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On orbits of unipotent flows on homogeneous spaces, II

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Abstract. We show that if (u_t) is a one-parameter subgroup of $\mathrm{SL}(n, \mathbb{R})$ consisting of unipotent matrices, then for any $\varepsilon > 0$ there exists a compact subset K of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ such that the following holds: for any $g \in \mathrm{SL}(n, \mathbb{R})$ either $m(\{t \in [0, T] \mid u_t g\, \mathrm{SL}(n, \mathbb{Z}) \in K\}) > (1-\varepsilon)T$ for all large T (m being the Lebesgue measure) or there exists a non-trivial $(g^{-1}u_t g)$-invariant subspace defined by rational equations.

Similar results are deduced for orbits of unipotent flows on other homogeneous spaces. We also conclude that if G is a connected semisimple Lie group and Γ is a lattice in G then there exists a compact subset D of G such that for any closed connected unipotent subgroup U, which is not contained in any proper closed subgroup of G, we have $G = D\Gamma U$. The decomposition is applied to get results on Diophantine approximation.

0. *Introduction*

Let (u_t) be a one-parameter subgroup in $\mathrm{SL}(n, \mathbb{R})$ consisting of unipotent matrices. In [3], strengthening a result of G. A. Margulis [**10**], it was proved that for any $x \in \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ there exists a compact set K such that the set $\{t \geq 0 \mid u_t x \in K\}$ has positive lower density. Recently while studying density of orbits of horospherical flows (cf. [7]) the author found that in certain contexts it is necessary to know whether the compact set K can be chosen so that the above assertion holds (simultaneously) for all x such that the (u_t)-orbit of x is not contained in a proper *closed* subset of the form $Hx\Gamma/\Gamma$, where $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ and H is a closed subgroup.

On the other hand in [6] it was proved that if G is a simple Lie group of $\mathbb{R}$-rank 1 and Γ is a lattice in G then given $\varepsilon > 0$ and a unipotent one-parameter subgroup (u_t) there exists a compact subset K of G/Γ such that for any $x \in G/\Gamma$ whose (u_t)-orbit is unbounded the lower density of $\{t \geq 0 \mid u_t x \in K\}$ exceeds $(1-\varepsilon)$. This raises the question of whether a similar stronger assertion about the lower density is possible in the above-mentioned case (and more generally for all arithmetic lattices).

The answers to both the questions turn out to be in the affirmative; namely, we have the following.

(3.1) THEOREM. *Let* $n \in \mathbb{N}$ *and* $\varepsilon > 0$ *be given. Then there exists a compact subset* K *of* $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ *such that for any unipotent one-parameter subgroup* (u_t) *and*

$g \in \mathrm{SL}(n, \mathbb{R})$ either

$$m(\{t \in [0, T] \mid u_t g\, \mathrm{SL}(n, \mathbb{Z}) \in K\}) > (1-\varepsilon)T,$$

for all large T (m being the Lebesgue measure), or there exists a $(g^{-1}u_t g)$-invariant proper subspace W of $\mathbb{R}^n$ defined over $\mathbb{Q}$ (by linear equations with rational coefficients).

It is easy to see that in the latter case there exists a proper closed subgroup H such that $Hg\Gamma$ is closed and contains the (u_t)-orbit of $g\Gamma$.

The result readily generalizes to arithmetic lattices in algebraic $\mathbb{R}$-groups and then, in the light of the results in [**6**], to all lattices in semisimple Lie groups (cf. theorem 3.5). A particular consequence of theorem 3.1 is that any (u_t)-orbit which does not intersect K is contained in a closed orbit of a proper closed subgroup H. We generalize this qualitative aspect to any connected unipotent subgroup, of any connected semisimple group G, acting on G/Γ, where Γ is any lattice in G (cf. theorem 3.7). That yields the following decomposition theorem.

(3.9) THEOREM. *Let G be a connected semisimple Lie group and Γ be a (not necessarily cocompact) lattice in G. Then there exists a compact subset D of G such that the following holds: if U is a closed connected unipotent subgroup which is not contained in any proper closed subgroup H such that $H\Gamma$ is closed and $H \cap \Gamma$ is a lattice in H then $G = D\Gamma U = U\Gamma D^{-1}$.*

In terms of G this means that for any 'generic' U as above all the elements of G are within a fixed maximum distance from the set $U\Gamma$ (with respect to a left invariant metric); the set $U\Gamma$ is actually expected to be dense (cf. conjecture II in [**5**]). In [**7**] we obtain a complete description of the closures of orbits of 'horospherical flows' verifying, in particular (the analogue of) conjecture II of [**5**] for that case; the proof makes crucial use of theorem 3.9 (cf. § 3 for some details). On the other hand, even in the general case the theorem enables us to make asymptotic comparison of G-orbits and Γ-orbits in certain linear actions yielding a result which has the flavour of a result on Diophantine approximation with matrix argument (cf. theorem 4.1). In particular the result enables us to obtain an abstract proof of proposition 3.2 in [**8**] which was obtained using classical results on Diophantine approximation together with certain special constructions (cf. corollary 4.2). We also give an application to conjugacy classes of unipotent elements in semisimple Lie groups (cf. corollary 4.3).

In view of the Mahler criterion theorem 3.1 is equivalent to theorem 2.1 whose proof is the subject of § 2. The proof of theorem 2.1 is in spirit very much similar to that of theorem 2.1 in [**3**]. However, unfortunately, it seems hard to explain to, and more so to convince, the reader the places where the latter is to be modified. I have therefore included a complete proof of theorem 2.1. An effort is also made to improve the exposition over that of the latter.

1. *Preliminaries*

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space equipped with the usual inner product $\langle\ ,\ \rangle$. For any non-zero discrete subgroup Δ of $\mathbb{R}^n$ we denote by $\Delta_{\mathbb{R}}$ the vector subspace spanned by Δ, and by $d(\Delta)$ the volume of the torus $\Delta_{\mathbb{R}}/\Delta$ (or equivalently, that of a fundamental domain for Δ in $\Delta_{\mathbb{R}}$) with respect to the inner product obtained by restriction of $\langle\ ,\ \rangle$ to $\Delta_{\mathbb{R}}$. As a convention we set $d(\{0\})=0$. We note the following (cf. [**3**, lemmas 1.1 and 1.5]).

(1.1) LEMMA. *Let $(u_t)_{t\in\mathbb{R}}$ be a one-parameter group of unipotent matrices in* SL $(n, \mathbb{R})$. *For any discrete subgroup Δ, $d^2(u_t\Delta)$ is a polynomial in t of degree at most $2n^2$. Further, the function ν defined for all $t\in\mathbb{R}$ by*

$$\nu(t)=\sup\{d(\Delta)^{-1}d(u_t\Delta)\,|\,\Delta \text{ any discrete subgroup of } \mathbb{R}\}$$

is continuous.

Let Λ be a lattice in $\mathbb{R}^n$. A subgroup Δ of Λ is said to be *complete* (in Λ) if $\Lambda\cap\Delta_{\mathbb{R}}=\Delta$. The set of all complete non-zero subgroups of Λ is denoted by $\mathcal{S}(\Lambda)$. We equip $\mathcal{S}(\Lambda)$ with the partial order given by the inclusion relation. For any totally ordered subset (possibly empty) S put

$$B(S)=\{\Delta\in\mathcal{S}(\Lambda)\,|\,\Delta\notin S \text{ and } S\cup\{\Delta\} \text{ is a totally ordered subset of } \mathcal{S}(\Lambda)\}.$$

Let $\|\cdot\|$ be the norm corresponding to the inner product $\langle\ ,\ \rangle$. The following lemma gives a lower bound for the norms of the non-zero elements of a lattice Λ in terms of the values of d on its subgroups.

(1.2) LEMMA. *Let α, β and θ be given positive real numbers. Let Λ be a lattice in $\mathbb{R}^n$ and S be a totally ordered subset of $\mathcal{S}(\Lambda)$. Suppose that*

(i) $\alpha\le d(\Delta)\le\beta$ *for all* $\Delta\in S$; *and*

(ii) $d(\Delta)\ge\theta$ *for all* $\Delta\in B(S)$.

Then $\|z\|\ge\min\{\alpha,\theta,\beta^{-1}\alpha,\beta^{-1}\theta\}$ *for all* $z\in\Lambda-(0)$.

Proof. Let $z\in\Lambda-(0)$ and let $\langle z\rangle$ denote the subspace spanned by z. If z is contained in every element of S (which includes the case when S is empty) then $\Delta=\Lambda\cap\langle z\rangle\in B(S)\cup S$ and hence $\|z\|\ge d(\Delta)\ge\min\{\alpha,\theta\}$. Now suppose that z is not contained in some element of S and let Δ be the largest such element. Then $\Delta'=\Lambda\cap(\Delta_{\mathbb{R}}+\langle z\rangle)$ is contained in $B(S)\cup S$ so that $d(\Delta')\ge\min\{\alpha,\theta\}$. It is easy to see by an explicit construction of fundamental domains that $d(\Delta')\le\|z\|d(\Delta)$. Hence $\|z\|\ge\beta^{-1}\min\{\alpha,\theta\}$, which proves the lemma.

As in [**10**] and [**3**] we also need the following lemmas on values of non-negative real-valued polynomials (cf. [**3**] for proofs). We denote by $\mathcal{P}_l$ the space of non-negative polynomials on $\mathbb{R}$ of degree at most l.

(1.3) LEMMA. *For any $k>1$ and $l\in\mathbb{N}$ there exist constants $\varepsilon_1(k,l)$ and $\varepsilon_2(k,l)$ such that the following holds: Let $c>0$ and $0\le t_1\le t_2$. If $P\in\mathcal{P}_l$ is such that $P(t)\le c$ for all $t\in[t_1,t_2]$ and $P(t_2)=c$ then there exists j, $0\le j\le l$ such that $P(t)\in[c\varepsilon_1(k,l), c\varepsilon_2(k,l)]$ for all $t\in[t_1+k^{2j+1}(t_2-t_1), t_1+k^{2j+2}(t_2-t_1)]$.*

(1.4) LEMMA. *For any $k>1$ and $l\in\mathbb{N}$ there exists a constant $\bar{\varepsilon}(k,l)$ such that the following holds: Let $c>0$ and $0\le t_1\le t_2$. If $P\in\mathcal{P}_l$ is such that $P(t)\ge c$ for some*

$t \in [t_1, t_2]$ and $P(t_2) < c\bar{\varepsilon}(k, l)$ then there exists $t \in [t_2, t_1 + k(t_2 - t_1)]$ such that $P(t) = c\bar{\varepsilon}(k, l)$.

2. *Orbits of lattices*

This section is devoted to the proof of the following theorem.

(2.1) THEOREM. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that for any unipotent one-parameter subgroup (u_t) in $\mathrm{SL}(n, \mathbb{R})$ and any lattice Λ in $\mathbb{R}^n$ with $d(\Lambda) = 1$ the following holds: either*

$$m(\{t \in [0, T] \mid u_t\Lambda \cap B_\delta = (0)\}) > (1-\varepsilon)T,$$

for all large T or there exists a (u_t)-invariant proper non-zero subspace W of $\mathbb{R}^n$ such that $W \cap \Lambda$ is a lattice in W.

It is evident that the two possibilities in the conclusion are not mutually exclusive. In fact, we shall also prove the following:

(2.2) THEOREM. *Let $n \in \mathbb{N}$, $\varepsilon > 0$ and a lattice Λ in $\mathbb{R}^n$ be given. Then there exists $\delta > 0$ such that for any unipotent one-parameter subgroup (u_t) in $\mathrm{SL}(n, \mathbb{R})$*

$$m(\{t \in [0, T] \mid u_t\Lambda \cap B_\delta = (0)\}) > (1-\varepsilon)T$$

for all large T.

For convenience, the proof is divided into several steps. Let $n \in \mathbb{N}$ and $\varepsilon > 0$, as in the theorems be fixed. We also fix two constants h and k such that $1 < h < k$ and

(2.3) $$(1-k^{-1})(h^{-1}-k^{-1})^n > (1-\varepsilon).$$

Let

(2.4) $$k_1 = 1 + (h-1)k^{-(4n^2+2)}$$

and

(2.5) $$\sigma = \sup\{t \mid \nu(s) < \sqrt{2} \text{ for all } s \in [-t, t]\},$$

where ν is the function given in lemma 1.1; observe that $\sigma > 0$. Also in the sequel for brevity we shall write $\varepsilon_1(k)$, $\varepsilon_2(k)$ and $\bar{\varepsilon}(k_1)$ for the constants $\varepsilon_1(k, 2n^2)$, $\varepsilon_2(k, 2n^2)$ and $\bar{\varepsilon}(k_1, 2n^2)$ respectively, the latter being given by lemmas 1.3 and 1.4; (though we have also fixed k and k_1 it would be convenient, in understanding the proof, not to suppress them from the notation).

(2.6) PROPOSITION. *Let Λ be a lattice in $\mathbb{R}^n$ and (u_t) be a unipotent one-parameter subgroup in $\mathrm{SL}(n, \mathbb{R})$. Let S be a totally ordered subset of $\mathcal{S}(\Lambda)$. Let a, $s \in \mathbb{R}$ and $\theta > 0$ be such that for any $\Delta \in B(S)$ there exists $t \in [a, s]$ such that $d^2(u_t\Delta) \geq \theta$. Then at least one of the following holds: either*

(i) *there exists $s' > s$ such that $d^2(u_t\Delta) > (\frac{1}{2})\theta\bar{\varepsilon}(k_1)$ for all $\Delta \in B(S)$ and $t \in [s, s']$; or*

(ii) *there exist a', $s' \in [s+\sigma, a+h(s-a)]$ such that $(s'-s) = k(a'-s)$ and the following conditions are satisfied:*

(a) *for any $\Delta \in B(S)$ there exists $t \in [s, a']$ such that $d^2(u_t\Delta) \geq (\frac{1}{2})\theta\bar{\varepsilon}(k_1)$; and*

(b) *there exists $\Delta_0 \in B(S)$ such that*

$$\theta\bar{\varepsilon}(k_1)\varepsilon_1(k) \leq d^2(u_t\Delta_0) \leq \theta\bar{\varepsilon}(k_1)\varepsilon_2(k)$$

for all $t \in [a', s'] = [a', s + k(a'-s)]$

Proof. Put

$$\mathscr{C}=\{\Delta\in B(S)\,|\,d^2(u_s\Delta)\le(\tfrac{1}{2})\theta\bar\varepsilon(k_1)\}.$$

First suppose that $\mathscr{C}$ is empty. Consider the set

$$E=\{t\in[s,a+h_0(s-a)]\,|\,d^2(u_t\Delta)>(\tfrac{1}{2})\theta\bar\varepsilon(k_1)\text{ for all }\Delta\in B(S)\}.$$

If $E=[s,a+h(s-a)]$ then we are through. Otherwise let $s'=\inf\{t\in[s,a+h(s-a)]\,|\,t\notin E\}$. Then there exists $\Delta\in B(S)$ such that $d^2(u_{s'}\Delta)=(\tfrac{1}{2})\theta\bar\varepsilon(k_1)$ (cf. [**3**, lemma 1.6]). Hence $s'>s$ and $[s,s')$ is contained in E, which shows that condition (i) holds.

Next suppose that $\mathscr{C}$ is non-empty. It is easy to see that $\mathscr{C}$ is a finite set (cf. [**3**, lemma 1.2]). Since for any $\Delta\in\mathscr{C}\subset B(S)$ there exists $t\in[a,s]$ such that $d^2(u_t\Delta)\ge\theta$ and $d^2(u_s\Delta)\le(\tfrac{1}{2})\theta\bar\varepsilon(k_1)<\theta\bar\varepsilon(k_1)$, by lemma 1.4 the set $H(\Delta)$ defined by

$$H(\Delta)=\{t\in[s,a+k_1(s-a)]\,|\,d^2(u_t\Delta)=\theta\bar\varepsilon(k_1)\}$$

is non-empty. For $\Delta\in\mathscr{C}$ let $t(\Delta)=\inf\{t\in H(\Delta)\}$ and put

$$y=\sup\{t(\Delta)\,|\,\Delta\in\mathscr{C}\}.$$

Since $\mathscr{C}$ is finite there exists $\Delta_0\in\mathscr{C}$ such that $y=t(\Delta_0)$. Observe that y is the smallest number such that for any $\Delta\in\mathscr{C}$ there exists $t\in[s,y]$ for which $d^2(u_t\Delta)\ge\theta\bar\varepsilon(k_1)$. We also note that since $d^2(u_s\Delta_0)\le(\tfrac{1}{2})\theta\bar\varepsilon(k_1)$ and $d^2(u_y\Delta_0)=\theta\bar\varepsilon(k_1)$, $y\ge s+\sigma$.

Since $d^2(u_t\Delta_0)\le\theta\bar\varepsilon(k_1)$ for all $t\in[s,y]$ and $d^2(u_y\Delta_0)=\theta\bar\varepsilon(k_1)$, by lemma 1.3 there exists j, $0\le j\le 2n^2$ such that $d^2(u_t\Delta_0)\in[\theta\bar\varepsilon(k_1)\varepsilon_1(k),\theta\bar\varepsilon(k_1)\varepsilon_2(k)]$ for all $t\in[s+k^{2j+1}(y-s),s+k^{2j+2}(y-s)]$. Put

$$a'=s+k^{2j+1}(y-s)\quad\text{and}\quad s'=s+k^{2j+2}(y-s).$$

Then $s+\sigma\le y<a'<s'\le s+k^{4n^2+2}(k_1-1)(s-a)=a+h(s-a)$. Also evidently $(s'-s)=k(a'-s)$.

Observe that condition (ii(b)) holds automatically because of our choice of a' and s'. Condition (ii(a)) follows from the fact that while for $\Delta\in\mathscr{C}$ there exists $t\in[s,y]\subset[s,a']$ such that $d^2(u_t\Delta)\ge\theta\bar\varepsilon(k_1)$ because of the choice of y, for $\Delta\in B(S)-\mathscr{C}$ we have $d^2(u_s\Delta)>(\tfrac{1}{2})\theta\bar\varepsilon(k_1)$ by the definition of $\mathscr{C}$. This proves the proposition.

Let $R=(0,\infty)\times(0,\infty)\times(1,\infty)$ and let $\psi:R\to R$ be the map defined by

$$\psi(\theta,\alpha,\beta)=((\tfrac{1}{2})\theta\bar\varepsilon(k_1),\min\{\alpha,\theta\bar\varepsilon(k_1)\varepsilon_1(k)\},\max\{\beta,\theta\bar\varepsilon(k_1)\varepsilon_2(k)\})$$

for all θ and α in $(0,\infty)$ and $\beta>1$. We next prove the following.

(2.7) PROPOSITION. *Let Λ be a lattice in $\mathbb{R}^n$ and let (u_t) be a unipotent one-parameter subgroup of* SL$(n,\mathbb{R})$. *Let S be a totally ordered subset of $\mathscr{S}(\Lambda)$ of cardinality p. Let $(\theta,\alpha,\beta)\in R$ and for $i=0,1,\ldots,n$ let $(\theta_i,\alpha_i,\beta_i)=\psi^i((\theta,\alpha,\beta))$ (in particular, $\theta_0=\theta$, $\alpha_0=\alpha$* and *$\beta_0=\beta$). Let*

$$\delta_p=\min_{0\le i\le n-p}\min\{\beta_i^{-1}\alpha_i,(\tfrac{1}{2})\beta_i^{-1}\theta_i\bar\varepsilon(k_1)\}.$$

Let $a,b\in\mathbb{R}$ be such that the following conditions are satisfied:

(a) *for any $\Delta\in B(S)$ there exists $t\in[a,b]$ such that $d^2(u_t\Delta)\ge\theta$; and*

(b) *for any $\Delta\in S$, $\alpha\le d^2(u_t\Delta)\le\beta$ for all $t\in[b,a+k(b-a)]$.*

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Then

$$m(\{t\in[b, a+k(b-a)] \,|\, \|u_t z\|\geq \delta_p \text{ for all } z\in\Lambda-(0)\})$$
$$\geq (k-1)(h^{-1}-k^{-1})^{n-p}(b-a).$$

Proof. We proceed by induction on $(n-p)$: If $p=n$ then by condition (b) and lemma 1.2 $\|u_t z\|\geq \beta^{-1}\alpha\geq\delta_p$ for all $t\in[b, a+k(b-a)]$ and hence our contention evidently holds. Now suppose that the proposition holds for all totally ordered subsets of cardinality $p+1$ (for all possible values of θ, α, β, a and b) and let S be a totally ordered subset of $\mathscr{S}(\Lambda)$ of cardinality p satisfying the hypothesis of the proposition (for certain values of θ, α, β, a and b). Put

$$X=\{t\in[b, a+k(b-a)] \,|\, \|u_t z\|\geq\delta_p \quad \text{for all } z\in\Lambda-(0)\}.$$

We shall first show that for any $s\in[b, a+kh^{-1}(b-a)]$ there exists s' in $[b, a+k(b-a)]$ such that either $s'>s$ and $[s, s')\subset X$ or $s'\geq s+\sigma$ and

$$m(X\cap[s, s'])\geq(1-k^{-1})(h^{-1}-k^{-1})^{n-p-1}(s'-s). \tag{2.8}$$

Let s as above be given. Then the conditions of proposition 2.6 are satisfied for the above values of a, s and θ. Hence one of the conclusions in that proposition must hold. Suppose that (i) holds for some $s'>s$; in that case, in view of condition (b) in the hypothesis and lemma 1.2 we get that for all $t\in[s, s']\cap[b, a+k(b-a)]$,

$$\|u_t z\|\geq\min\{\beta^{-1}\alpha, (\tfrac{1}{2})\beta^{-1}\theta\bar{\varepsilon}(k_1)\}\geq\delta_p$$

for all $z\in\Lambda-(0)$ so that $t\in X$, thus proving the claim. Next suppose that conclusion (ii) holds: Let a', s' and Δ_0 be as in that conclusion. Note that we have

$$s+\sigma\leq s'\leq a+h(s-a)\leq a+k(b-a).$$

Now let $S'=S\cup\{\Delta_0\}$. Since $\Delta_0\in B(S)$, S' is a totally ordered subset of cardinality $p+1$. We observe that the conditions of the proposition under discussion are satisfied for S' for θ_1, α_1, β_1, s and a' in the place of θ, α, β, a and b respectively: condition (a) follows from conclusion (ii(a)) of proposition 2.6 and condition (b) follows from conclusion (ii(b)) of that proposition together with condition (b) as in the hypothesis which is satisfied by S. Since S' has cardinality $p+1$, by our induction hypothesis the contention of the proposition holds for S' for the appropriate values of the constants. Since for any $i\geq 0$,

$$\psi^i(\theta_1, \alpha_1, \beta_1)=\psi^{i+1}(\theta, \alpha, \beta)=(\theta_{i+1}, \alpha_{i+1}, \beta_{i+1}),$$

the conclusion may be stated as follows: if $\delta'_p=\min_{1\leq i\leq n-p}\min\{\beta_i^{-1}\alpha_i, (\tfrac{1}{2})\beta_i^{-1}\theta_i\bar{\varepsilon}(k_1)\}$ and

$$X'=\{t\in[a', s+k(a'-s)] \,|\, \|u_t z\|\geq\delta'_p \quad \text{for all } z\in\Lambda-(0)\}$$

then

$$m(X')\geq(h^{-1}-k^{-1})^{n-p-1}(k-1)(a'-s). \tag{2.9}$$

Clearly X' is contained in $X\cap[s, s']$. Hence (2.9) together with the fact (from proposition 2.6) that $(s'-s)=k(a'-s)$ implies (2.8).

We define a finite sequence $s_0, s_1, \ldots s_r$ in $[b, a+k(b-a)]$ as follows: Let $s_0=b$ and suppose that for some $i\geq 0$, $s_0, s_1, \ldots, s_i$ have been chosen. If $s_i\notin[b, a+kh^{-1}(b-a)]$ then we choose $r=i$, thus terminating the sequence. If $s_i\in[b, a+kh^{-1}(b-a)]$ then by the above argument there exists $s_{i+1}\in[b, a+k(b-a)]$

such that

$$m(X\cap[s_i, s_{i+1}])\geq(1-k^{-1})(h^{-1}-k^{-1})^{n-p-1}(s_{i+1}-s_i)$$

and either $s_{i+1}\geq s_i+\sigma$ or $[s_i, s_{i+1})\subset X$ and $[s_i, s')$ is not contained in X for any $s'>s_{i+1}$. Observe that for any i, $s_{i+2}-s_i\geq\sigma$ so that the sequence necessarily terminates; that is, there exists $r\in\mathbb{N}$ such that $s_r\notin[b, a+kh^{-1}(b-a)]$.

Now we have

$$\begin{aligned}m(X)&\geq\sum_{i=0}^{r-1} m(X\cap[s_i, s_{i+1}])\\&\geq(1-k^{-1})(h^{-1}-k^{-1})^{n-p-1}(s_r-s_0)\\&\geq(k-1)(h^{-1}-k^{-1})^{n-p}(b-a),\end{aligned}$$

since $s_r\geq a+kh^{-1}(b-a)$ and $s_0=b$.

Proof of theorem 2.1. We choose and fix positive numbers θ, α and β satisfying the conditions $\alpha<\beta$, $\beta>1$ and $\theta\leq 1$. For $i=0,1,\dots,n$ let $(\theta_i, \alpha_i, \beta_i)=\psi^i((\theta,\alpha,\beta))$ and let

$$\delta=\min_{0\leq i\leq n}\min\{\beta_i^{-1}\alpha_i, (\tfrac{1}{2})\beta_i^{-1}\theta_i\bar{\varepsilon}(k_1)\}.$$

We note that $\delta>0$. Let Λ be a lattice in $\mathbb{R}^n$ such that $d(\Lambda)=1$ and (u_t) be a unipotent one-parameter subgroup in SL $(n,\mathbb{R})$. First suppose that there exists $b_0\geq 0$ such that for all $\Delta\in\mathcal{S}(\Lambda)$ there exists $t\in[0, b_0]$ such that $d^2(u_t\Delta)\geq\theta$. Then the conditions of proposition 2.7 are satisfied if we choose S as the empty subset, $a=0$ and $b\geq b_0$. Choosing $b=Tk^{-1}$ we conclude from the proposition that for all $T\geq kb_0$

$$m(\{t\in[0, T]\mid \|u_tz\|\geq\delta \text{ for all } z\in\Lambda-(0)\})\geq(k-1)(h^{-1}-k^{-1})^n k^{-1}T.$$

In view of (2.3), namely by our choice of h and k, the above inequality proves that the first possibility as in the conclusion of the theorem holds.

Next suppose that there does not exist any $b_0\geq 0$ for which the above mentioned condition is satisfied. It is easy to see that the set $\mathcal{J}=\{\Delta\in\mathcal{S}(\Lambda)\mid d^2(\Delta)<\theta\}$ is finite. If the condition is not to be satisfied for any b_0 then there must exist $\Delta\in\mathcal{J}$ such that $d^2(u_t\Delta)<\theta$ for all $t\geq 0$. Since $d^2(u_t\Delta)$ is a polynomial in t the last condition implies that $d^2(u_t\Delta)$ is constant. We note that since $d^2(\Delta)<\theta\leq 1=d^2(\Lambda)$, in particular, Δ is a proper subgroup of Λ; further since Δ is complete in Λ it is of rank $q<n$. Let $x_1, x_2,\dots,x_q$ be a free set of generators of Δ. There exists a norm $\|\cdot\|$ on $\bigwedge^q\mathbb{R}^n$ such that

$$d(u_t\Delta)=\left\|\left(\bigwedge^q u_t\right)(x_1\wedge x_2\wedge\cdots\wedge x_q)\right\|$$

for all t (cf. [3, lemma 1.4]). Since $d^2(u_t\Delta)$ is constant this means that the $(\bigwedge^q u_t)$-orbit of $x_1\wedge x_2\wedge\cdots\wedge x_q$ is bounded. Since (u_t) is a unipotent one-parameter subgroup, so is $(\bigwedge^q u_t)$ and therefore every bounded orbit of $(\bigwedge^q u_t)$ consists of a fixed point; that is, we have $(\bigwedge^q u_t)(x_1\wedge\cdots\wedge x_q)=x_1\wedge\cdots\wedge x_q$ for all t. This implies that the subspace W of $\mathbb{R}^n$ spanned by $x_1,\dots,x_q$ is invariant under (u_t). Also clearly $W\cap\Lambda=\Delta$ is a lattice in W. This proves the theorem.

Proof of theorem 2.2. In this case we choose θ so that for the given lattice Λ we have $d^2(\Delta)\geq\theta$ for all $\Delta\in\mathcal{S}(\Lambda)$. Existence of such a θ follows from lemma 1.4 in

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[3]. The theorem then follows from the same argument as in the first part of the above proof of theorem 2.1.

3. *Orbits of unipotent flows*

Using the well-known Mahler criterion (cf. [12, corollary 10.9]) theorem 2.1 can be reinterpreted as follows.

(3.1) THEOREM. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then there exists a compact subset K of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ such that for any unipotent one-parameter subgroup (u_t) in $SL(n, \mathbb{R})$ and any $g \in SL(n, \mathbb{R})$ the following holds: either for the point $x = g\, SL(n, \mathbb{Z})$ we have*

$$m(\{t \in [0, T] \mid u_t x \in K\}) > (1-\varepsilon)T,$$

for all large T or there exists a proper non-zero subspace W of $\mathbb{R}^n$ defined over $\mathbb{Q}$ (defined by linear equations with rational coefficients) which is invariant under $g^{-1}u_t g$ for all $t \in \mathbb{R}$.

Proof. $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ can be identified canonically with the space of lattices Λ in $\mathbb{R}^n$ for which $d(\Lambda) = 1$ via the (well-defined) map $g\, SL(n, \mathbb{Z}) \leftrightarrow g(\mathbb{Z}^n)$ for all $g \in SL(n, \mathbb{R})$. Let $\delta > 0$ be as in theorem 2.1 for the given values of n and ε. By the Mahler criterion, under the above correspondence the set of lattices Λ with $d(\Lambda) = 1$, which do not contain any non-zero element from B_δ correspond to a compact subset, say K, of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$. Hence by theorem 2.1 for any given unipotent one-parameter subgroup (u_t) and any element $g \in SL(n, \mathbb{R})$ either

$$m(\{t \in [0, T] \mid u_t g\, SL(n, \mathbb{Z}) \in K\}) > (1-\varepsilon)T,$$

for all large T or there exists a proper non-zero (u_t)-invariant subspace W such that $W \cap g(\mathbb{Z}^n)$ is a lattice in W. In the latter case $g^{-1}(W)$ is a $(g^{-1}u_t g)$-invariant subspace in which $g^{-1}(W) \cap \mathbb{Z}^n$ is a lattice; but any subspace of $\mathbb{R}^n$ intersecting $\mathbb{Z}^n$ in a lattice is defined by linear equations with rational coefficients.

Similarly theorem 2.2 implies the following.

(3.2) THEOREM. *Let $n \in \mathbb{N}$, $\varepsilon > 0$ and $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ be given. Then there exists a compact subset K of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ such that for any unipotent one-parameter subgroup (u_t) in $SL(n, \mathbb{R})$*

$$m(\{t \in [0, T] \mid u_t x \in K\}) > (1-\varepsilon)T$$

for all large T.

These theorems can easily be generalized to arithmetic lattices in algebraic groups.

(3.3) THEOREM. *Let G be the group of $\mathbb{R}$-elements of a Zariski-connected semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ and Γ be an arithmetic lattice in G. Then given $\varepsilon > 0$ there exists a compact subset C of G/Γ such that for any unipotent one-parameter subgroup (u_t) and $g \in G$ the following holds: either*

$$m(\{t \in [0, T] \mid u_t g\Gamma \in C\}) > (1-\varepsilon)T$$

for all large T or there exists a proper algebraic subgroup $\mathbf{H}$ defined over $\mathbb{Q}$ such that $g^{-1}u_t g \in \mathbf{H}_\mathbb{R}$ for all $t \in \mathbb{R}$.

Proof. Without loss of generality we may assume that $\mathbf{G}$ is simple over $\mathbb{Q}$; that is, there exists no normal algebraic subgroup defined over $\mathbb{Q}$. Further clearly we may also assume $\Gamma = \mathbf{G}_{\mathbb{Z}}$. Let $\rho : \mathbf{G} \to \mathbf{GL}(\mathbf{V})$ be a non-trivial irreducible representation of $\mathbf{G}$ defined over $\mathbb{Q}$. Then ρ induces a natural map $\bar{\rho} : G/\mathbf{G}_{\mathbb{Z}} \to \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$, where n is the dimension of $\mathbf{V}$. It is well known that $\bar{\rho}$ is a proper map. Let K be a compact subset of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ such that the contention of theorem 3.2 holds for the given $\varepsilon > 0$ and n as above. Put $C = \bar{\rho}^{-1}(K)$. Since $\bar{\rho}$ is proper C is a compact subset. Then, since $(\rho(u_t))$ is a unipotent one-parameter subgroup of $\mathrm{SL}(n, \mathbb{R})$, for any $g \in G$ theorem 3.2 implies that either $m(\{t \in [0, T] \mid u_t g\Gamma \in C\}) > (1-\varepsilon)T$ for all large T or there exists a proper non-zero subspace $\mathbf{W}$ of $\mathbf{V}$ defined over $\mathbb{Q}$ which is invariant under $\rho(g^{-1}u_t g)$ for all $t \in \mathbb{R}$. Suppose the latter holds. Let $\mathbf{H} = \{h \in G \mid \rho(h)(\mathbf{W}) = \mathbf{W}\}$. Then $\mathbf{H}$ is a subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ and $g^{-1}u_t g \in \mathbf{H}_{\mathbb{R}}$ for all $t \in \mathbb{R}$. Since $\mathbf{W}$ is proper and non-zero and $\bar{\rho}$ is irreducible we conclude that $\mathbf{H}$ is a proper subgroup, thus proving the theorem.

We would like to generalize theorem 3.3 to all (not necessarily arithmetic) lattices in semisimple Lie groups. For this purpose we first formulate a Lie group theoretic variant of the second possibility in that theorem.

(3.4) Proposition. *Let G be the connected component of the identity in the group of $\mathbb{R}$-elements of a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$. Let Γ be an arithmetic lattice in G and let U be a subgroup of G consisting of unipotent elements. Then the following conditions are equivalent:*

(a) *There exists a proper algebraic subgroup $\mathbf{H}$ of $\mathbf{G}$ defined over $\mathbb{Q}$ such that U is contained in $\mathbf{H}_{\mathbb{R}}$.*

(b) *There exists a proper closed connected subgroup H of G such that U is contained in H, $H\Gamma$ is closed and $H \cap \Gamma$ is a lattice in H.*

Proof. (a)$\Rightarrow$(b) Let $\mathbf{H}'$ be the intersection of the kernels of characters of $\mathbf{H}$ into $\mathbf{GL}(1)$ which are defined over $\mathbb{Q}$. Since U consists of unipotent elements U is contained in $\mathbf{H}'_{\mathbb{R}}$. Since $\mathbf{H}'$ does not admit any non-trivial character defined over $\mathbb{Q}$, by a theorem of Borel and Harish-Chandra (cf. [**1**, theorem 13.1]) $\mathbf{H}'_{\mathbb{R}} \cap \Gamma$ is a lattice in $\mathbf{H}'_{\mathbb{R}}$. Also by a well-known result (cf. [**5**, lemma 2.2], for instance) $\mathbf{H}'_{\mathbb{R}}\Gamma$ is closed. Now if H is the connected component of the identity in $\mathbf{H}'_{\mathbb{R}}$ then it has finite index in the latter and hence in view of the above, $H\Gamma$ is closed and $H \cap \Gamma$ is a lattice in H. Further, H contains U as it contains all the unipotent elements in $\mathbf{H}'_{\mathbb{R}}$.

(b)$\Rightarrow$(a) Let $\mathbf{H}$ be the Zariski closure of $H \cap \Gamma$. Since Γ is arithmetic $\mathbf{H}$ is an algebraic subgroup defined over $\mathbb{Q}$. Since $H \cap \Gamma$ is a lattice in H, by a version of Borel's density theorem all unipotent elements in H are contained in the Zariski closure of $H \cap \Gamma$ (cf. [**4**, corollary 4.2], for instance). In particular, $U \subset \mathbf{H}_{\mathbb{R}}$. We claim that $\mathbf{H}$ is proper. Suppose otherwise; then in particular H itself must be Zariski-dense in $\mathbf{G}$. Therefore the (solvable) radical of H is a normal subgroup of $\mathbf{G}_{\mathbb{R}}$. Since G is semisimple the radical must hence be trivial. Therefore H must be semisimple. But any semisimple Lie subalgebra is algebraic (cf. [**2**, Ch. II, theorem 15]) and this implies that H must be open in $\mathbf{G}_{\mathbb{R}}$. But this contradicts the hypothesis that H is a proper subgroup of G. Hence (b)$\Rightarrow$(a).

We note that in the proposition if U is connected then in (b) we may drop the connectedness condition for H, as the connected component of the identity in H has the desired properties.

Now let G be any connected semisimple Lie group. An element $u \in G$ is said to be unipotent if $\operatorname{Ad} u$ is unipotent and a subgroup U is said to be unipotent if it consists only of unipotent elements. If G is the connected component of the identity in $\mathbf{G}_{\mathbb{R}}$, where $\mathbf{G}$ is an algebraic group defined over $\mathbb{R}$ then the above notion of a unipotent subgroup U coincides with the usual notion in algebraic groups if either G has trivial centre or U is connected.

(3.5) THEOREM. *Let G be a connected semisimple Lie group and Γ be a lattice in G. Then given $\varepsilon > 0$ there exists a compact subset C of G/Γ such that for any unipotent one-parameter subgroup (u_t) and $g \in G$ the following holds: either*

$$m(\{t \in [0, T] \mid u_t g\Gamma \in C\}) > (1-\varepsilon)T \tag{3.6}$$

for all large T or there exists a proper closed subgroup H of G such that $H\Gamma$ is closed, $H \cap \Gamma$ is a lattice in H and $g^{-1}u_t g \in H$ for all $t \in \mathbb{R}$.

Further, there exists a countable family $\mathcal{F}$ of proper closed subgroups of G depending only on Γ (and not on (u_t) or g or ε) such that if there exists a subgroup H satisfying the above conditions (for some (u_t) and g as above) then we can choose one such from the family $\mathcal{F}$.

Proof. If Z is the centre of G then $Z\Gamma$ is closed and $Z\Gamma/\Gamma$ is finite (cf. [**12**, corollary 5.17]). In view of this, replacing G by G/Z if necessary, we may assume that G has trivial centre. Similarly we may also assume that G has no non-trivial compact normal subgroup.

There exist closed connected normal subgroups $G_1, G_2, \ldots, G_r$, $r \geq 1$, of G such that $G = G_1 \cdot G_2 \cdot \cdots \cdot G_r$ (direct product) and for $i = 1, 2, \ldots, r$, $\Gamma_i = \Gamma \cap G_i$ is an irreducible lattice in G_i (cf. [**12**, theorem 5.22]). Thus G/Γ is a quotient in a natural way of $\Pi G_i/\Gamma_i$. In view of this it is enough to prove the theorem for irreducible lattices Γ_i in the connected semisimple groups G_i. In other words, we may assume Γ to be irreducible. Then by Margulis's arithmeticity theorem (cf. [**11**]) one of the following (not mutually exclusive) conditions holds: either (a) G/Γ is compact or (b) G is of $\mathbb{R}$-rank 1 or (c) G has the structure of the connected component of the identity in the group of $\mathbb{R}$-elements of an algebraic group defined over $\mathbb{Q}$ and Γ is an arithmetic lattice. The contention of the theorem is obvious if condition (a) holds and follows from theorem 3.3 and proposition 3.4 if condition (c) holds; we also note that the class of algebraic subgroup $\mathbf{H}$ defined over $\mathbb{Q}$ is countable.

A proof of the theorem in the remaining case, namely when G is a simple Lie group of $\mathbb{R}$-rank 1, is essentially contained in [**6**]. Theorem 0.2 in [**6**] differs from the present theorem only in the following two ways: (i) The statement of the former does not emphasize that the compact set may be chosen independently of the unipotent one-parameter subgroup. A scrutiny of the proof will however show that the choice of the compact set is indeed independent of the unipotent one-parameter subgroup. (ii) The former gives boundedness of the (u_t)-orbit of $g\Gamma$ as the alternative to validity of (3.6). The proof actually shows that (3.6) holds unless for some $a \in \mathbb{R}$ and $\sigma \in \Sigma$, $u_t g\Gamma \in X(\sigma, s)$ for all $t > a$; (notation as in [**6**] and s as chosen in § 3 of

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that paper). Suppose the latter condition holds. Then by lemma 2.4 in [**6**] there exists $\gamma \in \Gamma$ such that $\{\rho(u_t g\gamma)v \mid t > a\}$ is a bounded subset. Since $(\rho(u_t))$ is a unipotent one-parameter subgroup any bounded orbit consists only of a fixed point; thus $\rho(u_t)$ must fix $\rho(g\gamma)v$. This implies that $(g^{-1}u_t g)$ is contained in the isotropy subgroup of $\rho(\gamma)v$, which is $\gamma\sigma^{-1}(K \cap Z) \cdot N\sigma\gamma^{-1}$; we note that K is a compact subgroup. Since $(g^{-1}u_t g)$ consists of unipotent elements it must actually be contained in $\gamma\sigma^{-1}N\sigma\gamma^{-1}$. But, by choice, N is such that $\Gamma \cap \sigma^{-1}N\sigma$ is a cocompact lattice in $\sigma^{-1}N\sigma$. Therefore $H = \gamma\sigma^{-1}N\sigma\gamma^{-1}$ has the desired properties. It is also clear that H comes from a fixed countable family (depending only on Γ) since in the above expression for H, N is fixed, σ varies in a finite set Σ, and γ varies in Γ which is countable.

(3.7) *Remark.* Let the notation be as in theorem 3.5. If the $\mathbb{R}$-rank of G is 1 then the subgroup H as in the second possible conclusion can also be chosen to be a maximal horospherical subgroup intersecting Γ in a lattice; this is obvious from the above proof.

Theorem 3.5 has a purely qualitative aspect: if $g \in G$ is such that $\{u_t g\Gamma \mid t \in \mathbb{R}\}$ is totally outside C then the latter condition must hold. Though this seems rather special it has certain interesting consequences. We first generalize the assertion to all connected unipotent subgroups.

(3.8) Theorem. *Let G be a connected semisimple Lie group and Γ be a lattice in G. Then there exists a compact subset C of G/Γ such that for any closed connected unipotent subgroup U of G and any $g \in G$ either $C \cap Ug\Gamma/\Gamma$ is non-empty or $g^{-1}Ug$ is contained in a proper closed subgroup H such that $H\Gamma$ is closed and $H \cap \Gamma$ is a lattice in H.*

Proof. Let $\theta \in (0, 1)$ be arbitrary and let C and $\mathscr{F}$ be the compact subset of G/Γ and the family of closed subgroups given by theorem 3.5. Let U be a closed connected unipotent subgroup and $g \in G$ be such that $C \cap Ug\Gamma/\Gamma$ is empty. Then by theorem 3.5 for any (unipotent) one-parameter subgroup $X = (u_t)$ in U there exists a closed subgroup $H(X) \in \mathscr{F}$ such that $g^{-1}Xg$ is contained in $H(X)$, $H(X)\Gamma$ is closed and $H(X) \cap \Gamma$ is a lattice in $H(X)$. Since U is a connected nilpotent Lie group every element of U is contained in a one-parameter subgroup of U (since the exponential map is surjective). Hence the last observation implies that $g^{-1}Ug$ is contained in $\bigcup H(X)$, where the union is over all one-parameter subgroups. Since a connected Lie group cannot be expressed as a union of countably many proper closed subgroups and since each $H(X)$ belongs to $\mathscr{F}$, which is a countable family, it follows that there exists a one-parameter subgroup X of U such that $g^{-1}Ug$ is contained in $H(X)$. Since $H(X)\Gamma$ is closed and $H(X) \cap \Gamma$ is a lattice in $H(X)$ this proves the theorem.

(3.9) Theorem. *Let G be a connected semisimple Lie group and Γ be a lattice in G. Then there exists a compact subset D of G such that the following holds: if U is a closed connected unipotent subgroup which is not contained in any closed subgroup H such that $H\Gamma$ is closed and $H \cap \Gamma$ is a lattice in H then $G = D\Gamma U = U\Gamma D^{-1}$.*

Proof. Let C be a compact subset of G/Γ as in the conclusion in theorem 3.8. Let D be a compact subset of G such that $D\Gamma/\Gamma = C$. Let U be as in the statement of the theorem and let $g \in G$ be arbitrary. Since $g^{-1}(gUg^{-1})g = U$ is not contained in any closed subgroup as in theorem 3.8 we deduce that $C \cap gU\Gamma/\Gamma$ is nonempty. Since $C = D\Gamma/\Gamma$ this implies that there exists $u \in U$ and $\gamma \in \Gamma$ such that $gu\gamma \in D$. Hence $g \in D\gamma^{-1}u^{-1} \in D\Gamma U$. Since $g \in G$ is arbitrary, it follows that $G = D\Gamma U$. Also $G = G^{-1} = U\Gamma D^{-1}$.

(3.10) *Remark.* Let G be the connected component of the identity in $\mathbf{G}_{\mathbb{R}}$ where $\mathbf{G}$ is a semisimple algebraic group defined over $\mathbb{Q}$ and let Γ be an arithmetic lattice in G. Let $P = \mathbf{P} \cap G$, where P is a proper parabolic subgroup defined over $\mathbb{Q}$. Then for any unipotent subgroup U of P and *any* compact subset D_1 of G, $D_1\Gamma U$ is a proper subset of G. This may be proved by observing that the conclusion of theorem 4.1 below does not hold for a $\mathbb{Q}$-representation of $\mathbf{G}$ on a vector space $\mathbf{V}$ defined over $\mathbb{Q}$, if we choose v to be a rational vector such that the one-dimensional subspace spanned by v is $\mathbf{P}$-invariant (cf. [**1**, proposition 7.8]). In view of proposition 3.4 this shows that the condition on U in theorem 3.9 is not redundant.

Let G and Γ be as above. For $g_0 \in G$ the subgroup $U = \{u \in G \mid g_0^j u g_0^{-j} \to e \text{ as } j \to \infty\}$, where e is the identity in G, is called the (*contracting*) *horospherical subgroup* corresponding to g_0. It is well-known that horospherical subgroups are closed connected unipotent subgroups. In [**7**] we prove that if $g_0 \in G$ is a semisimple element in G (that is, Ad g_0 is semisimple) acting ergodically on G/Γ (with respect to the G-invariant probability measure) and U is the corresponding horospherical subgroup then for $g \in G$, $Ug\Gamma/\Gamma$ is dense in G/Γ whenever there exists a sequence $\{u_j\}$ in $g^{-1}Ug$ such that the sequence $g_0^j u_j \Gamma$ has a convergent subsequence. By theorem 3.9 the latter is indeed the case whenever $g^{-1}Ug$ is not contained in a closed subgroup H such that $H\Gamma$ is closed, which, in fact, is also a necessary condition for $Ug\Gamma/\Gamma$ to be dense. Thus every non-dense orbit of U is contained in a closed orbit of a *proper closed subgroup*, with some further work we are then able to show that the closure of any orbit of U is actually the orbit of a closed connected subgroup containing U – in particular, a homogeneous space – thus verifying (analogue of) conjecture II in [**5**] for horospherical subgroups. The result generalizes the well-known theorem on minimality of ergodic horospherical flows on *compact* homogeneous spaces G/Γ.

Earlier in [**8**] we had proved the homogeneity of the closure of orbits in a very special case, viz. $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ and U the subgroup consisting of all elements fixing (under the natural action) $e_1, \ldots, e_{n-1}$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. In that case the role of theorem 3.9 was played by a result involving Diophantine approximation. Indeed, theorem 3.9 can also be interpreted as a result on Diophantine approximation. We shall do this in the next section. We conclude the present section with the following remarks.

(3.11) *Remark.* Let G and Γ be as in theorem 3.7 and let U be a not necessarily connected unipotent subgroup. Then still one can assert the existence of a compact set C *depending on* U such that for any $g \in G$ either $G \cap Ug\Gamma/\Gamma$ is non-empty or $g^{-1}Ug$ is contained in a proper closed connected subgroup H such that $H\Gamma$ is closed. This may be deduced from theorem 3.8 using the fact that if Z is the centre of G then UZ is a cocompact subgroup of $U'Z$ for a suitable closed connected unipotent subgroup U' (we could choose U' to be the unique subgroup such that $U'Z/Z$ is the Zariski closure of UZ/Z in G/Z (which is the adjoint group of G). However, it is not clear whether C can be chosen independent of U.

(3.12) *Remark.* The quantitative aspect of theorem 3.5 can also be generalized along the lines of [**3**] to all unipotent subgroups. In particular it can be proved that given $\varepsilon > 0$ there exists a compact subset K such that if U is a connected unipotent subgroup and π is an ergodic U-invariant measure then either $\pi(K) > 1-\varepsilon$ or there exists a closed subgroup H as in theorem 3.5 and such that π is supported on $H\Gamma/\Gamma$. As in remark 3.11 this also implies an analogous result for all not necessarily connected, unipotent subgroups if we allow K to depend on the subgroup. We shall, however, not go into the details of proofs of these assertions.

4. *Diophantine approximation*

Let G be a connected Lie group acting as a group of linear transformations of a real vector space V of dimension n. Let Γ be a lattice in G. Given $v \in V-(0)$ we ask whether there exists a sequence $\{\gamma_j\}$ in Γ such that $\gamma_j v \to 0$, or equivalently whether $\|\gamma v\| < \varepsilon$, where $\|\ \|$ is a certain norm on V, admits a solution $\gamma \in \Gamma$ for an arbitrary $\varepsilon > 0$. If $G = \mathbf{G}_{\mathbb{R}}$ where $\mathbf{G}$ is an algebraic group defined over $\mathbb{Q}$ and $\Gamma = \mathbf{G}_{\mathbb{Z}}$ then solving the above inequality is equivalent to finding for any $\varepsilon > 0$ solutions $x_1, \ldots, x_n$ of the Diophantine inequality $\|x \cdot v\| < \varepsilon$ with the additional condition that $x_1, \ldots, x_n$ form the rows of an element of $\mathbf{G}_{\mathbb{Z}}$. More generally, the above question may be viewed as a question of Diophantine approximation with the argument in a discrete group of matrices (rather than integers).

Propositions 3.2 and 3.3 in [**8**] provide satisfactory answers to the question in the particular cases of p-fold product action of $\mathrm{SL}(n, \mathbb{Z})$ on $(\mathbb{R}^n)^p$ for $1 \le p \le n-1$ and $\mathrm{Sp}(2n, \mathbb{Z})$ on $(\mathbb{R}^{2n})^p$ for $1 \le p \le n$. For instance, the former implies that given $\xi_1, \xi_2, \ldots, \xi_p \in \mathbb{R}^n$, $p \ge 1$, there exists a sequence $\{\gamma_j\}$ in $\mathrm{SL}(n, \mathbb{Z})$ such that $\gamma_j \xi_i \to 0$ for all $i = 1, 2, \ldots, p$ if and only if the subspace spanned by $\xi_1, \ldots, \xi_p$ does not contain any non-zero rational vector. The proofs of the theorems are based on classical results on Diophantine approximation of values of linear forms. Here we use theorem 3.9 to obtain a sufficient condition for affirmative answer for the general question and deduce the above-mentioned particular case. We also give an application to conjugacy classes of unipotent elements.

(4.1) Theorem. *Let G, Γ and V be as above. Let $v \in V-(0)$ be such that*:

(i) *there exists a sequence $\{g_j\}$ in G such that $g_j v \to 0$; and*

(ii) *the isotropy subgroup of v, namely the subgroup $\{g \in G \mid gv = v\}$, contains a connected unipotent subgroup U which is not contained in any closed subgroup H such that $H\Gamma$ is closed and $H \cap \Gamma$ is a lattice in H.*

Then there exists a sequence $\{\gamma_j\}$ in Γ such that $\gamma_j v \to 0$.

Proof. By theorem 3.9 there exists a sequence $\{d_j\}$ in a compact set D, $\{\gamma_j\}$ in Γ and $\{u_j\}$ in U such that $g_j = d_j \gamma_j u_j$. Let $\|\ \|$ be a norm on V (making it a normed vector space). Then in view of the boundedness of $\{d_j\}$ there exists a constant $M > 0$ such that $\|d_j^{-1}w\| \leq M\|w\|$ for all $w \in V$ and $j \in \mathbb{N}$. Hence

$$\|\gamma_j v\| = \|\gamma_j u_j v\| = \|d_j^{-1} g_j v\| \leq M\|g_j v\| \to 0,$$

as $j \to \infty$, since by hypothesis $g_j v \to 0$. Thus $\gamma_j v \to 0$.

(4.2) COROLLARY. *Let $p \geq 1$ and $\xi_1, \ldots, \xi_p \in \mathbb{R}^n$. Then there exists a sequence $\{\gamma_j\}$ in $\mathrm{SL}(n, \mathbb{Z})$ such that $\gamma_j \xi_i \to 0$ as $j \to \infty$ for all $i = 1, 2, \ldots, p$ if and only if the subspace spanned by $\xi_1, \ldots, \xi_p$ does not contain any non-zero rational vector.*

Proof. If $x = \sum_{i=1}^{p} \lambda_i \xi_i$ is a non-zero rational vector then $\{\gamma(x) | \gamma \in \mathrm{SL}(n, \mathbb{Z})\}$ is a discrete set not containing zero, so that the simultaneous convergence cannot hold. Now suppose that the subspace spanned by $\xi_1, \ldots, \xi_p$ does not contain any non-zero rational vector. There is no loss of generality in assuming that $\xi_1, \ldots, \xi_p$ are linearly independent and (by including more vectors if necessary) that $p = n-1$. Let $G = \mathrm{SL}(n, \mathbb{R})$ and U be the isotropy subgroup of $(\xi_1, \ldots, \xi_{n-1})$ under the $(n-1)$-fold product action. U is a connected unipotent subgroup conjugate to the subgroup $U_0 = \{g \in G | g(e_i) = e_i$ for all $i = 1, 2, \ldots, n-1\}$, $e_1, \ldots, e_n$ being the standard basis of $\mathbb{R}^n$. Since $\mathrm{SL}(n, \mathbb{Z})$ is a lattice in $\mathrm{SL}(n, \mathbb{R})$ (cf. [**12**, corollary 10.5]) and the $\mathrm{SL}(n, \mathbb{R})$-orbit of $(\xi_1, \ldots, \xi_{n-1})$ contains zero in its closure, by theorem 4.1 there exists a sequence $\{\gamma_j\}$ in $\mathrm{SL}(n, \mathbb{Z})$ such that $\gamma_j(\xi_1, \ldots, \xi_{n-1}) \to 0$ unless U is contained in a proper closed subgroup H such that $H\,\mathrm{SL}(n, \mathbb{Z})$ is closed. Suppose the latter condition holds. From the way we found the subgroup H, in the particular case of the lattice $\mathrm{SL}(n, \mathbb{Z})$ in $\mathrm{SL}(n, \mathbb{R})$ it is clear that there exists a proper non-zero H-invariant subspace W of $\mathbb{R}^n$ which is defined by linear equations with rational coefficients (cf. theorem 3.1 and the proof of theorem 3.8); actually even if we choose to forget how we arrived at H, it is possible to conclude the existence of such an invariant subspace W from the above data using certain results in algebraic groups. We shall however not go into the details of this. The subspace W is in particular invariant under U. However, it is easy to see that any proper U-invariant subspace consists of fixed points; this need only be proved for U_0 which is a conjugate of U. Hence, in particular, U must fix a non-zero rational point. However, evidently the space spanned by $\xi_1, \xi_2, \ldots, \xi_{n-1}$ is precisely the set of points fixed by U. Hence the latter must contain a non-zero rational vector – this contradicts the hypothesis; hence the corollary.

Proposition 3.2 in [**8**] actually shows that if $\xi_1, \ldots, \xi_{n-1} \in \mathbb{R}^n$ span a $(n-1)$-dimensional subspace not containing a non-zero rational vector then there exist sequences γ_j in $\mathrm{SL}(n, \mathbb{Z})$ and λ_j in $\mathbb{R}^+$ such that for all $i = 1, 2, \ldots, n-1$, $\gamma_j \xi_i \to 0$ and $\lambda_j \gamma_j \xi_i \to \eta_j$ where $\eta_1, \ldots, \eta_{n-1}$ are linearly independent vectors in $\mathbb{R}^n$. The latter kind of condition can be incorporated in theorem 4.1 if we can choose the sequence $\{g_j\}$ in G as in the hypothesis of the theorem to be contained in the normalizer of the unipotent subgroup U; this is indeed true in the special case.

One can also deduce an analogue of corollary 4.2 in the case of $\mathrm{Sp}(2n, \mathbb{Z})$ as in [**8**]. This, however, seems to involve considerably more work. We shall not go into the details since in any case this (as also corollary 4.2) follows from the stronger results in [**7**]. In either case the isotropy subgroups contain horospherical subgroups to which results of [7] are applicable. We shall next give an application in which the isotropy subgroup in question does *not* in general, contain a horospherical subgroup.

(4.3) COROLLARY. *Let G be a connected semisimple Lie group with trivial centre and Γ be a lattice in G. Let u be a unipotent element in G. Suppose that u is not contained in a closed subgroup H such that $H\Gamma$ is closed. Then the closure of the conjugacy class $\{\gamma u \gamma^{-1} \mid \gamma \in \Gamma\}$ contains the identity element.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G and consider the adjoint action of G on $\mathfrak{g}$. The unipotent element u can be expressed as $\exp X$ for some nilpotent element $X \in \mathfrak{g}$. Since G is semisimple, by the Jacobson-Morosov lemma (cf. [**9**, Ch. III, theorem 17]) there exists $Y \in \mathfrak{g}$ such that $\mathrm{Ad}\,(\exp jY)(X) \to 0$. The unipotent one-parameter subgroup $(\exp tX)$ fixes X under the adjoint action and it is not contained in any closed subgroup H such that $H\Gamma$ is closed. Hence by theorem 4.1 there exists a sequence $\{\gamma_j\}$ in Γ such that $\mathrm{Ad}\, \gamma_j(X) \to 0$; in other words, $\gamma_j u \gamma_j^{-1} \to$ the identity.

(4.4) *Remark.* Let G and Γ be as above. If G/Γ is compact, condition (ii) in theorem 4.1 is not needed in the proof; in that case the above argument then implies that for any unipotent element u the conjugacy class $\{\gamma u \gamma^{-1} \mid \gamma \in \Gamma\}$ has the identity in its closure. If G/Γ is non-compact, however, one cannot expect such an assertion since in that case Γ itself contains (non-trivial) unipotent elements; the conjugacy class of such an element would be discrete. Similarly if u is any unipotent element such that the one-parameter subgroup containing u intersects Γ non-trivially then $\{\gamma u \gamma^{-1} \mid \gamma \in \Gamma\}$ is discrete. The author is unable at this stage to characterize the set of unipotent elements u for which the closure of the $\{\gamma u \gamma^{-1} \mid \gamma \in \Gamma\}$ contains the identity and whether this is true whenever the set is non-discrete. Note that since all the unipotent elements contained in proper closed subgroups H such that $H\Gamma$ is closed are contained in a fixed countable family of proper closed subgroups (defined by algebraic subgroups defined over $\mathbb{Q}$ if Γ is arithmetic), corollary 4.3 answers the question for a generic unipotent element. This is in contrast with the fact that for a non-unipotent element of G the conjugacy class is always bounded away from the identity since the set of its eigenvalues is invariant under conjugation.

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