Proceedings of the Edinburgh Mathematical Society (2000) **43**, 167–175 ©

THE POSITIVE CONES OF K_0-GROUPS OF CROSSED PRODUCTS ASSOCIATED WITH FURSTENBERG TRANSFORMATIONS ON THE 2-TORUS

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(Received 18 June 1998)

Abstract Let θ be an irrational number in $(0,1)$ and f a real-valued continuous function on the 1-torus $\boldsymbol{T}$. Let $\phi_{\theta,f}$ be a Furstenberg transformation on the 2-torus $\boldsymbol{T}^2$ defined by $\phi_{\theta,f}^{-1}(t,s)=(t+\theta,s+\rho t+f(t))$ for any $(t,s)\in\boldsymbol{T}^2$, where ρ is a non-zero integer, and we identify a function on $\boldsymbol{T}$ or $\boldsymbol{T}^2$ with a function on $\boldsymbol{R}$ or $\boldsymbol{R}^2$ with period 1, respectively. Let $A_{\theta,f}$ be the crossed product associated with $\phi_{\theta,f}$. In this paper we will compute the positive cone of the K_0-group of $A_{\theta,f}$.

Keywords: crossed product; equivalence bimodule; full projection; Furstenberg transformation; K_0-group; positive cone

AMS 1991 *Mathematics subject classification:* Primary 46L80

1. Introduction

Let θ be an irrational number in $(0,1)$ and f a real-valued continuous function on the 1-torus $\boldsymbol{T}$. Let $\phi_{\theta,f}$ be a Furstenberg transformation on the 2-torus $\boldsymbol{T}^2$ defined by $\phi_{\theta,f}^{-1}(t,s)=(t+\theta,s+\rho t+f(t))$ for any $(t,s)\in\boldsymbol{T}^2$, where ρ is a non-zero integer, and we identify a function on $\boldsymbol{T}$ or $\boldsymbol{T}^2$ with a function on $\boldsymbol{R}$ or $\boldsymbol{R}^2$ with period 1, respectively. Let $C(\boldsymbol{T}^2)$ be the C^*-algebra of all complex-valued continuous functions on $\boldsymbol{T}^2$ and let $\tilde{\phi}_{\theta,f}$ be the automorphism of $C(\boldsymbol{T}^2)$ defined by $\tilde{\phi}_{\theta,f}(F)=F\circ\phi_{\theta,f}^{-1}$ for any $F\in C(\boldsymbol{T}^2)$. We also denote $\tilde{\phi}_{\theta,f}$ by $\phi_{\theta,f}$. Let $A_{\theta,f}$ be the crossed product associated with $\phi_{\theta,f}$ and $\boldsymbol{K}$ the C^*-algebra of all compact operators on a countably infinite-dimensional Hilbert space.

First we will apply results of Packer [**10, 11**] in order to construct automorphisms of $A_{\theta,f}\otimes\boldsymbol{K}$ in the same way as in [**6, 8, 9**]. And, using these automorphisms, we will compute the positive cone of $K_0(A_{\theta,f})$, the K_0-group of $A_{\theta,f}$.

Let B be a C^*-algebra and $M(B)$ the multiplier algebra of B and $\mathrm{Aut}(B)$ the group of all automorphisms of B. Let $K_0(B)$ be the K_0-group of B and $\mathrm{Aut}(K_0(B))$ the group of all automorphisms of $K_0(B)$. Let T_B be the homomorphism of $\mathrm{Aut}(B)$ to $\mathrm{Aut}(K_0(B))$

defined by $T_B(\alpha) = \alpha_*$ for any $\alpha \in \mathrm{Aut}(B)$, where α_* is the automorphism of $K_0(B)$ induced by α. Let range T_B be the image of $\mathrm{Aut}(B)$ by T_B.

For each $n \in \boldsymbol{N}$, let M_n be the $n \times n$-matrix algebra over $\boldsymbol{C}$ and $M_n(B)$ the $n \times n$-matrix algebra over B. We identify $M_n(B)$ with $B \otimes M_n$. Furthermore, we regard $\cup_{n \in \boldsymbol{N}} M_n(B)$ as a dense $*$-subalgebra of $B \otimes \boldsymbol{K}$.

2. Automorphisms of crossed products $A_{\theta,f}$

Let u and v be the unitary elements in $C(\boldsymbol{T}^2)$ defined by $u(t,s) = \mathrm{e}^{2\pi \mathrm{i} t}$, $v(t,s) = \mathrm{e}^{2\pi \mathrm{i} s}$ for any $(t,s) \in \boldsymbol{T}^2$. Let w be a unitary element in $A_{\theta,f}$ such that $\phi_{\theta,f} = Ad(w)$. Then $A_{\theta,f}$ is generated by u, v and w. Let $C^*(u,w)$ be the C^*-subalgebra of $A_{\theta,f}$ generated by u, w. Then $C^*(u,w) \cong A_\theta$, the irrational rotation C^*-algebra corresponding to θ.

Let τ be a tracial state on $A_{\theta,f}$ induced by the Lebesgue measure on $\boldsymbol{T}^2$ in the usual way. By Ji [4, Theorem 2.23], $\tau_* = \mathrm{tr}_*$ on $K_0(A_{\theta,f})$ for any tracial state on $A_{\theta,f}$, where τ_* and tr_* are the homomorphisms of $K_0(A_{\theta,f})$ to $\boldsymbol{R}$ induced by τ and tr.

Let $p(1,1)$ be a projection $M_2(C(\boldsymbol{T}^2))$ having trace 1 and twist -1, which is defined in [**5, 11, 16**]. Let p_θ be a Rieffel projection in $C^*(u,w)$ with $\tau(p_\theta) = \theta$ defined in Rieffel [**15**]. By the Pimsner–Voiculescu exact sequence, we see that

$$K_0(A_{\theta,f}) = \boldsymbol{Z}[p_\theta] \oplus \boldsymbol{Z}[1] \oplus \boldsymbol{Z}([1] - [p(1,1)]).$$

We express an automorphism of $K_0(A_{\theta,f})$ as an element in $GL(3,\boldsymbol{Z})$ using the above basis, where $GL(3,\boldsymbol{Z})$ is the group of 3×3-matrices over $\boldsymbol{Z}$ with determinant ± 1.

Lemma 2.1. *With the above notation,*

$$\mathrm{range}\, T_{A_{\theta,f}} \subset \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n & 0 & \epsilon \end{bmatrix} \,|\, n \in \boldsymbol{Z},\ \epsilon = \pm 1 \right\}.$$

Proof. Let $\alpha \in \mathrm{Aut}(A_{\theta,f})$. Since $\alpha(1) = 1$, we can suppose that

$$\alpha_*([p_\theta]) = a_{11}[p_\theta] + a_{21}[1] + a_{31}([1] - [p(1,1)]), \quad \alpha_*([1]) = [1],$$
$$\alpha_*([1] - [p(1,1)]) = a_{13}[p_\theta] + a_{23}[1] + a_{33}([1] - [p(1,1)]),$$

where $a_{ij} \in \boldsymbol{Z}$ $(i = 1,2,3,\ j = 1,3)$ represent the matrix coefficients of α_* corresponding to the basis described earlier. Since $\tau_* = \tau_* \circ \alpha_*$ on $K_0(A_{\theta,f})$ and θ is irrational, by a routine calculation: $a_{11} = 1$, $a_{21} = a_{13} = a_{23} = 0$. Since $\alpha_* \in GL(3,\boldsymbol{Z})$, we obtain the conclusion. □

Let σ be the automorphism of $A_{\theta,f}$ defined by $\sigma(u) = u$, $\sigma(v) = v$ and $\sigma(w) = wv$.

Lemma 2.2. *Let σ be as above. Then*

$$\sigma_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

on $K_0(A_{\theta,f})$.

Proof. In the same way as in the proof of Packer [**11**, Proposition 2.3], we can prove this lemma. □

3. Preliminaries on equivalence bimodules and crossed products

In this section we assume the reader to be familiar with equivalence bimodules and unitarily covariant systems (see [**2, 10, 14, 15**]).

We consider the following situation: let A be a unital C^*-algebra and p a full projection in some $M_k(A)$. We denote $M_k(A)$ by E. Let $\{d_{ij}\}_{ij=1}^k$ be matrix units of M_k and regard $(1 \otimes d_{11})Ep$ as an A–pEp-equivalence bimodule in the usual way. Let $(A, \boldsymbol{Z}, \alpha)$ be a unitarily covariant system with respect to $(1 \otimes d_{11})Ep$. Let U be an automorphism of $(1 \otimes d_{11})Ep$ chosen so that for any $x, y \in E$, $c \in A$,

(1) $\langle U(1 \otimes d_{11})xp, U(1 \otimes d_{11})yp\rangle_A = \alpha(\langle(1 \otimes d_{11})xp, (1 \otimes d_{11})yp\rangle_A)$,

(2) $UcU^{-1} = \alpha(c)$.

We note that there is an automorphism β of pEp such that for any $b \in pEp$, $\beta(b) = UbU^{-1}$ by Packer [**11**, Theorem 1.2]. Let W be the unitary element in M_k such that $d_{ij} = W^{-i+1}d_{11}W^{j-1}$ for $i, j = 1, 2, \ldots, k$, and $\bar{U}$ the automorphism of Ep defined by

$$\bar{U}(1 \otimes d_{jj})xp = (1 \otimes W^{-j+1})U(1 \otimes W^{j-1})(1 \otimes d_{jj})xp$$

for any $x \in E$ and $j = 1, 2, \ldots, k$. Let $\bar{\alpha}$ be the automorphism of E defined by $\bar{\alpha} = \alpha \otimes \mathrm{id}$ where id is the identity map of M_k.

Lemma 3.1. *With the above notation we regard Ep as an E–pEp-equivalence bimodule in the usual way. Then we have the following:*

(1) $\langle \bar{U}xp, \bar{U}yp\rangle_E = \bar{\alpha}(\langle xp, yp\rangle_E)$ *for any $x, y \in E$; and*

(2) $\bar{U}c\bar{U}^{-1} = \bar{\alpha}(c)$ *for any $c \in E$.*

Proof. This can be proved by routine calculations. □

Let $\widehat{1 \otimes d_{11}}$ and $\hat{p}$ be the E-valued functions on $\boldsymbol{Z}$ defined by

$$(\widehat{1 \otimes d_{11}})(n) = \begin{cases} 1 \otimes d_{11}, & \text{if } n = 0, \\ 0, & \text{elsewhere}, \end{cases} \qquad \hat{p}(n) = \begin{cases} p, & \text{if } n = 0, \\ 0, & \text{elsewhere}. \end{cases}$$

Let $\mathcal{A} = K(\boldsymbol{Z}, A)$ and $\mathcal{B} = K(\boldsymbol{Z}, pEp)$ be the $*$-algebras of functions with compact support from $\boldsymbol{Z}$ to A and pEp, respectively. Let $\mathcal{X} = K(\boldsymbol{Z}, (1 \otimes d_{11})Ep)$ be the set of functions with compact support from $\boldsymbol{Z}$ to $(1 \otimes d_{11})Ep$. We define the left and right actions of $\mathcal{A}$ and $\mathcal{B}$ and the $\mathcal{A}$-and $\mathcal{B}$-valued inner products on $\mathcal{X}$ in the same way as in Combes [2] and Packer [10]. Upon suitably completing $\mathcal{X}$ we obtain the equivalence bimodule, which shows that $A \times_\alpha \boldsymbol{Z}$ and $pEp \times_\beta \boldsymbol{Z}$ are strongly Morita equivalent by Packer [10, Theorem 2.6]. We denote their equivalence bimodule by $(1 \otimes d_{11})Ep \times_U \boldsymbol{Z}$. We will show that $(1 \otimes d_{11})Ep \times_U \boldsymbol{Z}$ is isomorphic to $(\widehat{1 \otimes d_{11}})(E \times_{\bar{\alpha}} \boldsymbol{Z})\hat{p}$ as left Hilbert $A \times_\alpha \boldsymbol{Z}$-modules.

Let $K(\boldsymbol{Z}, E)$ be the $*$-algebra of functions with compact support from $\boldsymbol{Z}$ to E and we identify $K(\boldsymbol{Z}, E)$ with the $*$-algebra of $k \times k$-matrices over $\mathcal{A}$, which is denoted by $M_k(\mathcal{A})$. Since for any $x \in M_k(\mathcal{A})$ and any $n \in \boldsymbol{Z}$,

$$((\widehat{1 \otimes d_{11}})x\hat{p})(n) = (1 \otimes d_{11})(x\hat{p})(n) = (1 \otimes d_{11})x(n)\bar{U}^n p\bar{U}^{-n},$$

we see that for any $n \in \boldsymbol{Z}$,

$$((\widehat{1 \otimes d_{11}})x\hat{p})(n)\bar{U}^n p = (1 \otimes d_{11})x(n)\bar{U}^n p \in (1 \otimes d_{11})Ep.$$

Let Φ be the map of $(\widehat{1 \otimes d_{11}})M_k(\mathcal{A})\hat{p}$ to $\mathcal{X}$ defined by

$$\Phi((\widehat{1 \otimes d_{11}})x\hat{p})(n) = ((\widehat{1 \otimes d_{11}})x\hat{p})(n)\bar{U}^n p$$

for any $x \in M_k(\mathcal{A})$. By an easy computation, Φ is a left $\mathcal{A}$-module map, where we identify $(\widehat{1 \otimes d_{11}})M_k(\mathcal{A})(\widehat{1 \otimes d_{11}})$ with $\mathcal{A}$.

Proposition 3.2. *With the above notation, Φ is an isomorphism of $(\widehat{1 \otimes d_{11}})(E \times_{\bar{\alpha}} \boldsymbol{Z})\hat{p}$ onto $(1 \otimes d_{11})Ep \times_U \boldsymbol{Z}$ as left Hilbert $A \times_\alpha \boldsymbol{Z}$-modules.*

Proof. Using Lemma 3.1, by a routine calculation we see that for any $n \in \boldsymbol{Z}$:

$$\langle \Phi((\widehat{1 \otimes d_{11}})x\hat{p}), \Phi((\widehat{1 \otimes d_{11}})y\hat{p})\rangle_{\mathcal{A}}(n) = \langle (\widehat{1 \otimes d_{11}})x\hat{p}, (\widehat{1 \otimes d_{11}})y\hat{p}\rangle_{(\widehat{1 \otimes d_{11}})M_k(\mathcal{A})\hat{p}}(n).$$

Hence we obtain the conclusion, that is:

$$(1 \otimes d_{11})(E \times_{\bar{\alpha}} \boldsymbol{Z})p \cong (1 \otimes d_{11})Ep \times_U \boldsymbol{Z}.$$

□

4. Automorphisms of stable algebras of crossed products $A_{\theta,f}$

Let $\{e_{ij}\}_{ij \in Z}$ be matrix units of $\boldsymbol{K}$. We know that

$$K_0(A_{\theta,f} \otimes \boldsymbol{K}) = \boldsymbol{Z}[p_\theta \otimes e_{00}] \oplus \boldsymbol{Z}[1 \otimes e_{00}] \oplus \boldsymbol{Z}([1 \otimes e_{00}] - [p(1,1) \otimes e_{00}]).$$

We express an automorphism of $K_0(A_{\theta,f} \otimes \boldsymbol{K})$ as an element in $GL(3, \boldsymbol{Z})$ using the above basis.

In this section we will construct an automorphism of $A_{\theta,f} \otimes \boldsymbol{K}$ from an equivalence $A_{\theta,f}$–$A_{\theta,f}$-bimodule. In the same way as in Packer [10, Example 2.8] we construct an equivalence $A_{\theta,f}$–$A_{\theta,f}$-bimodule.

Lemma 4.1. *Let q and r be relatively prime integers with $q, r > 0$ and a, b integers with $ar + bq = 1$. Let $X(q,a)$ be a left $C(\boldsymbol{T}^2)$-module defined in Rieffel [**16**, Notation 3.7]. Then $(C(\boldsymbol{T}^2), \boldsymbol{Z}, \phi_{\theta,f})$ is a unitarily covariant system with respect to $X(q,a)$.*

Proof. Let g be the continuous function on $\boldsymbol{R}$ defined by, for any $t \in \boldsymbol{R}$,

$$g(t) = \frac{a}{2q}\rho t^2 + \tfrac{1}{2}a\rho t + \frac{a}{q}tf(t).$$

Let Q be the linear map of $X(q,a)$ defined by, for any $h \in X(q,a)$,

$$(Qh)(t,s) = \mathrm{e}^{2\pi \mathrm{i} g(t)}h(t+\theta, s+\rho t + f(t)).$$

Then $Q \in \mathrm{Aut}(X(q,a))$ and the necessary calculations to prove Lemma 4.1 are similar to those in Packer [**10**, **11**]. We leave it to the reader. □

By the above lemma we can apply Packer [**10**, Theorem 2.6] and [**11**, Theorem 1.2] to the equivalence bimodule $X(q,a)$. By Rieffel [**16**, Theorem 3.1 and Proposition 3.8], $\mathrm{End}_{C(\boldsymbol{T}^2)}(X(q,a)) \cong A_\eta$, the rational rotation C^*-algebra corresponding to $\eta = (r/q)$. Hence, $A_{\theta,f}$ and $A_\eta \times_\gamma \boldsymbol{Z}$ are strongly Morita equivalent, where γ is the automorphism of A_η defined by, for any $b \in A_\eta$, $\gamma(b) = QbQ^{-1}$. We denote by $X(q,a) \times_Q \boldsymbol{Z}$ the $A_{\theta,f}$–$A_\eta \times_\gamma \boldsymbol{Z}$-equivalence bimodule obtained by Packer [**10**, Theorem 2.6].

Lemma 4.2. *With the above notation, let V and W be unitary generators in A_η with $WV = \mathrm{e}^{2\pi \mathrm{i}\eta}VW$. Then $\gamma(V) = \mathrm{e}^{2\pi \mathrm{i}(\theta/q)}V$, $\gamma(W) = \kappa \mathrm{e}^{2\pi \mathrm{i}(1/q)f(V^q)}V^\rho W$, where $\kappa = \mathrm{e}^{2\pi \mathrm{i}(\rho r/2q)(aq - ar + 2)}$.*

Proof. Since the proof is easy calculations, it is left to the reader. □

Proposition 4.3. *With the above notation let γ be the automorphism of A_η defined by $\gamma(V) = \mathrm{e}^{2\pi \mathrm{i}(\theta/q)}V$, $\gamma(W) = \kappa \mathrm{e}^{2\pi \mathrm{i}(1/q)f(V^q)}V^\rho W$, where $\kappa = \mathrm{e}^{2\pi \mathrm{i}(\rho r/2q)(aq-ar+2)}$. Then $A_{\theta,f}$ is strongly Morita equivalent to $A_\eta \times_\gamma \boldsymbol{Z}$.*

Proof. This is immediate by Lemmas 4.1 and 4.2. □

Since $X(q,a)$ is a finitely generated projective left $C(\boldsymbol{T}^2)$-module, there is a projection $p(q,a)$ in some $M_k(C(\boldsymbol{T}^2))$ such that $(1 \otimes d_{11})M_k(C(\boldsymbol{T}^2))p(q,a) \cong X(q,a)$ as left Hilbert $C(\boldsymbol{T}^2)$-modules, where $\{d_{ij}\}_{ij=1}^k$ are matrix units of M_k. Hence, $A_\eta \cong p(q,a)M_k(C(\boldsymbol{T}^2))p(q,a)$. If we identify $X(q,a)$ with $(1 \otimes d_{11})M_k(C(\boldsymbol{T}^2))p(q,a)$, we can regard Q and γ as an automorphism of $(1 \otimes d_{11})M_k(C(\boldsymbol{T}^2))p(q,a)$ and an automorphism of $p(q,a)M_k(C(\boldsymbol{T}^2))p(q,a)$, respectively. Therefore, by Proposition 4.3, $A_{\theta,f}$ is strongly Morita equivalent to $p(q,a)M_k(C(\boldsymbol{T}^2))p(q,a) \times_\gamma \boldsymbol{Z}$ and $(1 \otimes d_{11})M_k(C(\boldsymbol{T}^2))p(q,a) \times_Q \boldsymbol{Z}$ is their equivalence bimodule. By Proposition 3.2,

$$(1 \otimes d_{11})M_k(C(\boldsymbol{T}^2))p(q,a) \times_Q \boldsymbol{Z} \cong (1 \otimes d_{11})M_k(A_{\theta,f})p(q,a),$$

as left Hilbert $A_{\theta,f}$-modules. Thus we obtain that $p(q,a)M_k(A_{\theta,f})p(q,a) \cong A_\eta \times_\gamma \boldsymbol{Z}$. Put $q = r = 1$. Then $a + b = 1$ and $\eta = (r/q) = 1$ and $\kappa = 1$. Thus $A_\eta = C(\boldsymbol{T}^2)$ and $\gamma(V) = \mathrm{e}^{2\pi\mathrm{i}\theta}V$, $\gamma(W) = \mathrm{e}^{2\pi\mathrm{i}f(V)}V^\rho W$. Hence, $A_\eta \times_\gamma \boldsymbol{Z} \cong C(\boldsymbol{T}^2) \times_{\phi_{\theta,f}} \boldsymbol{Z}$. Therefore, $p(1,a)M_k(A_{\theta,f})p(1,a) \cong A_{\theta,f}$, where a is any integer and $b = 1 - a$.

In the same way as in [**6**, **8**, **9**], we construct an automorphism of $A_{\theta,f} \otimes \boldsymbol{K}$ from the projection $p(1,a)$. Since $p(1,a)$ is a full projection in $M_k(A_{\theta,f})$, by Brown [**1**, Lemma 2.5], there is a partial isometry $w \in M(M_k(A_{\theta,f}) \otimes \boldsymbol{K})$ with $w^*w = p(1,a) \otimes 1$, $ww^* = 1 \otimes I_k \otimes 1$, where I_k is the unit element in M_k. Then $Ad(w)$ is an isomorphism of $(p(1,a) \otimes 1)(M_k(A_{\theta,f}) \otimes \boldsymbol{K})(p(1,a) \otimes 1)$ onto $M_k(A_{\theta,f}) \otimes \boldsymbol{K}$. Let ψ be an isomorphism of $M_k(A_{\theta,f}) \otimes \boldsymbol{K}$ onto $A_{\theta,f} \otimes \boldsymbol{K}$ with $\psi_* = \mathrm{id}$ of $K_0(M_k(A_{\theta,f}) \otimes \boldsymbol{K})$ onto $K_0(A_{\theta,f} \otimes \boldsymbol{K})$. Let χ be an isomorphism of $A_{\theta,f}$ onto $p(1,a)M_k(A_{\theta,f})p(1,a)$. Let β_a be an automorphism of $A_{\theta,f} \otimes \boldsymbol{K}$ defined by $\beta_a = \psi \circ Ad(w) \circ \chi \otimes \mathrm{id}$, where id is the identity map of $\boldsymbol{K}$.

Theorem 4.4. *With the above notation,*

$$\beta_{a*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n & -a & \epsilon \end{bmatrix}$$

on $K_0(A_{\theta,f} \otimes \boldsymbol{K})$, where $n \in \boldsymbol{Z}$, $\epsilon = 1$ or -1.

Proof. We suppose that $\beta_{a*} = [a_{ij}] \in GL(3, \boldsymbol{Z})$. By the definition of β_a, in $K_0(A_{\theta,f})$

$$\beta_{a*}([1 \otimes e_{00}]) = \psi_*([w(\chi(1) \otimes e_{00})w^*]) = [p(1,a) \otimes e_{00}].$$

By the definition of $X(1,a)$, we see that $[p(1,a)] = [1] - a([1] - [p(1,1)])$ in $K_0(C(\boldsymbol{T}^2))$. Since $K_0(C(\boldsymbol{T}^2))$ is embedded injectively in $K_0(A_{\theta,f})$, in $K_0(A_{\theta,f})$

$$\beta_{a*}([1 \otimes e_{00}]) = [1 \otimes e_{00}] - a([1 \otimes e_{00}] - [p(1,1) \otimes e_{00}]).$$

Thus, $a_{12} = 0$, $a_{22} = 1$ and $a_{32} = -a$. In the same way as in the proof of [**6**, Theorem 2], we see that $\tau_* = (\tau \otimes \mathrm{Tr})_* \circ \beta_{a*}$ on $K_0(A_{\theta,f})$, where Tr is the canonical trace on $\boldsymbol{K}$. Since $\tau_*([p_\theta]) = (\tau \otimes \mathrm{Tr})_* \circ \beta_{a*}([p_\theta])$, $\theta = a_{11}\theta + a_{21}$. Hence $a_{11} = 1$ and $a_{21} = 0$. Similarly, since $\tau_*([1] - [p(1,1)]) = (\tau \otimes \mathrm{Tr})_* \circ \beta_{a*}([1] - [p(1,1)])$, $a_{13} = a_{23} = 0$. Thus,

$$\beta_{a*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & -a & a_{33} \end{bmatrix}.$$

Since $\beta_{a*} \in GL(3, \boldsymbol{Z})$, $a_{33} = \pm 1$. Therefore, we obtain the conclusion. □

Remark 4.5. By the definition of $p(1,a)$, we see that we can choose any integer a. Hence, by Theorem 4.4, for any $a \in \boldsymbol{Z}$, there is a $\beta_a \in \mathrm{Aut}(A_{\theta,f} \otimes \boldsymbol{K})$ such that

$$\beta_{a*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n & a & \epsilon \end{bmatrix}$$

on $K_0(A_{\theta,f} \otimes \boldsymbol{K})$, where $n \in \boldsymbol{Z}$ depends on the integer a and $\epsilon = 1$ or -1.

https://doi.org/10.1017/S0013091500020782 Published online by Cambridge University Press

5. The positive cones of K_0-groups of $A_{\theta,f}$

In this section, we will compute the positive cone of $K_0(A_{\theta,f})$.

Lemma 5.1. *With the same notation as in § 4, for any $x, y \in \boldsymbol{Z}$, there is a $\beta(x,y) \in \mathrm{Aut}(A_{\theta,f} \otimes \boldsymbol{K})$ such that*

$$\beta(x,y)_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & \epsilon \end{bmatrix}$$

on $K_0(A_{\theta,f} \otimes \boldsymbol{K})$ and that $\epsilon = 1$ or -1.

Proof. By Remark 4.5. for any $y \in \boldsymbol{Z}$, there is a $\beta_y \in \mathrm{Aut}(A_{\theta,f} \otimes \boldsymbol{K})$ such that

$$\beta_{y*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n & y & \epsilon \end{bmatrix}$$

on $K_0(A_{\theta,f} \otimes \boldsymbol{K})$, where $n \in \boldsymbol{Z}$ depends on the integer y and $\epsilon = 1$ or -1. And, by Lemma 2.2, there is an $\alpha_{x-n} \in \mathrm{Aut}(A_{\theta,f})$ such that

$$\alpha_{x-n*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x-n & 0 & 1 \end{bmatrix}$$

on $K_0(A_{\theta,f})$. Let $\beta(x,y) = \alpha_{x-n} \otimes \mathrm{id} \circ \beta_y$ where id is the identity map of $\boldsymbol{K}$. Then

$$\beta(x,y)_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & \epsilon \end{bmatrix}$$

on $K_0(A_{\theta,f} \otimes \boldsymbol{K})$. Therefore we obtain the conclusion. □

Theorem 5.2. *With the above notation let $a[p_\theta]+b[1]+c([1]-[p(1,1)])$ be any element in $K_0(A_{\theta,f})$ where $a, b, c \in \boldsymbol{Z}$. Then there are an $r \in \boldsymbol{N}$ and a non-zero projection $q \in M_r(A_{\theta,f})$ such that $[q] = a[p_\theta] + b[1] + c([1] - [p(1,1)])$ if and only if $a\theta + b > 0$.*

Proof. One direction is obvious, so we concentrate on the reverse implication. We suppose that $a\theta + b > 0$. First we follow the method of Packer [**11**, Lemma 2.9]. Let d be the greatest (positive) common divisor of a, b and c and write $(a,b,c) = d(l,m,n)$, where l, m, n have no common factor. Let j be the greatest (positive) common divisor of l and m, and write $(a,b,c) = d(jg, jh, n)$, where $(g,h) = 1$. We note that $g\theta + h > 0$ since $a\theta + b = djg\theta + djh > 0$, and that $(j,n) = 1$. Since $K_0(C^*(u,w))$ is embedded injectively in $K_0(A_{\theta,f})$ and $g\theta + h > 0$, there is a non-zero projection $q(g,h,0) \in A_{\theta,f} \otimes \boldsymbol{K}$ such that $[q(g,h,0)] = g[p_\theta \otimes e_{00}] + h[1 \otimes e_{00}]$ in $K_0(A_{\theta,f} \otimes \boldsymbol{K})$. Since $(g,h) = 1$, there are

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$x, y \in \boldsymbol{Z}$ such that $xg + yh = 1$. By Lemma 5.1, there is a $\beta(kx, ky) \in \mathrm{Aut}(A_{\theta,f} \otimes \boldsymbol{K})$ such that

$$\beta(kx, ky)_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ kx & ky & \epsilon \end{bmatrix}$$

on $K_0(A_{\theta,f} \otimes \boldsymbol{K})$, where $\epsilon = 1$ or -1. Let $q(g,h,k) = \beta(kx,ky)(q(g,h,0))$ for any $k \in \boldsymbol{Z}$. Then $q(g,h,k)$ is a non-zero projection in $A_{\theta,f} \otimes \boldsymbol{K}$ and in $K_0(A_{\theta,f} \otimes \boldsymbol{K})$ $[q(g,h,k)] = \beta(kx,ky)_*([q(g,h,0)]) = {}^{\mathrm{T}}[g,h,k]$ since $xg + yh = 1$. If $j = 1$, let $\bar{q} = \oplus_1^d q(g,h,n)$. Then $\bar{q}$ is a non-zero projection in $M_d(A_{\theta,f} \otimes \boldsymbol{K})$ and $[\bar{q}] = a[p_\theta \otimes e_{00}] + b[1 \otimes e_{00}] + c([1 \otimes e_{00}] - [p(1,1) \otimes e_{00}])$ since $(a,b,c) = d(g,h,n)$. Thus, there are an $r \in \boldsymbol{N}$ and a non-zero projection $q \in M_r(A_{\theta,f})$ such that in $K_0(A_{\theta,f})$, $[q] = a[p_\theta] + b[1] + c([1] - [p(1,1)])$. We suppose that $j \geqslant 2$. Then $(a,b,c) = d\{((j-1)g, (j-1)h, 0) + (g,h,n)\}$. Since $(j-1)g\theta + (j-1)h > 0$, there is a non-zero projection $q((j-1)g, (j-1)h, 0) \in A_{\theta,f} \otimes \boldsymbol{K}$ such that in $K_0(A_{\theta,f} \otimes \boldsymbol{K})$, $[q((j-1)g,(j-1)h,0)] = (j-1)g[p_\theta \otimes e_{00}] + (j-1)h[1 \otimes e_{00}]$. Let $\bar{q} = \oplus_1^d \{q((j-1)g,(j-1)h,0) \oplus q(g,h,n)\}$. Then $\bar{q}$ is a non-zero projection in $M_{2d}(A_{\theta,f} \otimes \boldsymbol{K})$ and $[\bar{q}] = a[p_\theta \otimes e_{00}] + b[1 \otimes e_{00}] + c([1 \otimes e_{00}] - [p(1,1) \otimes e_{00}])$. Thus, there are an $r \in \boldsymbol{N}$ and a non-zero projection $q \in M_r(A_{\theta,f})$ such that $[q] = a[p_\theta] + b[1] + c([1] - [p(1,1)])$ in $K_0(A_{\theta,f})$. Therefore we obtain the conclusion. □

Acknowledgements. The author thanks the referee for a number of valuable suggestions for improvement of the manuscript.

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