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SOME METRIC PROPERTIES OF LÜROTH EXPANSIONS OVER THE FIELD OF LAURENT SERIES

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J. Knopfmacher and A. Knopfmacher have previously produced some metric results concerning the coefficients of the Lüroth expansions of elements in the field of Laurent series with coefficients from a finite field. In this paper, we obtain analogous metric results for subsequences of the coefficients of the expansions.

1. INTRODUCTION

In [5] and [4], the authors produce various metric results on the coefficients of the Lüroth expansions of Laurent series with coefficients from a finite field. In this paper, we generalise these to similar subsequence results.

In Section 2, we prove that the coefficients of the Lüroth expansions are independent, identically distributed random variables. In Section 3, we apply various classical theorems from probability theory to this setting.

2. MAIN CONSTRUCTION

Let $\mathbb{F}_q$ be the finite field of order q. Further, we let

$$\mathbb{L} = \left\{ \sum_{i=-n}^{\infty} \alpha_{-i} X^{-i} : n \in \mathbb{Z}, \alpha_{-i} \in \mathbb{F}_q, \alpha_n \neq 0 \right\} \tag{1}$$

be the field of Laurent expansions with coefficients from $\mathbb{F}_q$. We equip $\mathbb{L}$ with the norm $\|A\| = q^n$ of A, where $\alpha_n \neq 0$ is the leading coefficient in the expansion. We shall refer to the n in this definition as the *degree of* A written as $\deg(A)$. It is well-known that if we let d denote the metric induced by the norm above, $(\mathbb{L}, d)$ is a complete metric space (see [3]). It is also known that the norm satisfies:

$$\|AB\| = \|A\| \, \|B\|$$
$$\|A + B\| \leqslant \max(\|A\|, \|B\|) \text{ with equality when } \|A\| \neq \|B\|.$$

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We define the ring of integers $\mathbb{J}$ in $\mathbb{L}$ to be $\{A \in \mathbb{L} : \|A\| \leqslant 1\}$. Clearly, this is a ring, and the set $\mathbb{I} = \{A \in \mathbb{L} : \|A\| < 1\}$ is a maximal ideal in $\mathbb{J}$. Thus, we define the integral part $[A]$ of an element $A = \sum_{i=-n}^{\infty} \alpha_{-i} X^{-i} \in \mathbb{L}$ to be $\sum_{i=-n}^{0} \alpha_{-i} X^{-i}$ when $n \geqslant 0$ and 0 otherwise.

We know (see [**6**]) that when we define the measure μ on the balls $B(c, q^{-r})$ in $\mathbb{L}$ by $\mu\big(B(c, q^{-r})\big) = q^{-r+1}$, we have a characterisation of the Haar measure on $\mathbb{L}$, since the Haar measure is unique up to scaling. Consequently, since $\mu(\mathbb{I}) = 1$, μ induces a probability measure on $\mathbb{I}$. In the following, we shall solely be concerned with this measure.

Let $A \in \mathbb{L}$ and define the sequences (a_n) and (A_n) recursively as follows:

$$a_0 = [A], \qquad A_1 = A - a_0,$$

and for the following elements,

$$a_n = \left[\frac{1}{A_n}\right], \qquad A_{n+1} = (a_n - 1)(a_n A_n - 1),$$

unless we reach a point where $a_n = 0$ or $A_n = 0$, in which case the recursion stops. It can be shown (see [**2**]) that this leads to a unique expansion of A,

$$A = a_0 + \frac{1}{a_1} + \sum_{i=2}^{\infty} \frac{1}{a_1(a_1 - 1) \cdots a_{i-1}(a_{i-1} - 1) a_i}, \tag{2}$$

where the $a_i \in \mathbb{F}_q[X]$. This expansion is called the *Lüroth expansion of* A.

We need a dynamical interpretation of this construction. We shall only consider the ideal $\mathbb{I}$, since all our results extend to $\mathbb{L}$ by translation. On this ideal, we define operators $a : \mathbb{I} \setminus \{0\} \to \mathbb{F}_q[X]$ and $T : \mathbb{I} \to \mathbb{I}$ by

$$a(x) = \left[\frac{1}{x}\right], \qquad Tx = \begin{cases} 0 & \text{if } x = 0 \\ (a(x) - 1)(x a(x) - 1) & \text{otherwise.} \end{cases} \tag{3}$$

It is clear that a does indeed map $\mathbb{I}$ into $\mathbb{F}_q[X]$. Simple calculations yield

$$\|Tx\| = \|x\|^{-1} \|x\| \left\| a(x) - \frac{1}{x} \right\| < 1$$

for $x \in \mathbb{I}$. Hence, T maps $\mathbb{I}$ into $\mathbb{I}$.

When $x \in \mathbb{I}$, clearly a_0 in (2) is zero. For $r \geqslant 1$, the definitions in (3) gives the relationship $a_r = a(T^{r-1}x)$. The following is our main theorem:

Theorem 1. *The coefficients $a_i, i \in \mathbb{N}$ in the Lüroth series expansion of a Laurent series in $\mathbb{I}$ are independent, identically distributed random variables.*

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PROOF: Consider the sets

$$I_n = I_n(k_1, \dots, k_n) = \{x \in \mathbb{I} : a_1(x) = k_1, \dots, a_n(x) = k_n\}, \tag{4}$$

together with the set $I_0 = \mathbb{I}$. We shall refer to these as the *Lüroth cylinders.*

Let $x \in I_n$ for some Lüroth cylinder I_n. Since $x \in \mathbb{I}$, $a_0(x) = 0$. We now see that the Lüroth expansion of x has the form

$$x = c_n + d_n \sum_{i=n+1}^{\infty} \frac{1}{a_{n+1}(a_{n+1}-1)\cdots a_{i-1}(a_{i-1}-1)a_i}, \tag{5}$$

where

$$d_0 = 1, \quad d_i = \frac{1}{k_1(k_1-1)\cdots k_i(k_i-1)}, \quad c_n = \sum_{i=1}^{n} \frac{d_{i-1}}{k_i}.$$

But the sum appearing in (5) is nothing but the tail of the Lüroth series of x. Hence, if we define the function $\phi_n : \mathbb{I} \to I_n$ by the equation $\phi_n(y) = c_n + d_n y$, we see that $x = \phi_n(T^n x)$, so ϕ_n has T^n as an inverse map.

Clearly, ϕ_n is surjective. Hence,

$$I_n = \phi_n(\mathbb{I}) = c_n + d_n\mathbb{I} = B(c_n, q^{-1}\|d_n\|),$$

which implies,

$$\mu(I_n) = \frac{1}{\|k_1(k_1-1)\cdots k_n(k_n-1)\|} = \frac{1}{\|k_1\cdots k_n\|^2}. \tag{6}$$

It follows directly from the definition of the Lüroth cylinders and (6) that the a_i are independent and identically distributed. In particular,

$$\mu\{x \in \mathbb{I} : a_r(x) = k\} = \frac{1}{\|k\|^2}. \tag{7}$$

□

3. APPLICATIONS

With Theorem 1 in place, we deduce a number of results about the a_i. Weaker results are given in [4] and [5]. The Strong Law of Large Numbers together with Theorem 1 immediately implies several results.

PROPOSITION 2. *Let $(n_i) \subseteq \mathbb{N}$ be a strictly increasing sequence and let $k \in \mathbb{F}_q[X]$.*

$$\lim_{i\to\infty} \frac{1}{i}\left|\{r \leqslant i : a_{n_r}(x) = k\}\right| = q^{-2\deg(k)} \tag{8}$$

for almost every $x \in \mathbb{I}$.

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PROOF: Apply the Strong Law of Large Numbers [1, Theorem 3.30] to the independent and identically distributed random variables $X_i = \chi_{\{x\in\mathbb{I}:a_{n_i}=k\}}$, where χ_A denotes the indicator function of A. □

PROPOSITION 3. *Let $(n_i) \subseteq \mathbb{N}$ be a strictly increasing sequence. For almost every $x \in \mathbb{I}$,*

$$\lim_{i\to\infty} \frac{1}{i}\sum_{r=1}^{i} \deg(a_{n_r}) = \frac{q}{q-1}. \tag{9}$$

PROOF: Apply the Strong Law of Large Numbers to the independent and identically distributed random variables $Y_i = \deg(a_{n_i})$. □

Noting that $\deg(a_i) = \log_q \|a_i\|$, we obtain the following corollary:

COROLLARY 4. *Let $(n_i) \subseteq \mathbb{N}$ be a strictly increasing sequence. For almost every $x \in \mathbb{I}$,*

$$\lim_{i\to\infty} \|a_{n_1}\cdots a_{n_i}\|^{1/i} = q^{q/(q-1)}.$$

By using stronger probabilistic theorems, we can obtain stronger results on the coefficients. First, we obtain generalisations of Propositions 2 and 3.

PROPOSITION 5. *Let $(n_i) \subseteq \mathbb{N}$ be a strictly increasing sequence. Define the random variables $Z_{r,k} = \left|\{i \leqslant r : a_{n_i} = k\}\right|$.*

$$\limsup_{r\to\infty} \frac{Z_{r,k} - r\|k\|^{-2}}{\sqrt{r\log\log r}} = \sqrt{2\|k\|^{-2}\left(1-\|k\|^{-2}\right)}$$

for almost all $x \in \mathbb{I}$. Furthermore, for any $s \in \mathbb{R}$:

$$\lim_{r\to\infty} \mu\left\{x\in\mathbb{I} : Z_{r,k}(x) - r\|k\|^{-2} < \frac{s}{\|k\|}\sqrt{r(1-\|k\|^{-2})}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} e^{-u^2/2}du.$$

PROOF: We consider the independent and identically distributed random variables $X_i = \chi_{\{x\in\mathbb{I}:a_{n_i}=k\}}$. We calculate the first and second moments of these. The random variables are indicator functions, so $X_i{}^2 = X_i$. Since

$$\mathbb{E}X_i = \int_{\mathbb{I}} X_i d\mu = \int_{\mathbb{I}} \chi_{\{x\in\mathbb{I}|a_{n_i}=k\}} d\mu = \|k\|^{-2},$$

we have

$$\sigma^2(X_i) = \mathbb{E}(X_i{}^2) - \left(\mathbb{E}(X_i)\right)^2 = \|k\|^{-2}\left(1-\|k\|^{-2}\right).$$

Now, the proposition follows directly from the Law of the Iterated Logarithm [1, Theorem 13.25] and the Central Limit Theorem [1, Corollary 8.23]. □

PROPOSITION 6. *Let $(n_i) \subseteq \mathbb{N}$ be a strictly increasing sequence.*

$$\limsup_{r\to\infty} \frac{\sum_{i=1}^{r} \deg\big(a_{n_i}(x)\big) - (q/q-1)r}{\sqrt{r \log\log r}} = \frac{\sqrt{2q}}{q-1}$$

for almost every $x \in \mathbb{I}$.

PROOF: Once again, we define random variables,

$$Y_i'(x) = \begin{cases} \deg\big(a_{n_i}(x)\big) & \text{for } \|a_{n_i}(x)\| \leqslant i^2 \\ 0 & \text{otherwise.} \end{cases}$$

A simple calculation yields

$$\mathbb{E}Y_i' = \sum_{r:q^r \leqslant i^2} \frac{q^i(q-1)r}{q^{2r}} = \mathbb{E}\big(\deg\big(a_{n_i}(x)\big)\big) + O\left(\frac{\log i}{i^2}\right).$$

Similarly,

$$\mathbb{E}(Y_i'^2) = \mathbb{E}\big(\deg\big(a_{n_i}(\cdot)\big)^2\big) + O\left(\frac{\log^2 i}{i^2}\right).$$

Hence, we can calculate the variance of the random variables

$$\operatorname{var}(Y_i') = \operatorname{var}\big(\deg\big(a_{n_i}(\cdot)\big)\big) + O\left(\frac{\log^2 i}{i^2}\right).$$

Letting

$$B_i = \sum_{j=1}^{i} \operatorname{var}(Y_j') = \sum_{j=1}^{i} \operatorname{var}\big(\deg\big(a_{n_j}(\cdot)\big)\big) + O(1) = \frac{qi}{(q-1)^2} + O(1),$$

and noting that

$$Y_i'(x) \leqslant 2\log_q i = O\left(\sqrt{\frac{B_i}{\log\log B_i}}\right),$$

we see that the Law of the Iterated Logarithm gives

$$\limsup_{i\to\infty} \frac{\sum_{j=1}^{i} Y_i' - \sum_{j=1}^{i} \deg\big(a_{j_i}(\cdot)\big)}{\sqrt{2\big(q/(q-1)^2\big) i \log\log i}} = \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} Y_j' - \sum_{j=1}^{i} \mathbb{E}(Y_i')}{\sqrt{2B_i \log\log B_i}} = 1 \tag{10}$$

for almost every $x \in \mathbb{I}$, since asymptotically $B_i \log\log B_i \asymp \frac{q}{(q-1)^2} i \log\log i$.

We define the sets $U_i = \left\{x \in \mathbb{I} : \deg\big(a_{n_i}(x)\big) \neq Y_i'(x)\right\}$.

$$\sum_{i=1}^{\infty} \mu(U_i) = \sum_{i=1}^{\infty} \sum_{\|k\| \geqslant i^2} \|k\|^{-2} < \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.$$

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By the Borel–Cantelli Lemma, for almost every $x \in \mathbb{I}$, there exists an $i_0(x) \in \mathbb{N}$ such that $Y_i'(x) = \deg\bigl(a_{n_i}(x)\bigr)$ for $i \geqslant i_0(x)$. Inserting this in (10) and applying our previous estimates yields the proposition. ∎

As we obtained Corollary 4 from Proposition 3, we get:

Corollary 7. *Let $(n_i) \subseteq \mathbb{N}$ be a strictly increasing sequence. For almost every $x \in \mathbb{I}$,*

$$\|a_{n_1} \cdots a_{n_r}\|^{1/r} = q^{q/(q-1)} + O\left(\frac{\sqrt{\log\log r}}{r}\right).$$

Our final result tells us something about the series of the norms of the partial coefficients in the Lüroth expansion of a given Laurent series.

Proposition 8. *Let $(n_i) \subseteq \mathbb{N}$ be a strictly increasing series. For any $\varepsilon > 0$*

$$\lim_{r\to\infty} \mu\left\{x \in \mathbb{I} : \frac{1}{r\log_q r}\left|\sum_{i=1}^{r}\|a_{n_i}(x)\| - (q-1)\right| > \varepsilon\right\} = 0.$$

Proof: We split the interesting random variable up into two for any given i. Hence, we define random variables for $r \leqslant i$:

$$V_r(x) = \begin{cases} \|a_{n_r}\| & \text{for } \|a_{n_r}\| \leqslant i\log_q i \\ 0 & \text{otherwise.} \end{cases}$$

$$W_r(x) = \begin{cases} 0 & \text{for } \|a_{n_r}\| \leqslant i\log_q i \\ \|a_{n_r}\| & \text{otherwise.} \end{cases}$$

With these definitions, we see that

$$\begin{aligned} \mu\left\{x \in \mathbb{I} : \frac{1}{i\log_q i}\left|\sum_{r=1}^{i}\|a_{n_r}\| - (q-1)\right| > \varepsilon\right\} \\ \leqslant \mu\Bigl\{x \in \mathbb{I} : \bigl|V_1 + \cdots + V_i - (q-1)i\log_q i\bigr| > \varepsilon i\log_q i\Bigr\} \\ + \mu\{x \in \mathbb{I} : W_1 + \cdots + W_i \neq 0\}. \end{aligned}$$

We consider each summand separately. By Theorem 1, $\mathbb{E}(V_1 + \cdots + V_i) = i\mathbb{E}(V_1)$ and $\operatorname{var}(V_1 + \cdots + V_i) = i\operatorname{var}(V_1)$, so

$$\begin{aligned} \mathbb{E}(V_1) = \int_{\mathbb{I}} V_1 d\mu &= \sum_{\|k\|\leqslant i\log_q i} \int_{\{x\in\mathbb{I}|a_{n_1}=k\}} \|a_{n_1}\|\, d\mu \\ &= \sum_{\|k\|\leqslant i\log_q i} \|k\|^{-1} = \sum_{q^r\leqslant i\log_q i} q^{-r}(q-1)q^r = (q-1)\bigl[\log_q(i\log_q i)\bigr], \end{aligned}$$

and

$$\operatorname{var}(V_1) < \mathbb{E}(V_1^2) = \sum_{\|k\| \leqslant i \log_q i} 1 = \sum_{q^r \leqslant i \log_q i} (q-1)q^r < qi \log_q i.$$

Using Chebychev's Inequality [**1**, Proposition 1.7],

$$\begin{aligned}\mu\Big\{x \in \mathbb{I} : \big|V_1 + \cdots + V_i - i\mathbb{E}(V_1)\big| > \varepsilon i \mathbb{E}(V_1)\Big\} &\leqslant \frac{\mathbb{E}\big(V_1 + \cdots + V_i - i\mathbb{E}(V_1)\big)}{\big(\varepsilon i \mathbb{E}(V_1)\big)^2} \\ &= \frac{i \operatorname{var}(V_1)}{\big(\varepsilon i \mathbb{E}(V_1)\big)^2} < \frac{q i^2 \log_q i}{\big(\varepsilon (q-1) i \big[\log_q (i \log_q i)\big]\big)^2},\end{aligned}$$

which tends to zero as i tends to infinity. Since $\mathbb{E}(V_1)$ approximates $(q-1)\log_q i$ for $i \to \infty$, the first summand tends to zero.

For the second summand, simply observe that

$$\begin{aligned}\mu\{x \in \mathbb{I} : W_1 + \cdots + W_i \neq 0\} &\leqslant i\mu\Big\{x \in \mathbb{I} : \big\|a_{n_1}(x)\big\| > i \log_q i\Big\} \\ &= i \sum_{\|k\| > i \log_q i} \|k\|^{-2} < \frac{1}{\log_q i},\end{aligned}$$

which also tends to zero as i tends to infinity. This completes the proof. ∎

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