\underline{R})}^*/I_n \xrightarrow{\psi} W_n^{\text{HK},*}$$

where  $(W^{\text{HK},i})''$  is defined as a quotient of<sup>14</sup>

$$(W_n(R) \otimes \wedge^i M^{\text{gp}}/N^{\text{gp}}) \oplus (W_n(R) \otimes \wedge^i M^{\text{gp}}/N^{\text{gp}}).$$

By Remark 4.8, we have a graded ring homomorphism  $F : \omega_{W_n(\underline{R})}^* \rightarrow \omega_{W_{n-1}(\underline{R})}^*$ . Unwinding its construction,  $F$  sends the ideal  $I_n$  into  $I_{n-1}$ , thus inducing a graded ring homomorphism

$$F : \omega_{W_n(\underline{R})}^*/I_n \rightarrow \omega_{W_{n-1}(\underline{R})}^*/I_{n-1}.$$

We claim that the following diagram (on the right) commutes:

$$\begin{array}{ccccc} (W_n^{\text{HK},*})'' & \xrightarrow{t} & \omega_{W_n(\underline{R})}^*/I_n & \xrightarrow{\psi} & W_n^{\text{HK},*} \\ \downarrow F & & \downarrow F & & \downarrow F \\ (W_{n-1}^{\text{HK},*})'' & \xrightarrow{t} & \omega_{W_{n-1}(\underline{R})}^*/I_{n-1} & \xrightarrow{\psi} & W_{n-1}^{\text{HK},*} \end{array}$$

<sup>14</sup>The complex  $(W^{\text{HK},i})'$  that appears in [9, Proposition 4.6] is incorrect (and the proof given there is incomplete); instead, we should take the modification  $(W^{\text{HK},i})''$  given in [18, Section 7]. With this correction, the map  $s$  is indeed an isomorphism, by [18, Theorem 7.5].

where  $F : (W_n^{\text{HK},i})'' \rightarrow (W_{n-1}^{\text{HK},i})''$  is defined as follows: for any  $x \in W_n(R)$  and  $y \in W_n(R)^\times$ , let us write  $y = [y_0] + Vy'$ , and send

$$\begin{aligned} (x \otimes m_1 \wedge \cdots \wedge m_i, 0) &\mapsto (F(x) \otimes m_1 \wedge \cdots \wedge m_i, 0) \\ (0, y \otimes m_2 \wedge \cdots \wedge m_i) &\mapsto ([y_0]^p \otimes \beta^{-1}([y_0]) \wedge m_2 \wedge \cdots \wedge m_i, 0) + (0, y' \otimes m_2 \wedge \cdots \wedge m_i). \end{aligned}$$

$\beta$  here denotes the log structure  $M \xrightarrow{\alpha} R \xrightarrow{[\ ]} W(R)$ , where the first arrow  $\alpha$  is a log structure by assumption, so there exists a unique element  $\beta^{-1}([y_0]) \in M$  that is sent to  $[y_0]$ .

The desired commutativity follows from [18, Remark 7.7.4] and the commutativity of the left square, which we check by hand on the generators of  $(W_n^{\text{HK},i})''$ . Clearly,  $F \circ t = t \circ F$  on  $(x \otimes m_1 \wedge \cdots \wedge m_i, 0)$ ; for  $(0, y \otimes m_2 \wedge \cdots \wedge m_i)$ , we simply observe that

$$Fdy = Fd([y_0] + Vy') = [y_0]^p d \log [y_0] + dy'.$$

This finishes the proof. □

**Lemma 7.5.**  $W^{\text{HK}}\omega_{\underline{R}/\underline{k}}^*$  is a saturated Dieudonné algebra.

**Proof.** Retain notations from the proof of Lemma 7.4. By [9, Corollary 4.5 (2)],  $W^{\text{HK}}\omega^*$  is  $p$ -torsion free. Because  $\underline{R}/\underline{k}$  is log-smooth of log-Cartier type, by the same proof of [10, Isomorphism I.3.11.4] in the case  $n = 0$ , the Cartier isomorphism (cf. Proposition A.10) implies that

$$\ker(d : W_1^{\text{HK}}\omega^i \rightarrow W_1^{\text{HK}}\omega^{i+1}) = \text{im}(F : W_2^{\text{HK}}\omega^i \rightarrow W_1^{\text{HK}}\omega^i).$$

This in turn implies that  $d^{-1}(p \cdot W^{\text{HK}}\omega^*) = F(W^{\text{HK}}\omega^*)$ . To see this, suppose that  $dx \in p \cdot W^{\text{HK}}\omega^*$ ; then by the equality above we have  $x = Fx'_1 + Vx'_2 + dVy'_2 = Fx_1 + Vx'_2$  where  $x_1 = x'_1 + dV^2(y'_2)$ . This implies that  $dVx'_2 \in p \cdot W^{\text{HK}}\omega^*$ , so  $dx'_2 = FdVx'_2 \in p \cdot W^{\text{HK}}\omega^*$  and we may write  $x'_2 = Fx_2 + Vx'_3$ . By repeating this procedure we may write  $x = F(x_1 + Vx_2 + V^2x_3 + \cdots)$ . This; together with the injectivity of  $F$  (because  $W^{\text{HK}}\omega^*$  is  $p$ -torsion free); implies that  $W^{\text{HK}}\omega^*$  is saturated (as a Dieudonné complex).

It remains to show that  $F(x) \equiv x^p \pmod{p}$  for all  $x \in W^{\text{HK}}\omega^0$ . Because  $W^{\text{HK}}\omega^*$  is saturated, in fact it suffices to check that  $F(x) \equiv x^p \pmod{V}$  for all  $x \in W^{\text{HK}}\omega^0$ . But this follows from the commutative diagram in the proof of Lemma 7.4, because the middle vertical map is constructed to extend the Witt vector Frobenius and the map  $\psi$  preserves the  $V$ -filtration (by Proposition 7.1 and diagram 7.5.6 in [18]). □

**Corollary 7.6.**  $W^{\text{HK}}\omega_{\underline{R}/\underline{k}}^*$  admits a natural structure of a log Dieudonné algebra, which is strict (in particular saturated) and  $p$ -compatible.

**Proof.** For each  $n$ , the log structure  $[\alpha]_n : M \xrightarrow{\alpha} R \xrightarrow{[\ ]} W_n(R)$  induces a log derivation  $(d, \delta_n : M \rightarrow \omega_{W_n(\underline{R})}^1)$ . By composing with projection, we get a monoid map  $\delta_n : M \rightarrow \omega_{W_n(\underline{R})}^1/I_n$ . Both  $[\alpha_n]$  and  $\delta_n$  are compatible under the canonical projections. Now the map  $\psi$  in

$$(W_n^{\text{HK},*})'' \xrightarrow{t} \omega_{W_n(\underline{R})}^*/I_n \xrightarrow{\psi} W_n^{\text{HK},*}$$

(from the proof of Lemma 7.4) is also compatible with the canonical projections. Therefore, we have monoid morphisms (which we still denote by the same symbols):  $[\alpha]_n : M \rightarrow W_n^{\text{HK}}\omega^0$  and  $\delta_n : M \rightarrow W_n^{\text{HK}}\omega^1$ . In the inverse limit they give rise to

$$[\alpha] : M \rightarrow W^{\text{HK}}\omega^0, \quad \delta : M \rightarrow W^{\text{HK}}\omega^1.$$

This makes  $W^{\text{HK}}\omega^*$  a  $p$ -compatible log Dieudonné algebra. It is saturated by Lemma 7.5, and it is strict because for each  $n \geq 1$  and any  $k \geq 0$ , we have the following exact sequence:

$$0 \rightarrow \text{im} \left( W_k^{\text{HK}}\omega^* \xrightarrow{V^n+dV^n} W_{n+k}^{\text{HK}}\omega^* \right) \longrightarrow W_{n+k}^{\text{HK}}\omega^* \xrightarrow{p^f} W_n^{\text{HK}}\omega^* \rightarrow 0$$

by [9, Section 4.9]. □

Now it is straightforward to prove Theorem 7.2. As discussed previously, it suffices to work locally on charts. In the affine setup, the identity map

$$\underline{R} \longrightarrow W_1^{\text{HK}}\omega_{\underline{R}/\underline{k}}^0 \cong \underline{R}$$

induces a map of strict log Dieudonné algebras

$$\gamma : \mathcal{W}\omega_{\underline{R}/\underline{k}}^* \longrightarrow W^{\text{HK}}\omega_{\underline{R}/\underline{k}}^*.$$

Theorem 7.2 follows from the next proposition.

**Proposition 7.7.**  *$\gamma$  is an isomorphism of log Dieudonné algebras*

$$\gamma : \mathcal{W}\omega_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} W^{\text{HK}}\omega_{\underline{R}/\underline{k}}^*.$$

**Proof.** The induced map  $\bar{\gamma} : \mathcal{W}_1\omega_{\underline{R}/\underline{k}}^* \rightarrow W_1^{\text{HK}}\omega_{\underline{R}/\underline{k}}^*$  is an isomorphism by Theorem 5.8. The theorem then follows from Corollary 2.16 part (3). □

**7.2. Comparison with the constructions of Matsuue**

As before, we are under the assumption that  $\underline{R}/\underline{k}$  is log-smooth of log-Cartier type. Matsuue’s construction of log de Rham–Witt complex is an extension of the construction of Langer–Zink [14]. We start with a review of the definition of the category  $\mathcal{C}_{FV}$  of ( $R$ -framed) log F- $V$ -procomplexes.

**Definition 7.8** ([15] Definition 3.4). Let  $\underline{R} = (R, M)$  be a log algebra over  $\underline{k} = (k, N)$ . An  $R$ -framed F- $V$ -procomplex over  $\underline{R}/\underline{k}$  is a projective system

$$\dots \rightarrow E_{m+1}^* \xrightarrow{R_m} E_m^* \xrightarrow{R_{m-1}} \dots \rightarrow E_1^* \rightarrow E_0^* = 0$$

where each  $E_m^* = (E_m^*, d_m, \delta_m)$  is a log cdga over  $W_m(\underline{R})/W_m(\underline{k})$ , together with

- a collection  $F : E_{m+1}^* \rightarrow E_m^*$  of graded ring homomorphisms;
- a collection  $V : E_m^* \rightarrow E_{m+1}^*$  of graded abelian group homomorphisms.

These data are required to satisfy the following conditions:

- (1)  $R = \{R_m\}$  is compatible with  $\delta = \{\delta_m\}$ ; that is,  $\delta_m = R_m \circ \delta_{m+1}$  for  $m \geq 0$ .
- (2)  $R$  is compatible with the collection of maps  $F$  and  $V$ .
- (3) The structure maps  $\beta_m : W_m(R) \rightarrow E_m^0$  are compatible with  $F$  and  $V$ .

(4) The collections of  $F, V, d, \delta$  satisfy relations

$$\begin{aligned}
 FV = p \quad Fd_{m+1}V = d_m \quad F\delta_{m+1} = \delta_m \\
 (Vx)y = V(xFy) \quad Fd_{m+1}[x] = [x]^{p-1}d_m[x].
 \end{aligned}$$

Here, by the structure maps we mean the collection of ring homomorphisms  $\beta_m : W_m(R) \rightarrow E_m^0$  given as part of the data of the  $W_m(\underline{R})/W_m(\underline{k})$ -cdga  $(E_m^*, d_m, \delta_m)$  where  $(d_m \circ \beta_m, \delta_m : M \rightarrow E_m^1)$  forms a log derivation of  $W_m(\underline{R})/W_m(\underline{k})$  into  $E^1$  and satisfies  $d_m \circ \delta_m = 0$ .

Denote the category of log F-V-procomplexes over  $\underline{R}/\underline{k}$  by  $\mathcal{C}_{FV, \underline{R}}$ . In [15], Matsue constructed an initial object  $\{W_m \Lambda_{\underline{R}/\underline{k}}^*\}$  in  $\mathcal{C}_{FV, \underline{R}}$ , where each  $W_m \Lambda_{\underline{R}/\underline{k}}^*$  is constructed as a certain quotient of  $\omega_{W_m(\underline{R})/W_m(\underline{k})}^*$ . The complexes  $W \Lambda_{\underline{R}/\underline{k}}^* = \lim_n W_n \Lambda_{\underline{R}/\underline{k}}^*$  are then globalised to a sheaf  $W \Lambda_{\underline{X}/\underline{k}}^*$  on  $X_{\text{ét}}$ . As in [14], the construction in [15] works more generally over nonperfect bases, but we have restricted to the case of perfect base (in fact a perfect field) to compare with our version of saturated log de Rham–Witt complexes. The following lemma is clear.

**Lemma 7.9.** *The saturated log de Rham–Witt complexes  $\mathcal{W} \omega_{\underline{R}/\underline{k}}^*$  give rise to an  $R$ -framed F-V-procomplex  $\{W_m \omega_{\underline{R}/\underline{k}}^*\}$  over  $\underline{R}/\underline{k}$ .*

Therefore, by the universal property of  $W \Lambda^*$  we have a map  $\gamma' : W \Lambda_{\underline{R}/\underline{k}}^* \rightarrow \mathcal{W} \omega_{\underline{R}/\underline{k}}^*$  of cochain complexes that is compatible with  $F, V$  and  $R$ .

**Proposition 7.10.**  *$\gamma'$  is an isomorphism.*

**Proof.** The map  $\gamma'$  induces commutative diagrams

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Fil}^n W_{n+1} \Lambda_{\underline{R}/\underline{k}}^* & \longrightarrow & W_{n+1} \Lambda_{\underline{R}/\underline{k}}^* & \longrightarrow & W_n \Lambda_{\underline{R}/\underline{k}}^* \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Fil}^n \mathcal{W}_{n+1} \omega_{\underline{R}/\underline{k}}^* & \longrightarrow & \mathcal{W}_{n+1} \omega_{\underline{R}/\underline{k}}^* & \longrightarrow & \mathcal{W}_n \omega_{\underline{R}/\underline{k}}^* \longrightarrow 0
 \end{array}$$

of short exact sequences, where the filtrations are standard filtrations, given by  $\text{Fil}^n W_{n+1} \Lambda_{\underline{R}/\underline{k}}^* = V^n W_1 \Lambda_{\underline{R}/\underline{k}}^* + dV^n W_1 \Lambda_{\underline{R}/\underline{k}}^{*-1}$  and likewise for  $\mathcal{W}_{n+1} \omega_{\underline{R}/\underline{k}}^*$ . By Theorem 5.8, we know that  $\gamma'$  induces an isomorphism  $W_1 \Lambda_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} \mathcal{W}_1 \omega_{\underline{R}/\underline{k}}^*$  and hence  $\text{Fil}^n W_{n+1} \Lambda_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} \text{Fil}^n \mathcal{W}_{n+1} \omega_{\underline{R}/\underline{k}}^*$  for each  $n$ . The proposition therefore follows by induction.  $\square$

**Corollary 7.11.** *For a quasi-coherent log scheme  $\underline{X}$  that is log-smooth of log-Cartier type over  $\underline{k}$ , the isomorphism  $\gamma'$  globalises to an isomorphism of sheaves  $\gamma' : W \Lambda_{\underline{X}/\underline{k}}^* \xrightarrow{\sim} \mathcal{W} \omega_{\underline{X}/\underline{k}}^*$ .*

**Remark 7.12.** Another way to obtain  $\mathcal{W} \omega_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} W \Lambda_{\underline{R}/\underline{k}}^*$  is to use the comparison with Hyodo–Kato complexes in Subsection 7.1 and prove that the Hyodo–Kato complexes agree with  $W \Lambda_{\underline{R}/\underline{k}}^*$ . The latter is claimed in Proposition 3.1 of [6], which asserts that there is an isomorphism  $W \Lambda_{\underline{R}/\underline{k}}^* \xrightarrow{\sim} W^{\text{HK}} \omega_{\underline{R}/\underline{k}}^*$  identifying the log structures  $F, V$  and  $R$  on both

sides. The key assertion used in their proof is that  $W_m^{\text{HK}}\omega_{\underline{R}/\underline{k}}^*$  form an  $R$ -framed F-V-procomplex, which is claimed without justification. It seems to the author that proving their claim would involve proving most of the arguments in Subsection 7.1, which seems nontautological (though probably known to experts).

### 8. The monodromy operator

In this section we explain a construction of the monodromy operator on the log crystalline cohomology for log schemes of ‘generalised semistable type’ in the following sense: let  $\underline{k} = (k, \mathbb{N})$  be the standard log point,  $\underline{X}/\underline{k}$  be a log scheme such that étale locally,  $\underline{X}$  is naively étale (cf. Definition 6.1) over a finite fibre product of log schemes arising from log algebras of the form

$$\begin{array}{ccc} \underline{R}^\square & = & \left( \mathbb{N}^{r+1} \xrightarrow{\alpha} k[X_0, \dots, X_r, \dots, X_d] / \prod_{0 \leq i \leq r} X_i \right) \\ & & \uparrow \scriptstyle{1 \mapsto (1, \dots, 1)} \qquad \qquad \qquad \uparrow \\ \underline{k} & = & \left( \mathbb{N} \longrightarrow k \right). \end{array}$$

Here the log structure  $\alpha$  on  $R^\square$  is given by  $e_i \mapsto X_i$  for  $0 \leq i \leq r$ , where  $e_i = (0, \dots, 1, \dots, 0)$  with 1 at the  $i$ th position. More precisely,  $\underline{X}$  can be covered by charts that are naively étale over the spectrum of log algebras of the form  $\underline{S}^\square = (M, S^\square)$ , where

$$S^\square = k[X_{l,h}]_{\substack{1 \leq l \leq s \\ 0 \leq h \leq d_l}} / \left( \prod_{1 \leq i \leq d_1} X_{1,i}, \dots, \prod_{1 \leq i \leq d_s} X_{s,i} \right)$$

and  $M = \bigsqcup_{1 \leq l \leq s} \mathbb{N}^{r_l+1}$  is the pushout of the diagonal maps  $\mathbb{N} \rightarrow \mathbb{N}^{r_l+1}$  for  $1 \leq l \leq s$ . Clearly,  $R^\square$  (respectively  $S^\square$ ) is fine and log-smooth over  $\underline{k}$  of log-Cartier type.

**Lemma 8.1.** *Let  $\underline{R}^\square$  be as above and let  $k^\circ = (k, 0)$  be the trivial log point; then  $\underline{R}^\square/k^\circ$  satisfies conditions (1) and (2) in Remark 5.10. In particular, there is a canonical isomorphism*

$$\omega_{\underline{R}^\square/k^\circ} \xrightarrow{\sim} \mathcal{W}_1\omega_{\underline{R}^\square/k^\circ}.$$

The same conclusion holds for  $\underline{S}^\square$ .

**Proof.** For notational simplicity we check the claim for  $\underline{R}^\square$ . For (1) we may take the lift

$$A^\square = W(k)[X_0, \dots, X_d] / \prod_{0 \leq i \leq r} X_i$$

with log structure  $\mathbb{N}^{r+1} \xrightarrow{e_i \mapsto X_i} A^\square$ . For the lift of Frobenius we take the canonical lift  $\sigma$  on  $W(k)$  and  $X_i \mapsto X_i^p$ , and the Frobenius on  $\mathbb{N}^{r+1}$  is given by multiplication by  $p$ . The log differential  $\omega_{A^\square/k}^1$  is free over  $A$  with generators

$$d\log X_0, \dots, d\log X_r, dX_{r+1}, \dots, dX_d$$



(see Subsection A.2.2); hence, both conditions (1) and (2) in Remark 5.10 are satisfied.  $\square$

The canonical map  $\underline{R}^\square/k^\circ \rightarrow \underline{R}^\square/\underline{k}$  induces a morphism of saturated log de Rham–Witt complexes

$$\Theta : \mathcal{W}\omega_{\underline{R}^\square/k^\circ}^* \rightarrow \mathcal{W}\omega_{\underline{R}^\square/\underline{k}}^*.$$

The cochain complex  $\ker \Theta$  is concentrated in (cohomological) degree  $\geq 1$ .

**Example 8.2.** For a simple working example, let us consider the case where  $\underline{R}^\square = k[x, y]/(xy)$ . For this example,  $\mathcal{W}\omega_{\underline{R}^\square/\underline{k}}^*$  is isomorphic to the  $V$ -completion  $W(A_{\text{sat}}^0) \xrightarrow{d} W(A_{\text{sat}}^1)$  of the saturated log Dieudonné algebra

$$A_{\text{sat}}^0 = \{\omega \in A^0[F^{-1}] : d\omega \in A^1[F^{-1}]\} \xrightarrow{d} A_{\text{sat}}^1 = A^1[F^{-1}]$$

described in Example 3.16, and  $\mathcal{W}\omega_{\underline{R}^\square/k^\circ}^*$  is isomorphic to the  $V$ -completion of

$$B_{\text{sat}}^0 \xrightarrow{d} B_{\text{sat}}^1 \xrightarrow{d} B_{\text{sat}}^2$$

described in Example 3.17. The map  $\Theta$  constructed above can be described (before taking  $V$ -completions) as follows:

- $\Theta : B_{\text{sat}}^0 \xrightarrow{\sim} A_{\text{sat}}^0$  is an isomorphism and sends  $x^{\frac{1}{p^m}} \mapsto x^{\frac{1}{p^m}}$  and  $y^{\frac{1}{p^m}} \mapsto y^{\frac{1}{p^m}}$ .
- $\Theta : B_{\text{sat}}^1 \rightarrow A_{\text{sat}}^1$  sends

$$\left(f \frac{dx}{x}, g \frac{dy}{y}\right) \mapsto (f - g) \frac{dx}{x}.$$

This map is clearly surjective with kernel

$$\ker(\Theta : B_{\text{sat}}^1 \rightarrow A_{\text{sat}}^1) = \left\{ \left(f \frac{dx}{x}, f \frac{dy}{y}\right) : \text{where } f \in A_{\text{sat}}^0(x) \cup_{\mathbb{Z}_p} A_{\text{sat}}^0(y) = A_{\text{sat}}^0 \right\}.$$

In other words, we obtain the following short exact sequence of saturated complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{\text{sat}}^0 & \longrightarrow & A_{\text{sat}}^1 & & \\ & & \downarrow f \mapsto f\left(\frac{dx}{x}, \frac{dy}{y}\right) & & \downarrow z \mapsto z \wedge \left(\frac{dx}{x} + \frac{dy}{y}\right) & & \\ 0 & \longrightarrow & B_{\text{sat}}^0 & \longrightarrow & B_{\text{sat}}^1 & \longrightarrow & B_{\text{sat}}^2 \\ & & \downarrow \wr & & \downarrow & & \\ 0 & \longrightarrow & A_{\text{sat}}^0 & \longrightarrow & A_{\text{sat}}^1 & & \end{array}$$

After taking  $V$ -completion, we obtain a short exact sequence

$$0 \rightarrow \mathcal{W}\omega_{\underline{R}^\square/\underline{k}}^*[-1] \rightarrow \mathcal{W}\omega_{\underline{R}^\square/k^\circ}^* \xrightarrow{\Theta} \mathcal{W}\omega_{\underline{R}^\square/\underline{k}}^* \rightarrow 0.$$

This phenomenon continues to hold for all  $\underline{R}^\square$  of generalised semistable type.

**Proposition 8.3.** *The cochain complex  $(\ker \Theta)[1]$  admits a structure of a Dieudonné complex, which is saturated and strict. Moreover, there is a short exact sequence*

$$0 \rightarrow \mathcal{W}\omega_{\underline{R}/k}^*[-1] \rightarrow \mathcal{W}\omega_{\underline{R}/k^\circ}^* \xrightarrow{\Theta} \mathcal{W}\omega_{\underline{R}/k}^* \rightarrow 0.$$

*The same conclusion holds when  $\underline{R}$  is replaced by  $\underline{S}$ . In particular,  $(\ker \Theta)[1]$  carries a (log) Dieudonné algebra structure.*

**Proof.** Again we check the proposition for  $\underline{R}$  because the only complication for  $\underline{S}$  is notational. Write  $K^* = \ker \Theta$ , which is a  $p$ -torsion-free sub-Dieudonné complex of  $\mathcal{W}\omega_{\underline{R}/k^\circ}^*$  (here we ignore the algebra structure). It is straightforward to check that

$$\phi_F : K^* \rightarrow \eta_p(K^*), \quad x \in K^i \mapsto p^i F(x)$$

is an isomorphism; therefore,  $K^*[1]$  is a saturated Dieudonné complex as multiplication by  $p$  gives an isomorphism  $\eta_p(K^*[1]) \xrightarrow{\sim} \eta_p(K^*)[1]$ . By Lemma 8.4, we have a short exact sequence

$$0 \rightarrow W_m(K^*) \rightarrow \mathcal{W}_m\omega_{\underline{R}/k^\circ}^* \rightarrow \mathcal{W}_m\omega_{\underline{R}/k}^* \rightarrow 0$$

for each  $m$ . Taking the inverse limit, we see that  $K^*[1]$  is strict.

Next we construct a map of cochain complexes  $\omega_{\underline{A}/k}^* \rightarrow \omega_{\underline{A}/k^\circ}^*[1]$  preserving Frobenius on both sides. For this we define an element  $\theta \in \omega_{\underline{A}/k^\circ}^1$  from the log differential  $d\log : \mathbb{N}^{r+1} \rightarrow \omega_{\underline{A}/k^\circ}^1$  by

$$\theta := d\log(e_0) + \dots + d\log(e_r).$$

(More generally, for the product  $\underline{S} = (S^\square, M)$ , we have a map  $\mathbb{N} \rightarrow M$  coming from each diagonal map  $\mathbb{N} \rightarrow \mathbb{N}^{r_i+1}$ . Let  $e \in M$  be the image of  $1 \in \mathbb{N}$ . Then  $\theta$  is the element  $d\log(e)$ .) We abuse notation and use the same symbol to denote the image of  $\theta$  in  $\mathcal{W}\omega_{\underline{R}/k^\circ}^*$ ; then clearly  $\theta \in K^1$ . Because  $\omega_{\underline{A}/k}^*$  is generated over  $A^\square$  by elements of the form  $(\wedge_i d\log X_i) \wedge (\wedge_j dX_j)$ , we construct a map of complexes  $\Psi : \omega_{\underline{A}/k}^* \rightarrow \omega_{\underline{A}/k^\circ}^*[1]$  by

$$(\wedge_i d\log X_i) \wedge (\wedge_j dX_j) \mapsto (\wedge_i d\log X_i) \wedge (\wedge_j dX_j) \wedge \theta.$$

This map is well defined, because

$$\sum_{0 \leq i \leq r} d\log X_i \mapsto \left( \sum_{0 \leq i \leq r} d\log X_i \right) \wedge \theta = \theta \wedge \theta = 0.$$

Moreover, because  $d(\theta) = 0$  (by  $d \circ d\log = 0$ ) and  $F(\theta) = \theta$ , the map  $\Psi$  preserves differentials and Frobenius structures on both sides.

By passing to the ( $p$ -adic) completion and strictification (complete saturation), we get a map  $\Psi : \mathcal{W}\omega_{\underline{R}/k}^* \rightarrow \mathcal{W}\omega_{\underline{R}/k^\circ}^*[1]$ . The image of  $\Psi$  lies in  $K^*[1]$ , so we have now constructed a map of strict Dieudonné complexes

$$\Psi : \mathcal{W}\omega_{\underline{R}/k}^* \rightarrow K^*[1].$$

To finish the proof of the proposition, it remains to prove that this map is an isomorphism. By [2, Corollary 2.7.4],<sup>15</sup> it suffices to show that

$$W_1(\Psi) : \mathcal{W}_1\omega_{\underline{R}^\square/k}^* \longrightarrow W_1(K^*[1])$$

is an isomorphism. By Lemma 8.4 and Lemma 8.1, we need to check that

$$\omega_{\underline{R}^\square/k}^* \xrightarrow{\wedge^{\theta_1}} \ker(\omega_{\underline{R}^\square/k^\circ}^* \rightarrow \omega_{\underline{R}^\square/k}^*)[1]$$

is an isomorphism of cochain complexes, which follows from their explicit descriptions.  $\square$

The following lemma is used in the proof above.

**Lemma 8.4.** *Let  $\Theta : A^* \rightarrow B^*$  be a map between two saturated Dieudonné complexes. Let  $K^* = \ker \Theta$  be the sub-Dieudonné complex (which is saturated). Define the standard filtration on  $A^*$  (respectively on  $B^*$  and  $K^*$ ) as in Subsection 3.4 by  $\text{Fil}^i(A^*) = V^i(A^*) + dV^i(A^*)$ . Then*

$$\text{Fil}^i(K^*) = \ker(\text{Fil}^i(A^*) \xrightarrow{\Phi} \text{Fil}^i(B^*)).$$

**Proof.** We need to show that  $\ker(\text{Fil}^i(A^*) \xrightarrow{\Phi} \text{Fil}^i(B^*)) \subset \text{Fil}^i(K^*)$ . In other words, if  $x = V^i y + dV^i z$ , and  $\Phi(x) = 0$ , we claim that  $x \in \text{Fil}^i(K^*)$ . For notational simplicity we write  $\bar{x} = \Phi(x)$  for the image of  $x \in A^*$  in  $B^*$ . By assumption we know that

$$d\bar{z} = F^i dV^i \bar{z} = -F^i(V^i \bar{y}) = -p^i \bar{y} \in B^*.$$

Because  $B^*$  is saturated, we know that  $\bar{z} = F^i \bar{w}$  for some  $\bar{w} \in B^*$ , so there exists  $w \in A^*$  and  $\alpha \in K^*$  such that

$$x = V^i y + dV^i(F^i w + \alpha) = V^i(y + F^i dw) + dV^i \alpha.$$

Therefore, it suffices to prove the claim for  $x$  of the form  $x = V^i y$ , but this is clear because  $V^i \bar{y} = 0$  implies that  $\bar{y} = 0$  as  $B^*$  is  $p$ -torsion free.  $\square$

We now prove Theorem 9 in the Introduction.

**Corollary 8.5.** *Let  $\underline{X}/k$  be a log scheme of ‘generalised semistable type’; then there is a short exact sequence of cochain complexes*

$$0 \rightarrow \mathcal{W}\omega_{\underline{X}/k}^*[-1] \longrightarrow \mathcal{W}\omega_{\underline{X}/k^\circ}^* \longrightarrow \mathcal{W}\omega_{\underline{X}/k}^* \rightarrow 0$$

that preserves the Frobenius structures. As a consequence, this gives rise to a connecting homomorphism on the log crystalline cohomology

$$N : H_{\log\text{-cris}}^*(X/W(\underline{k})) \longrightarrow H_{\log\text{-cris}}^*(X/W(\underline{k})).$$

This operator satisfies  $N\varphi = p\varphi N$ .

**Proof.** Let  $R$  be a  $k$ -algebra with an étale coordinate; namely, an étale morphism  $S^\square \rightarrow R$  where  $S^\square$  has the form described as in the beginning of this section. Equip  $R$  with the

<sup>15</sup>See also Corollary 2.16. Note that the proof we cite from [2] is stated for Dieudonné complexes.

log structure from  $M \rightarrow S^\square$ . Then for each  $m$  we have

$$0 \rightarrow \mathcal{W}_m \omega_{\underline{R}/\underline{k}}^*[-1] \rightarrow \mathcal{W}_m \omega_{\underline{R}/k^\circ}^* \rightarrow \mathcal{W}_m \omega_{\underline{R}/\underline{k}}^* \rightarrow 0$$

from the étale base change of the corresponding sequence for  $\mathcal{W}_m \omega_{\underline{S}^\square/\underline{k}}^*$ . To globalise from these étale coordinates, we observe that the element  $\theta \in \mathcal{W} \omega_{\underline{R}/k^\circ}^1$  is independent of the choice of coordinates, because it can be described as the image of  $\text{dlog}(e)$  in  $\omega_{W(\underline{R})/W(k^\circ)}^1 \rightarrow W(\omega_{W(\underline{R})/W(k^\circ)}^*)_{\text{sat}}^1 =: \mathcal{W} \omega_{\underline{R}/k^\circ}^1$ , so the sequences above glue to an exact sequence of sheaves

$$0 \rightarrow \mathcal{W}_m \omega_{\underline{X}/\underline{k}}^*[-1] \rightarrow \mathcal{W}_m \omega_{\underline{X}/k^\circ}^* \rightarrow \mathcal{W}_m \omega_{\underline{X}/\underline{k}}^* \rightarrow 0.$$

Because there is no higher Rlim terms, taking the inverse limit we reach the conclusion. The relation  $N\varphi = p\varphi N$  follows from  $dF = pFd$  because  $\varphi$  is induced by  $\phi_F : x \in \mathcal{W} \omega_{\underline{X}/\underline{k}}^i \mapsto p^i F(x) \in \mathcal{W} \omega_{\underline{X}/\underline{k}}^i$ . □

**Remark 8.6.** When  $\underline{X}$  is of semistable type, our construction of the monodromy operator agrees with the monodromy operator constructed in [9, Section 1.5]. This follows from Theorem 7.2 and the description of  $(W_n \tilde{\omega}_Y^q)'$  in [9, Section 4.20]. Essentially, in this case, we give a more functorial description of Hyodo–Kato’s complex  $W_n \tilde{\omega}_Y^q$  as the saturated log de Rham–Witt complex of  $\underline{R}^\square$  over a trivial log point.

### 9. The Nygaard filtration and slopes of Frobenius

The goal of this section is to study slopes of iterated Frobenius  $\varphi^n$  acting on the cohomology  $\mathbb{H}^d(X_{\text{ét}}, \mathcal{W} \omega_{\underline{X}/\underline{k}}^*)$ , where  $\varphi$  is induced by  $\phi_F$  in Definition 2.3 and  $n \in \mathbb{Z}_{\geq 1}$ , by analysing the Nygaard filtration associated to the saturated log de Rham–Witt complex. In particular, we follow the idea of [19] to show that, under a mild finiteness assumption (see Subsection 9.4), the saturated Newton polygon  $\text{New}(\varphi)$  always lies on or above the saturated Hodge polygon  $\text{Hdg}(\underline{X}, n)$  (Definition 9.7). The latter recovers the usual Hodge polygon of (logarithmic) Hodge numbers when  $\underline{X}$  is log smooth of log-Cartier type. Our main result in this section is Theorem 9.8, which is a moderate generalisation of [19, Theorem 2.5] to log schemes. This, of course, specialises to the well-known result of Mazur and Ogus on Katz’s conjecture when  $n = 1$  (cf. [17, 1]. See [16] for an excellent introduction to this problem.)

#### 9.1. The Nygaard complexes and filtrations

Let  $(M^*, d, F)$  be a saturated Dieudonne complex concentrated in degree  $\geq 0$ . Following [19, Definition 1.2], we define the level  $n$  Nygaard complex  $M(r, n)^*$  associated to  $M^*$  to be

$$M(r, n)^* := (M^0 \xrightarrow{d} M^1 \xrightarrow{d} \dots \xrightarrow{d} M^{r-1} \xrightarrow{dV^n} M^r \xrightarrow{d} \dots).$$

Note that when  $r = 0$ , this recovers  $M^*$ . These Nygaard complexes are endowed with two natural maps

$$\tilde{F} : M(r, n)^* \rightarrow M(r + 1, n)^*, \tag{9}$$

$$\tilde{V} : M(r, n)^* \longrightarrow M(r - 1, n)^*, \tag{10}$$

defined by

$$\begin{array}{ccccccc}
 M(r, n)^* & = & & & & & \\
 \tilde{F} \downarrow & & & & & & \\
 M(r+1, n)^* & = & & & & & \\
 \\
 M^0 & \xrightarrow{d} & \dots & \xrightarrow{d} & M^{r-1} & \xrightarrow{dV^n} & M^r & \xrightarrow{d} & M^{r+1} & \xrightarrow{d} & \dots \\
 \downarrow \text{id} & & & & \downarrow \text{id} & & \downarrow F^n & & \downarrow p^n & & \downarrow p^n \\
 M^0 & \xrightarrow{d} & \dots & \xrightarrow{d} & M^{r-1} & \xrightarrow{d} & M^r & \xrightarrow{dV^n} & M^{r+1} & \xrightarrow{d} & \dots
 \end{array}$$

and

$$\begin{array}{ccccccc}
 M(r, n)^* & = & & & & & \\
 \tilde{V} \downarrow & & & & & & \\
 M(r-1, n)^* & = & & & & & \\
 \\
 M^0 & \xrightarrow{d} & \dots & \xrightarrow{d} & M^{r-2} & \xrightarrow{d} & M^{r-1} & \xrightarrow{dV^n} & M^r & \xrightarrow{d} & \dots \\
 \downarrow p^n & & & & \downarrow p^n & & \downarrow V^n & & \downarrow \text{id} & & \downarrow \text{id} \\
 M^0 & \xrightarrow{d} & \dots & \xrightarrow{d} & M^{r-2} & \xrightarrow{dV^n} & M^{r-1} & \xrightarrow{d} & M^r & \xrightarrow{d} & \dots
 \end{array}$$

We will be concerned mostly with the second family of maps  $\tilde{V}$ .

**Definition 9.1.** Let  $(M^*, d, F)$  be a saturated Dieudonné complex. We define its  $r$ th Nygaard filtration  $\mathcal{N}_n^r = \mathcal{N}_n^r(M^*)$  of level  $n$  to be the image of

$$\tilde{V}^r : M(r, n)^* \longrightarrow M^*.$$

In other words, we define

$$\mathcal{N}_n^r(M^*) := (p^{n(r-1)}V^n M^0 \rightarrow p^{n(r-2)}V^n M^1 \rightarrow \dots \rightarrow V^n M^{r-1} \rightarrow M^r \rightarrow \dots)$$

as a subcomplex of  $M^*$ .

**9.2. Relation to the Hodge and conjugate filtrations**

The Nygaard filtration of level  $n$  is closely related to the Hodge and conjugate (i.e., canonical) filtrations on  $W_n M^*$  by the following lemma and its corollary. We continue to assume that  $M^*$  is saturated.

**Lemma 9.2.** (1) (Conjugate). The natural ‘divided Frobenius’ map  $\varphi_r^n = \frac{\varphi^n}{p^{nr}}$

$$\varphi_r^n : \mathcal{N}_n^r(M^*) \rightarrow M^* \rightarrow M^*/p^n$$

sending  $x \in \mathcal{N}_n^r(M^i) \mapsto \varphi(x)/p^{nr} = F^n(x)/p^{n(r-i)}$  induces an isomorphism

$$\varphi_r^n : Gr_{\mathcal{N}_n}^r(M^*) \xrightarrow{\sim} \tau^{\leq r}(M^*/p^n)$$

where  $\tau^{\leq *}$  denotes the canonical truncation on cochain complexes.

(2) (Hodge). The multiplication by  $p^n$  map induces a short exact sequence of complexes

$$0 \rightarrow M^*/\mathcal{N}_n^r \xrightarrow{p^n} M^*/\mathcal{N}_n^{r+1} \rightarrow (M^{*\leq r}/p^n)/(V^n M^r[-r]) \rightarrow 0$$

where  $M^{*\leq r}$  denotes the stupid truncation and the last term denotes the following quotient of  $M^{*\leq r}/p^n$ :

$$M^0/p^n \rightarrow M^1/p^n \rightarrow \dots \rightarrow M^{r-1}/p^n \rightarrow M^r/V^n \rightarrow 0 \rightarrow \dots$$

**Proof.** Both claims are straightforward from definitions. We leave details to the reader. For the special case where  $n = 1$  in part (1), also see [2, Proposition 8.2.1]. □

**Corollary 9.3.** (1) (Conjugate). The divided Frobenius  $\varphi_r^n$  induces a quasi-isomorphism

$$\varphi_r^n : gr_{\mathcal{N}_n}^r(M^*) \xrightarrow{\sim} \tau^{\leq r} W_n M^*.$$

(2) (Hodge). Multiplication by  $n$  induces the following cofibre sequence:

$$M^*/\mathcal{N}_n^r \xrightarrow{p^n} M^*/\mathcal{N}_n^{r+1} \rightarrow W_n M^{*\leq r}.$$

**Proof.** By Proposition 2.11 (3), the natural projection

$$M^*/p^n \xrightarrow{\sim} W_n M^* \tag{11}$$

is a quasi-isomorphism for each  $n$ ; the first claim thus follows from part (1) of Lemma 9.2. For the second claim, we need to show that the projection of cochain complexes

$$(M^{*\leq r}/p^n)/V^n M^r[-r] \rightarrow W_n M^{*\leq r} = M^{*\leq r}/(V^n + dV^n)$$

is a quasi-isomorphism. Using the quasi-isomorphism (11) above, the only nontrivial term to check is the second last term ( $H^{r-1}$ ). Now, using (11) one more time, it suffices to show that

$$\frac{\ker(d : M^{r-1}/p^n \rightarrow M^r/V^n)}{\text{im}(d : M^{r-2}/p^n \rightarrow M^{r-1}/p^n)} \stackrel{?}{\cong} \frac{\ker(d : M^{r-1}/p^n \rightarrow M^r/p^n)}{\text{im}(d : M^{r-2}/p^n \rightarrow M^{r-1}/p^n)}. \tag{12}$$

But we can rewrite the respective kernels of  $d$  as follows:

$$\begin{aligned} \ker(d : M^{r-1}/p^n \rightarrow M^r/p^n) &= d^{-1}(p^n M^r)/p^n M^{r-1} \\ \ker(d : M^{r-1}/p^n \rightarrow M^r/V^n) &= d^{-1}(V^n M^r)/p^n M^{r-1}. \end{aligned}$$

The claim thus follows from part (1) of Lemma 2.6. This finishes the proof of the corollary. □

### 9.3. The Hodge and conjugate filtrations, part II

Following [19, Definition 1.3], we let  $\alpha(r, n)$  be the map of complexes

$$\alpha(r, n) : M(r, n)^* / \widetilde{F}M(r - 1, n)^* \longrightarrow W_n M^{*\geq r} \tag{13}$$

induced by the canonical projection and let  $\beta(r, n)$  denote the map

$$\beta(r, n) : M(r, n)^* / \widetilde{V}M(r + 1, n)^* \longrightarrow \tau^{*\leq r} W_n M^* \tag{14}$$

defined by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^0/p^n & \xrightarrow{d} & \dots & \xrightarrow{d} & M^{r-1}/p^n & \xrightarrow{dV^n} & M^r/V^n & \longrightarrow & 0 \\ & & \downarrow \text{pr} & & & & \downarrow \text{pr} & & \downarrow F^n & & \\ 0 & \longrightarrow & W_n M^0 & \xrightarrow{d} & \dots & \xrightarrow{d} & W_n M^{r-1} & \xrightarrow{d} & Z^r(W_n M^*) & \longrightarrow & 0. \end{array}$$

In both diagrams, ‘pr’ denotes the map induced by the canonical projection.

**Proposition 9.4** ([19, Theorem 1.5]). *Suppose that  $M^*$  is saturated; then both  $\alpha(r, n)$  and  $\beta(r, n)$  defined above are quasi-isomorphisms.*

**Remark 9.5.** This observation, essentially due to Nygaard, is the key to his proof of the relation between Frobenius slopes and the Hodge filtration. Nygaard’s original proof in [19] uses induction on  $r$  and eventually reduces to the following two facts about a saturated Dieudonné complex  $M^*$ :

- $d^{-1}(p^n M^r) = F^n M^{r-1}$ .
- $F^n$  induces an isomorphism  $W_n M^r \xrightarrow{\sim} \mathcal{H}^r(W_n M^*)$  (see Corollary 2.12).

We give a slightly simpler argument below, again relying on these two facts.

**Proof.** First we consider  $\alpha(r, n)$ . By the quasi-isomorphism (11), we need to show that

- (1)  $dV^n : M^{r-1}/F^n \rightarrow M^r/p^n$  is injective.
- (2) The following sequence is a short exact sequence:

$$0 \rightarrow \text{im}(dV^n) \hookrightarrow d^{-1}(p^n M^{r+1})/p^n M^r \xrightarrow{\text{pr}} \ker(d : W_n M^r \rightarrow W_n M^{r+1}) \rightarrow 0. \tag{15}$$

The first claim is clear by saturatedness. For the second claim, first suppose that  $x \in M^r$  satisfies  $dx = V^n y + dV^n z$ ; then by Lemma 2.6 we know that  $x - V^n z \in d^{-1}(V^n M^{r+1}) = d^{-1}(p^n M^r)$ . This shows that the projection map pr in the sequence (15) is surjective. Now suppose that  $x \in d^{-1}(p^n M^{r+1})$  satisfies  $\text{pr}(x) = 0$ , so we have  $x = V^n y + dV^n z$  for some  $y, z$ . After applying differential, we know that  $dV^n y = dx \in p^n M^{r+1}$ , so by the first claim we have  $y = F^n y'$  for some  $y' \in M^r$ . In other words, we have  $x = dV^n z + p^n y'$ ; thus, the kernel of pr is precisely  $\text{im}(dV^n)$ .

Next we turn to  $\beta(r, n)$ . Again using (11), it suffices to show

- (1)  $\ker(dV^n : M^{r-1}/p^n \rightarrow M^r/V^n) = \ker(d : M^{r-1}/p^n \rightarrow M^r/p^n)$ .
- (2)  $F^n$  induces an isomorphism  $F^n : M^r/(V^n + dV^n) = W_n M^r \xrightarrow{\sim} H^r(W_n M^*)$ .

The first claim is straightforward, because  $d$  factors as

$$d = F^n dV^n : M^{r-1}/p^n \xrightarrow{dV^n} M^r/V^n \xleftarrow{F^n} M^r/p^n.$$

The second claim is precisely Corollary 2.12. □

**9.4. Finiteness of cohomology**

In the rest of this section we study the slopes of power of Frobenius acting on the cohomology of the saturated log de Rham–Witt complexes. As in the Introduction, we let  $\underline{X}$  be a coherent proper log scheme over  $\underline{k}$ . Furthermore, we make the following assumption on the log scheme  $\underline{X}$ .

**Assumption.**

$\underline{X}$  is étale locally of the form  $\underline{R}$  that satisfies the two conditions in Remark 5.10. (\*\*)

Log-smooth schemes over  $\underline{k}$  of log-Cartier type clearly satisfy this assumption. Other examples include normal crossings  $(k[x,y]/(xy), \mathbb{N}^2)$  with the standard log structure over the trivial log point  $\underline{k}^\circ = (k, 0)$  (see Section 8). Assumption (\*\*) ensures the following finiteness results on the saturated log de Rham–Witt cohomology.

**Proposition 9.6.** *Let  $\underline{X}$  be a coherent proper log scheme satisfying Assumption (\*\*).*

- (1) *Each  $H^i(X, \mathcal{W}_n \omega_{\underline{X}/\underline{k}}^j)$  is of finite type over  $W_n(k)$ .*
- (2) *The canonical maps*

$$\begin{aligned} R\Gamma(X, \mathcal{W} \omega_{\underline{X}/\underline{k}}^j) &\longrightarrow \mathop{\mathrm{R}}\lim_{\leftarrow} R\Gamma(X, \mathcal{W}_n \omega_{\underline{X}/\underline{k}}^j) \\ H^i(X, \mathcal{W} \omega_{\underline{X}/\underline{k}}^j) &\longrightarrow \mathop{\mathrm{lim}}_{\leftarrow} H^i(X, \mathcal{W}_n \omega_{\underline{X}/\underline{k}}^j) \end{aligned}$$

*are (quasi-)isomorphisms.*

- (3) *The canonical map*

$$R\Gamma(X, \mathcal{W} \omega_{\underline{X}/\underline{k}}^* \otimes^{\mathbb{L}} W_n(k)) \longrightarrow R\Gamma(X, \mathcal{W}_n \omega_{\underline{X}/\underline{k}}^*)$$

*is a quasi-isomorphism. As a consequence,  $R\Gamma(X, \mathcal{W} \omega_{\underline{X}/\underline{k}}^*) \in D(W(k))$  is a perfect complex. In particular, the saturated log de Rham–Witt cohomology  $\mathbb{H}^d(X, \mathcal{W} \omega_{\underline{X}/\underline{k}}^*)$  is finitely generated as an  $W(k)$ -module.*

**Proof.** By Remark 5.10, under Assumption (\*\*) we have an isomorphism

$$\mathcal{W}_1 \omega_{\underline{X}/\underline{k}}^j \cong \omega_{\underline{X}/\underline{k}}^j$$

for each  $j \geq 0$ . Thus,  $\mathcal{W}_1 \omega_{\underline{X}/\underline{k}}^j$  is finitely generated as an  $\mathcal{O}_X$ -module. By Corollary 2.23, the graded pieces of the  $\bar{V}$ -filtration on  $\mathcal{W} \omega_{\underline{X}/\underline{k}}^*$  are extensions of quotients of  $\mathcal{W}_1 \omega_{\underline{X}/\underline{k}}^*$  as follows:

$$0 \rightarrow \mathcal{W}_1 \omega_{\underline{X}/\underline{k}}^j / \mathcal{B}_n(\mathcal{W}_1 \omega_{\underline{X}/\underline{k}}^j) \xrightarrow{V^n} \mathrm{gr}_V^n \mathcal{W}_{\underline{X}/\underline{k}}^j \xrightarrow{\beta} \mathcal{W}_1 \omega_{\underline{X}/\underline{k}}^{j-1} / \mathcal{Z}_n(\mathcal{W}_1 \omega_{\underline{X}/\underline{k}}^{j-1}) \rightarrow 0.$$



Here  $\mathcal{B}_n$  and  $\mathcal{Z}_n$  are obtained from globalisation of 2.21. Therefore, each  $\mathcal{W}_n\omega_{\underline{X}/k}^j$  is of finite type over  $W_n(\mathcal{O}_X)$ . Because  $X$  is proper, it then follows that each  $H^i(X, \mathcal{W}_n\omega_{\underline{X}/k}^j)$  is of finite type over  $W_n(k)$ . For the second part of the proposition, first note that  $\mathcal{W}\omega_{\underline{X}/k}^j \xrightarrow{\sim} \mathop{\text{Rlim}}_{\leftarrow} \mathcal{W}_n\omega_{\underline{X}/k}^j$ , so the first arrow is a quasi-isomorphism because derived pushforwards of maps between ringed topoi commute with derived limits. The claim on the cohomology group follows from the finiteness of each  $H^i(X, \mathcal{W}_n\omega_{\underline{X}/k}^j)$ , because  $\text{R}^q\text{lim}_n H^i(X, \mathcal{W}_n\omega_{\underline{X}/k}^j) = 0$  for all  $q > 0$  by the Mittag–Leffler condition. Finally, the last claim follows from Proposition 2.11 (3) (the quasi-isomorphism in (11)), from which we know that

$$\mathcal{W}\omega_{\underline{X}/k}^* \otimes^{\mathbb{L}} W_n(k) = \mathcal{W}\omega_{\underline{X}/k}^* \otimes W(k)/p^n \xrightarrow{\sim} \mathcal{W}_n\omega_{\underline{X}/k}^*$$

is a quasi-isomorphism. □

### 9.5. Newton above Hodge

In this subsection we prove a generalised version of Katz’s conjecture for saturated log de Rham–Witt complexes. We keep the assumption on  $\underline{X}$  from the previous subsection and fix an integer  $d \geq 0$ . By 9.6, the torsion-free quotient of the cohomology group  $\mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/k}^*)_{\text{tf}}$  is a finite free  $W(k)$ -module. This allows us to consider the F-crystal

$$(\mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/k}^*)_{\text{tf}}, \varphi)$$

where  $\varphi$  denotes the Frobenius operator. We denote by  $\text{New}(\varphi)$  (respectively  $\text{Hdg}(\varphi)$ ) the Newton (respectively Hodge) polygon of this  $F$ -crystal and refer to it as the **saturated Newton (respectively Hodge) polygon**. For an excellent exposition on the notion of Newton and Hodge polygons, we refer the reader to [16].

Slightly more generally, we study iterates of Frobenius  $\varphi^n$  on  $\mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/k}^*)_{\text{tf}}$  for any integer  $n \geq 1$  and consider the **saturated Hodge polygon**

$$\frac{1}{n}\text{Hdg}(\varphi^n)$$

**of level  $n$** , which is obtained from the Hodge polygon of  $(\mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/k}^*)_{\text{tf}}, \varphi^n)$ , except all slopes are divided by  $n$ . Note that one may define similarly  $(1/n)\text{New}(\varphi^n)$  but we have  $(1/n)\text{New}(\varphi^n) = \text{New}(\varphi)$  for all  $n$ .

Following [19], we also make the following definition.

**Definition 9.7.** Retain the setup from above.

- (1) Define the saturated Hodge numbers of level  $n$  of  $\underline{X}$  to be the integers

$$\mathfrak{h}^j(n) := \text{lg } H^{d-j}(X, \mathcal{W}_n\omega_{\underline{X}/k}^j)$$

indexed by  $j \in \{0, \dots, d\}$ , where  $\text{lg} = \text{lg}_{W(k)}$  denotes the length of a  $W(k)$ -module.

(2) Define the geometric Hodge polygon of level  $n$  of  $\underline{X}$  to be the Hodge polygon of the numbers

$$\left\{ \frac{\mathfrak{h}^0(n)}{n}, \frac{\mathfrak{h}^1(n)}{n}, \dots, \frac{\mathfrak{h}^d(n)}{n} \right\};$$

namely, the polygon where the slope  $j$  segment has horizontal length  $\mathfrak{h}^j(n)/n$ . This Hodge polygon is denoted by  $\text{Hdg}(\underline{X}, n)$ , or simply  $\text{Hdg}(n)$  if  $\underline{X}$  is understood.

The main theorem we prove in this section is as follows.

**Theorem 9.8.** *For any coherent proper log scheme  $\underline{X}$  over  $\underline{k}$  satisfying Assumption (\*\*), its saturated Hodge polygon  $(1/n)\text{Hdg}(\varphi^n)$  always lies on or above the Hodge polygon  $\text{Hdg}(\underline{X}, n)$ . Using notations from the Introduction, we have*

$$(1/n)\text{Hdg}(\varphi^n) \succcurlyeq \text{Hdg}(\underline{X}, n).$$

**Remark 9.9.** Mazur [16] has shown that the Newton polygon of an F-crystal always lies on or above the Hodge polygon of the F-crystal; therefore, the theorem implies that

$$\text{New}(\varphi) = (1/n)\text{New}(\varphi^n) \succcurlyeq (1/n)\text{Hdg}(\varphi^n) \succcurlyeq \text{Hdg}(\underline{X}, n).$$

**Remark 9.10.** One way to prove Theorem 9.8 is to faithfully follow the proof of [19, Theorem 2.5], which crucially uses Proposition 9.4. We find it more clarifying to phrase the argument using only the Nygaard filtration (instead of the Nygaard complexes that give rise to the filtration), which instead relies on Corollary 9.3.

Let us start with the following elementary lemma of Nygaard.

**Lemma 9.11.** *Let  $n \geq 1$  be an integer and let*

$$A = \{a^0, a^1, \dots\}, \quad B = \left\{ \frac{b^0}{n}, \frac{b^1}{n}, \dots \right\}$$

*be two series of nonnegative numbers. Let  $(1/n)\text{Hdg}(A)$  be the Hodge polygon of  $A$  with slopes divided by  $n$ . Then  $(1/n)\text{Hdg}(A)$  lies on or above  $\text{Hdg}(B)$  if (and only if) for any  $r \geq 1$  the following inequality holds:*

$$nr \cdot a^0 + (nr - 1) \cdot a^1 + \dots + a^{nr-1} \leq r \cdot b^0 + (r - 1) \cdot b^1 + \dots + b^{r-1}.$$

**Proof.** This is [19, Lemma 2.4]. □

**Lemma 9.12.** *The map  $\varphi^n$  sends  $\mathbb{H}^d(X, \mathcal{N}_n^r \mathcal{W}\omega_{\underline{X}/\underline{k}}^*)$  into  $p^{nr} \cdot \mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*)$  for all  $r \geq 1$ . Therefore, if we let  $\overline{\mathbb{H}}^d(X, \mathcal{N}_n^r \mathcal{W}\omega_{\underline{X}/\underline{k}}^*)$  denote the image of the map on cohomology induced by the inclusion  $\mathcal{N}_n^r \mathcal{W}\omega_{\underline{X}/\underline{k}}^* \hookrightarrow \mathcal{W}\omega_{\underline{X}/\underline{k}}^*$ , then we have*

$$\overline{\mathbb{H}}^d(X, \mathcal{N}_n^r \mathcal{W}\omega_{\underline{X}/\underline{k}}^*) \subset \varphi^{-n} (p^{nr} \cdot \mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*)).$$

**Proof.** From the description of  $\mathcal{N}_n^r \mathcal{W}\omega_{\underline{X}/\underline{k}}^*$ , the map  $\varphi^n$  lands in  $p^{nr} \cdot \mathcal{W}\omega_{\underline{X}/\underline{k}}^*$  and the claim follows immediately. □

Now we are ready to prove the theorem.

**Proof of Theorem 9.8.** For notational simplicity, write  $H = \mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*)_{\text{tf}}$ , and let  $\{a_0, a_1, \dots\}$  be the Hodge numbers associated to the  $F^n$ -crystal  $(H, \varphi^n)$ . By Lemma 9.11, it suffices to show that for any  $r \geq 1$ , we have inequality

$$nr \cdot a^0 + (nr - 1) \cdot a^1 + \dots + a^{nr-1} \leq r \cdot \mathfrak{h}^0(n) + (r - 1) \cdot \mathfrak{h}^1(n) + \dots + \mathfrak{h}^{r-1}(n). \tag{16}$$

It is easy to see that the left-hand side of inequality (16) is precisely the following length:

$$\text{lg} (H/\varphi^{-n}(p^{nr}H))$$

as a  $W(k)$ -module. Now by Lemma 9.12 we know that

$$\begin{aligned} \text{lg} H/\varphi^{-n}(p^{nr}H) &\leq \text{lg} \mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*)/\varphi^{-n}(p^{nr}\mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*)) \\ &\leq \text{lg} \mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*)/\overline{\mathbb{H}}^d(X, \mathcal{N}_n^r \mathcal{W}\omega_{\underline{X}/\underline{k}}^*) \\ &\leq \text{lg} \mathbb{H}^d(X, \mathcal{W}\omega_{\underline{X}/\underline{k}}^*/\mathcal{N}_n^r \mathcal{W}\omega_{\underline{X}/\underline{k}}^*). \end{aligned}$$

Now by induction on  $r$  (using the fibre sequence  $\text{gr}^{r-1}(\mathcal{N}_n^*) \rightarrow \mathcal{W}\omega_{\underline{X}/\underline{k}}^*/\mathcal{N}_n^r \rightarrow \mathcal{W}\omega_{\underline{X}/\underline{k}}^*/\mathcal{N}_n^{r-1}$ ), to prove the inequality (16) it suffices to show that

$$\text{lg} \mathbb{H}^d(X, \text{gr}_{\mathcal{N}_n}^{r-1}(\mathcal{W}\omega_{\underline{X}/\underline{k}}^*)) \leq \mathfrak{h}^0(n) + \mathfrak{h}^1(n) + \dots + \mathfrak{h}^{r-1}(n). \tag{17}$$

But now we can apply apply Corollary 9.3, which implies that

$$\begin{aligned} \text{lg} \mathbb{H}^d(X, \text{gr}_{\mathcal{N}_n}^{r-1}(\mathcal{W}\omega_{\underline{X}/\underline{k}}^*)) &\leq \text{lg} \mathbb{H}^d(X, \tau^{\leq r} \mathcal{W}_n \omega_{\underline{X}/\underline{k}}^*) \\ &\leq \text{lg} \mathbb{H}^d(X, \mathcal{H}^0(\mathcal{W}_n \omega_{\underline{X}/\underline{k}}^*)) + \dots + \text{lg} \mathbb{H}^{d-(r-1)}(X, \mathcal{H}^{r-1}(\mathcal{W}_n \omega_{\underline{X}/\underline{k}}^*)) \\ &= \text{lg} \mathbb{H}^d(X, \mathcal{W}_n \omega_{\underline{X}/\underline{k}}^0) + \dots + \text{lg} \mathbb{H}^{d-(r-1)}(X, \mathcal{W}_n \omega_{\underline{X}/\underline{k}}^{r-1}) \\ &= \mathfrak{h}^0(n) + \mathfrak{h}^1(n) + \dots + \mathfrak{h}^{r-1}(n). \end{aligned}$$

In the second last equality, we have used Corollary 2.12. This finishes the proof.

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**Competing Interest** None.

**Appendix A. Log geometry**

In this section we fix notations we use in log geometry, following [13], which are slightly different from notations in literature.<sup>16</sup> Most of the material (if not everything) presented in this section is considered standard, but we make some remarks that might only be obvious for experts.

**A.1. Log schemes (after Fontaine–Illusie–Kato)**

A pre-log structure on a scheme  $X$  is a morphism  $\alpha : M \rightarrow \mathcal{O}_X$  of sheaves of monoids on  $X_{\text{ét}}$ . The pre-log structure  $\alpha$  is a log structure if  $\alpha : \alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$  is an isomorphism. We will suppress notation and write  $(X, M)$  or  $\underline{X}$  for (pre-) log schemes. For a pre-log structure  $M$  on  $X$ , we often denote by  $M^a$  (or  $\mathcal{M}^a$ ) its associated log structure, which is the pushout of  $\alpha^{-1}(\mathcal{O}_X^\times) \rightarrow M$  and  $\alpha^{-1}(\mathcal{O}_X^\times) \rightarrow \mathcal{O}_X^\times$  in the category of sheaves of monoids on  $X_{\text{ét}}$ . We make the same definitions for formal schemes. A log structure is *fine* if it is coherent and integral. In this article we work with fine log schemes, but in the application to  $A_{\text{inf}}$ -cohomology in [20] we relax the condition of coherence to quasi-coherence. A chart for a log scheme  $\underline{X} = (X, M)$  is a morphism  $P_X \rightarrow M$  from a constant pre-log structure  $P_X$  to  $M$  that induces an isomorphism  $P_X^a \xrightarrow{\sim} M$ . Charts exist étale locally for fine log schemes. A chart for a morphism  $f : (X, M) \rightarrow (Y, N)$  between log schemes is a triple  $(P_X \xrightarrow{\psi_X} M, Q_Y \xrightarrow{\psi_Y} N, Q \xrightarrow{\beta} P)$  where  $\psi_X, \psi_Y$  are charts for  $(X, M), (Y, N)$ , respectively, and  $\beta$  a morphism of monoids making the obvious diagram commute.

**Lemma A.1.** *Let  $X$  be a scheme and  $L_X \xrightarrow{\alpha} \mathcal{O}_X$  be a constant pre-log structure with underlying monoid  $L$ . Let  $\mathcal{L} = (L_X)^a$  denote the log structure associated to  $L_X$  and  $L^{\text{sh}} := \Gamma(X, \mathcal{L})$  be the global sections of  $\mathcal{L}$ ; then its associated log structure  $\mathcal{L}^{\text{sh}} := (L_X^{\text{sh}})^a$  is identified with  $\mathcal{L}$ .*

**Proof.** We view  $L^{\text{sh}}$  as a constant pre-log structure  $\beta : L_X^{\text{sh}} \rightarrow \mathcal{O}_X$  by composing the natural map  $L_X^{\text{sh}} \rightarrow \mathcal{L}$  with the induced log structure  $\alpha^a : \mathcal{L} \rightarrow \mathcal{O}_X$ . We have the following diagram:

$$\begin{array}{ccc}
 & L_X^{\text{sh}} & \xrightarrow{\iota_\beta} & \mathcal{L}^{\text{sh}} \\
 h \nearrow & & & \searrow i \\
 L_X & \xrightarrow{\iota_\alpha} & & \mathcal{L}
 \end{array}$$

of pre-log structures on  $X_{\text{ét}}$ , where  $h$  is given by  $\iota_\alpha$  valued on  $X$ . By the universal property,  $\iota_\beta \circ h : L_X \rightarrow \mathcal{L}^{\text{sh}}$  induces a morphism  $f : \mathcal{L} \rightarrow \mathcal{L}^{\text{sh}}$  of log structures, and  $i$  induces  $g : \mathcal{L}^{\text{sh}} \rightarrow \mathcal{L}$ . The maps  $f$  and  $g$  are inverses to each other. To see that  $f \circ g = \text{id}$ , note that under the isomorphism  $\text{Hom}_{\text{log}}(\mathcal{L}^{\text{sh}}, \mathcal{L}) \cong \text{Hom}_{\text{pre-log}}(L_X^{\text{sh}}, \mathcal{L})$ ,  $g$  corresponds to  $i$ ; thus, after composing by  $f : \mathcal{L} \rightarrow \mathcal{L}^{\text{sh}}$ , we know that  $f \circ g$  corresponds to  $f \circ i$  under the isomorphism  $\text{Hom}_{\text{log}}(\mathcal{L}^{\text{sh}}, \mathcal{L}^{\text{sh}}) \cong \text{Hom}_{\text{pre-log}}(L_X^{\text{sh}}, \mathcal{L}^{\text{sh}})$ . But  $f \circ i = \iota_\beta$  by construction, which corresponds to  $\text{id} : \mathcal{L}^{\text{sh}} \rightarrow \mathcal{L}^{\text{sh}}$ ; hence  $f \circ g = \text{id}$ . The other verification is similar.  $\square$

<sup>16</sup>For example, we always use  $X$  to denote a scheme and  $\underline{X}$  a (pre-) log scheme, which is opposite to the convention adopted by some algebraic geometers.

**Lemma A.2.** Let  $(\text{Spec } R, \mathcal{M}^a)$  be the log scheme associated to a log algebra  $\underline{R} = (R, \alpha : M \rightarrow R)$  and  $R \rightarrow S$  be an étale morphism; then  $(\text{Spec } S, \mathcal{M}^a|_{\text{Spec } S})$  is the log scheme associated to  $(S, L \rightarrow S)$ . In other words, the composition of monoid maps  $L \rightarrow R \rightarrow S$  gives rise to a chart for  $\mathcal{M}^a|_{\text{Spec } R}$ .

**Proof.** Let  $U = \text{Spec } R$  and  $V = \text{Spec } S$ . By definition,  $\mathcal{M}^a$  is the sheafification of the presheaf pushout of  $\alpha^{-1}(\mathcal{O}_U^\times) \rightarrow L$  and  $\alpha^{-1}(\mathcal{O}_U^\times) \rightarrow \mathcal{O}_U^\times$ , so its restriction  $\mathcal{M}^a|_V$  can be computed by first restricting the presheaf pushout to  $V$  and then taking sheafification as presheaves on  $V_{\text{ét}}$ . Unwinding definitions, the latter is the log structure associated to  $L \rightarrow \mathcal{O}_V$  on  $V_{\text{ét}}$ .  $\square$

**Definition A.3.** A morphism  $i = (i, \psi) : \underline{X} \rightarrow \underline{Y}$  between fine log schemes  $\underline{X} = (X, \alpha : M \rightarrow \mathcal{O}_X)$  and  $\underline{Y} = (Y, \beta : N \rightarrow \mathcal{O}_Y)$  is a closed immersion if

- (1)  $i : X \rightarrow Y$  is a closed immersion; and
- (2)  $\psi^a : i^*N \rightarrow M$  is surjective (here  $\psi^a$  is induced from  $\psi : i^{-1}(N) \rightarrow M$ ).

A closed morphism  $i : \underline{X} \rightarrow \underline{Y}$  is exact if, moreover, in (2) above  $\psi^a$  is an isomorphism. More generally, a morphism  $f : (X, M) \rightarrow (Y, N)$  is exact if the following diagram is Cartesian:

$$\begin{array}{ccc} f^{-1}(N) & \longrightarrow & M \\ \downarrow & & \downarrow \\ f^{-1}(N)^{\text{gp}} & \longrightarrow & M^{\text{gp}} \end{array}$$

A log thickening (of order  $n$ ) of a fine log scheme  $(T, L)$  is an exact closed morphism

$$\iota : (T, L) \rightarrow (T', L')$$

such that  $T$  is defined in  $T'$  by an ideal  $I$  where  $I^{n+1} = 0$ .

**Example A.4.** The standard log point over a field  $k$  is  $\text{Spec } k$  with pre-log structure  $\mathbb{N} \rightarrow k$  by sending  $0 \mapsto 1$  and everything else to 0. Let  $(X, M)$  be a log scheme on which  $p$  is nilpotent. For each integer  $r \geq 1$ , the  $r$ th Witt log scheme  $W_r(X, M)$  consists of the underlying scheme  $W_r(X)$  and the pre-log structure  $W_r(\alpha) : M \rightarrow W_r(\mathcal{O}_X)$  given by  $m \mapsto [\alpha(m)]$ , here  $[\alpha(m)]$  is the  $r$ th Teichmüller lift of  $\alpha(m)$ . Another frequently encountered example is the divisorial log structure, defined by  $M_D(U) = \{f \in \mathcal{O}_X(U) : f|_{U \setminus D} \in \mathcal{O}_X^\times(U \setminus D)\}$  where  $D \subset X$  is a divisor.

### A.2. Log differentials

Let  $f = (f, \psi) : \underline{X} \rightarrow \underline{Y}$  be a morphism between fine log schemes  $\underline{X} = (X, \alpha : M \rightarrow \mathcal{O}_X)$  and  $\underline{Y} = (Y, \beta : N \rightarrow \mathcal{O}_Y)$ .

**A.2.1. Log derivations.** A log derivation of  $\underline{X}/\underline{Y}$  into a sheaf of  $\mathcal{O}_X$ -modules  $D$  is a pair  $(d, \delta)$  where  $d : \mathcal{O}_X \rightarrow D$  is a derivation of  $X/Y$  into  $D$ , and  $\delta : M \rightarrow D$  is a morphism of sheaf of monoids, such that  $d(\alpha(m)) = \alpha(m)\delta(m)$  for any local section  $m$  of  $M$  and  $\delta(\psi(n)) = 0$  for any local section  $n$  of  $N$ .

**Remark A.5.** Let  $f : (X, \alpha : M \rightarrow \mathcal{O}_X) \rightarrow (Y, \beta : N \rightarrow \mathcal{O}_Y)$  be a morphism of pre-log schemes. We define a pre-log derivation in the same way as above. Then a pre-log derivation  $(d, \delta)$  of  $(X, M)/(Y, N)$  into  $D$  extends uniquely to a log derivation  $(\tilde{d}, \tilde{\delta})$  of  $(X, M^a)/(Y, N^a)$ .

**A.2.2. Log differentials.** The relative log differential  $\omega_{\underline{X}/\underline{Y}}^1$  equipped with  $(d, \text{dlog})$  is the universal log derivation of  $\underline{X}/\underline{Y}$  representing the functor  $D \mapsto \text{Der}_D^{\text{log}}$ , where  $\text{Der}_D^{\text{log}}$  is the set of log derivations of  $\underline{X}/\underline{Y}$  into the sheaf of  $\mathcal{O}_X$  module  $D$ . More concretely,  $\omega_{\underline{X}/\underline{Y}}^1$  is the quotient of

$$\Omega_{X/Y}^1 \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{\text{gp}})$$

by (the  $\mathcal{O}_X$  module generated by) local sections  $(d\alpha(m), 0) - (0, \alpha(m) \otimes m)$  and  $(0, 1 \otimes \psi(n))$ . In the construction,  $\text{dlog}(m) := (0, 1 \otimes m)$ .

**Remark A.6.** If  $f : \underline{X} \rightarrow \underline{Y}$  is a morphism of pre-log schemes, then we can make the same definitions. Moreover, we have

$$\omega_{(X, M)/(Y, N)}^1 \xrightarrow{\sim} \omega_{(X, M^a)/(Y, N)}^1 \xrightarrow{\sim} \omega_{(X, M^a)/(Y, N^a)}^1.$$

**A.2.3. Log smoothness.** A morphism  $f = (f, \psi) : \underline{X} \rightarrow \underline{Y}$  of fine log schemes is log-smooth (respectively log étale) if  $X \xrightarrow{f} Y$  is locally of finite presentation and for any commutative diagram

$$\begin{array}{ccc} (T, L) & \xrightarrow{s} & (X, M) \\ \downarrow \iota & \nearrow g & \downarrow f \\ (T', L') & \xrightarrow{t} & (Y, N) \end{array}$$

where  $\iota$  is a first order log thickening, étale locally there exists (respectively there exists a unique)  $g : (T', L') \rightarrow (X, M)$  making all diagrams commute.

As in the classical case, if  $f : \underline{X} \rightarrow \underline{Y}$  is log-smooth, then the log differential  $\omega_{\underline{X}/\underline{Y}}^1$  is locally free of finite type.

**Proposition A.7** ([13] 3.5). *Let  $f : \underline{X} \rightarrow \underline{Y}$  be a morphism between fine log schemes as above and  $Q \rightarrow N$  a chart for  $\underline{Y}$ . Then  $f$  is log-smooth if and only if étale locally on  $X$  there exists a chart  $(P_X \rightarrow M, Q_Y \rightarrow N, \beta : Q \rightarrow P)$  of  $f$  extending the chart for  $Y$  such that*

- (1)  $\ker(\beta^{gp})$  and  $\text{coker}(\beta^{gp})_{\text{tor}}$  are finite groups of order invertible on  $X$ ;
- (2) the induced morphism  $X \rightarrow Y \times_{\text{Spec}(\mathbb{Z}[Q])} \text{Spec} \mathbb{Z}[P]$  is étale.

Similarly,  $f$  is log étale if and only if a similar condition holds with the torsion part of the kernel  $\text{coker}(\beta)_{\text{tor}}$  replaced by  $\text{coker}(\beta)$  in (1) above.

### A.3. Log-Cartier type

**A.3.1. Integral morphisms.** A log-smooth morphism might fail to be flat. For example, consider  $\underline{X} = (\text{Spec} \mathbb{Z}[x, y], \mathbb{N}^2)$  with log structure  $(1, 0) \mapsto x; (0, 1) \mapsto y$ . The

morphism  $\underline{X} \rightarrow \underline{X}$  given by  $x \mapsto x, y \mapsto xy$  is log-smooth (even log-étale) but not flat. This leads to the following definition.

**Definition A.8.** A morphism  $f : \underline{X} \rightarrow \underline{Y}$  is *integral* if for any  $\underline{Y}' \rightarrow \underline{Y}$  where  $\underline{Y}'$  is a fine log scheme, the base change  $\underline{X} \times_{\underline{Y}} \underline{Y}'$  is a fine log scheme.

This is equivalent to requiring that étale locally on  $X$  and  $Y$ ,  $f$  has charts given by  $\beta : Q \rightarrow P$  such that the induced morphism  $\mathbb{Z}[Q] \rightarrow \mathbb{Z}[P]$  is flat. If  $f : \underline{X} \rightarrow \underline{Y}$  is log-smooth and integral, then the underlying morphism  $f : X \rightarrow Y$  is flat.

**A.3.2. log-Cartier type.** Let  $\underline{X} = (X, M)$  be a log scheme in characteristic  $p$ , the absolute ( $p$ -power) Frobenius  $F_{\underline{X}}$  is given by the usual absolute Frobenius on  $X$  and  $M \xrightarrow{\times p} M$ . Note that we have implicitly identified  $F_{\underline{X}}^{-1}(M) \cong M$  on  $X_{\text{ét}}$ .

**Definition A.9.** A morphism  $f : \underline{X} \rightarrow \underline{Y}$  over  $\mathbb{F}_p$  is of log-Cartier type if  $f$  is integral and the relative Frobenius  $F_{\underline{X}/\underline{Y}}$  in the diagram below is exact:

$$\begin{array}{ccccc}
 & & F_{\underline{X}} & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \underline{X} & \xrightarrow{\quad F_{\underline{X}/\underline{Y}} \quad} & \underline{X}^{(p)} & \xrightarrow{h} & \underline{X} \\
 & & \downarrow & \square & \downarrow f \\
 & & \underline{Y} & \xrightarrow{F_{\underline{Y}}} & \underline{Y}
 \end{array}$$

The most important feature for a log-smooth morphism of log-Cartier type is that the Cartier isomorphism holds. This will be a key step to relate our log de Rham–Witt complex with the de Rham complex of a log Frobenius lift.

**Proposition A.10** ([9] 2.12). *Let  $f$  be a log-smooth morphism of log-Cartier type; then there exists a (Cartier) isomorphism*

$$F = C^{-1} : \omega_{\underline{X}^{(p)}/\underline{Y}}^k \cong \mathcal{H}^k(\omega_{\underline{X}/\underline{Y}}^*),$$

which, on local sections  $a \in \mathcal{O}_X, m_1, \dots, m_k \in M$ , is given by

$$x \, d\log(h^* m_1) \wedge \dots \wedge d\log(h^* m_k) \mapsto F_{\underline{X}/\underline{Y}}(x) \, d\log(m_1) \wedge \dots \wedge d\log(m_k).$$

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