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EXTENSIONS OF CERTAIN RESULTS IN WALSH-TYPE EQUICONVERGENCE

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Two sequences of rational functions are constructed from different expansions of $(t^{qn}-1)^{-1}$ and extensions of certain known results in the theory of Walsh-type equiconvergence are sought.

1. INTRODUCTION

Let π_s denote the class of all polynomials of degree $\leqslant s$ over the field of complex numbers. For a given $\sigma > 1$ and a fixed integer $m \geqslant -1$, let $\mathcal{R}_{n+m}$ denote the class of all rational functions of the form $p(z)/(z^n - \sigma^n)$, $p(z) \in \pi_{n+m}$. We denote by A_ρ $\rho > 1$, the class of all functions analytic in $|z| < \rho$ but not in $|z| \leqslant \rho$, and consider the following minimisation problems:

$$\text{(P1)} \quad [6] \quad \min_{r(z)\in\mathcal{R}_{n+m}} \int_{|z|=1} |f(z) - r(z)|^2 \, |dz|$$

$$\text{(P2)} \quad [2] \quad \min_{r(z)\in\mathcal{R}_{n+m}} \sum_{k=0}^{qn-1} \left|f(\omega^k) - r(\omega^k)\right|^2$$

where $q \geqslant 2$ and $\omega = \exp(2\pi iqn)$.

It is known that for any $f \in A_\rho$, the elements $r_{n+m,n}(z, f)$ and $R^*_{n+m,n}(z, f)$ of $\mathcal{R}_{n+m}$ which respectively solves (P1) and (P2) are given by ((4.1) – (4.4), [2])

$$r_{n+m,n}(z, f) = \frac{1}{2\pi i} \int_\Gamma \frac{(t^n - \sigma^n) f(t)}{(z^n - \sigma^n)(t - z)} \sum_{j=1}^{3} A_j(t, z)\,dt \tag{1.1}$$

and

$$R^*_{n+m,n}(z, f) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{(z^n - \sigma^n)(t - z)(t^{qn} - 1)} \sum_{j=1}^{3} A_j(t, z) B_j(t, \sigma)\,dt \tag{1.2}$$

where Γ is the circle $|t| = R$, $1 < R < \rho$ and

$$\begin{cases} A_1(t, z) = \dfrac{t^{m+1} - z^{m+1}}{t^{m+1}}, \quad A_2(t, z) = \dfrac{z^{m+1}\left(t^{n-m-1} - z^{n-m-1}\right)}{t^n - \sigma^{-n}} \\ A_3(t, z) = z^n\left(t^{m+1} - z^{m+1}\right)/t^{m+1}\left(t^n - \sigma^{-n}\right) \end{cases}, \tag{1.3}$$

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and

$$(1.4)\qquad \begin{cases} B_1(t,\sigma) = \sigma^{-(q-2)n}B(t,\sigma) - t^{qn}\sigma^n, \\ B_2(t,\sigma) = \big(t^{qn} - \sigma^{-qn}\big)\big(\sigma^{-n} - \sigma^n\big)/\big(1+\sigma^{-qn}\big), \\ B_3(t,\sigma) = \big(t^n - \sigma^{-n}\big)\big(t^{qn} - \sigma^n B(t,\sigma)\big), \end{cases}$$

with

$$B(t,\sigma) = \frac{t^n\big(t^{(q-1)n} - \sigma^{-(q-1)n}\big)\big(1-\sigma^{-2n}\big)}{\big(t^n - \sigma^{-n}\big)\big(1-\sigma^{-2(q-1)n}\big)}.$$

It has been shown in [2] that

$$(1.5)\qquad \lim_{n\to\infty}\{R^*_{n+m,n}(z,f) - r_{n+m,n}(z,f)\} = 0, \begin{cases} |z| < \rho^{1+q} & \text{if } \sigma \geqslant \rho^{1+q} \\ |z| \neq \sigma & \text{if } \sigma < \rho^{1+q} \end{cases}$$

and that (1.5) extends the following theorem due to Rivilin [5]:

Theorem A. *Let* $f(z) := \sum_{j=0}^{\infty} a_j z^j \in A_\rho$, $\rho > 1$ *and* $S_{n-1}(z,f) = \sum_{j=0}^{n-1} a_j z^j$. *Let* $p_{n-1,q}(z,f)$, $q \geqslant 2$, *denote the polynomial of degree* $n-1$ *of least squares approximation to* f *on the* (nq)*-th roots of unity. Then*

$$(1.6)\qquad \lim_{n\to\infty}\{p_{n-1,q}(z,f) - S_{n-1}(z,f)\} = 0, \qquad \forall\, |z| < \rho^{1+q}.$$

Another generalisation of (1.6) is that for any positive integer $\ell \geqslant 1$ and $f \in A_\rho$, we have

$$(1.7)\qquad \lim_{n\to\infty}\Big\{p_{n-1,q}(z,f) - \sum_{k=0}^{\ell-1} S_{n-1,k}(z,f)\Big\} = 0, \qquad \forall\, |z| < \rho^{1+\ell q},$$

where $S_{n-1,k}(z,f) = \sum_{j=0}^{n-1} a_{j+kq} z^j$, $k = 0, 1, \ldots, \ell - 1$.

It may be noted that a classical theorem of Walsh ([9] p.153) which deals with equiconvergence of certain sequences of polynomials is a special case for each of the results (1.5) – (1.7). For further information on this topic we refer the interested reader to [**1, 3, 7**].

Our object in this paper is to extend (1.5) in the spirit of (1.7). For this, we construct two different sequences of *help rational functions* which lead us to obtain a larger region of equiconvergence. These extensions are obtained from two different expansions of $(t^{qn} - 1)^{-1}$.

2. Extension I

Our first extension is based on the following identity:

$$(2.1)\qquad (t^{qn}-1)^{-1} = [t^{qn}-\sigma^{-qn}-(1-\sigma^{-qn})]^{-1} = \sum_{\nu=1}^{\infty}\widetilde{F}_\nu(t,\sigma)$$

where
$$\widetilde{F}_\nu(t,\sigma) = \frac{(1-\sigma^{-qn})^{\nu-1}}{(t^{qn}-\sigma^{-qn})^{\nu}},\qquad \nu=1,2,\dots .$$

We define the rational functions

$$(2.2)\qquad \widetilde{r}_{n+m,n}(z,f,\nu) := \sum_{j=0}^{n+m}\left(\widetilde{c}_j(\nu)z^j/z^n-\sigma^n\right),\qquad \nu=1,2,3,\dots,$$

where

$$(2.3)\qquad \widetilde{c}_j(\nu) := \begin{cases} \frac{1}{2\pi i}\int_\Gamma \frac{K_1(t,\sigma)}{t^{j+1}}\widetilde{F}_\nu(t,\sigma)f(t)dt, & 0\leqslant j\leqslant m,\\ \frac{1}{2\pi i}\int_\Gamma \frac{K_2(t,\sigma)\widetilde{F}_\nu(t,\sigma)}{t^{m-n+j+2}\left(t^n-\sigma^{-n}\right)}f(t)dt, & m+1\leqslant j\leqslant n-1,\\ \frac{1}{2\pi i}\int_\Gamma \frac{K_3(t,\sigma)\widetilde{F}_\nu(t,\sigma)}{t^{j+1}\left(t^n-\sigma^{-n}\right)}f(t)dt, & n\leqslant j\leqslant n+m.\end{cases}$$

with ([2], (4.6))

$$K_j(t,\sigma) = B_j(t,\sigma)-(t^{qn}-1)(t^n-\sigma^n),\qquad j=1,2,3$$

where $B_j(t,\sigma)$ are given in (1.4). For $\nu=0$, we let

$$(2.4)\qquad \widetilde{r}_{n+m,n}(z,f,0) := r_{n+m,n}(z,f).$$

Remark 1. From (2.1) we can rewrite

$$\widetilde{r}_{n+m,n}(z,f,\nu) = \frac{1}{z^n-\sigma^n}\left\{\sum_{j=0}^{m}\widetilde{c}_j(\nu)z^j + \sum_{j=m+1}^{n-1}\widetilde{c}(\nu)z^j + \sum_{j=n}^{n+m}\widetilde{c}_j(\nu)z^j\right\},$$

$(\nu=1,2,3,\dots)$, so that using (1.3) we have

$$(2.5)\qquad \widetilde{r}_{n+m,n}(z,f,\nu) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)\widetilde{F}_\nu(t,\sigma)}{(z^n-\sigma^n)(t-z)}\sum_{j=1}^{3}A_j(t,z)K_j(t,\sigma)dt.$$

If we define

$$(2.6)\qquad \widetilde{W}_{n+m,n}(z,f,\ell) := \sum_{\nu=0}^{\ell-1}\widetilde{r}_{n+m,n}(z,f,\nu),$$

we have the first extension of (1.5) (see [2], Theorem 2.1) given by:

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Theorem 1. *Let $m \geqslant -1$, $q \geqslant 2$ and $\ell \geqslant 1$ be three fixed integers and let $\sigma > 1$. If $f \in A_\rho$, $1 < \rho < \infty$, then*

$$\lim_{n\to\infty}\{R^*_{n+m,n}(z, f) - \widetilde{W}_{n+m,n}(z, f, \ell)\} = 0, \begin{cases} |z| < \rho^{\ell q+1} & \text{if } \sigma \geqslant \rho^{\ell q+1} \\ |z| \neq \sigma & \text{if } \sigma < \rho^{\ell q+1} \end{cases} \tag{2.7}$$

the convergence being uniform and geometric on any compact subset of the regions defined above. Moreover, the result is sharp in the sense that for each $|z| = \rho^{1+\ell q}$, there is an $\widehat{f} \in A_\rho$ for which (2.7) does not hold.

Proof: The difference in (2.7) can be written as

$$\begin{aligned} &R^*_{n+m,n}(z, f) - \widetilde{W}_{n+m,n}(z, f, \ell) \\ &\qquad = R^*_{n+m,n}(z, f) - \widetilde{r}_{n+m,n}(z, f, 0) - \sum_{\nu=1}^{\ell-1} \widetilde{r}_{n+m,n}(z, f, \nu). \end{aligned}$$

Applying (1.1), (1.2), (2.4) and (2.5) to the above relation we obtain

$$\begin{aligned} &R^*_{n+m,n}(z, f) - \widetilde{W}_{n+m,n}(z, f, \ell) \\ &\qquad = \frac{1}{2\pi i}\int_\Gamma \frac{\sum_{j=1}^{3} A_j(t, z)K_j(t, \sigma)}{(z^n - \sigma^n)(t - z)} \sum_{\nu=\ell}^{\infty} \widetilde{F}_\nu(t, \sigma) f(t)dt. \end{aligned} \tag{2.8}$$

Since $\sum\limits_{\nu=\ell}^{\infty} \widetilde{F}_\nu(t, \sigma) = (1 - \sigma^{-qn})^\ell \Big/ \left((t^{qn} - \sigma^{-qn})^{\ell-1}(t^{qn} - 1)\right)$, we conclude (2.7) from (2.8) after some computation. As usual, the function $\widehat{f}(z) = \left(z - \rho e^{i\theta}\right)^{-1}$, $0 \leqslant \theta \leqslant 2\pi$, shows that the result is sharp. ∎

3. Extension II

Here we rearrange a double series in order to construct another sequence of *help rational functions*. First, note that for an absolutely convergent series $\sum\limits_{s=1}^{\infty}\sum\limits_{\lambda=1}^{\infty} g(s, \lambda)$ and a fixed integer $q \geqslant 1$, we have

$$\begin{aligned} \sum_{s=1}^{\infty}\sum_{\lambda=1}^{\infty} g(s, \lambda) &= \sum_{s=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\lambda=1}^{q} g(s, (j-1)q + \lambda) \\ &= \sum_{\lambda=1}^{q}\sum_{s=1}^{\infty}\sum_{j=1}^{\infty} g(j, (s-j)q + \lambda); \end{aligned} \tag{3.1}$$

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the last expression follows on writing the series $\sum_{s=1}^{\infty} \sum_{j=1}^{\infty} g(s, (j-1)q+\lambda)$, for each fixed λ, as shown below, and then on adding the terms along transverse diagonals as shown below

$$
\begin{array}{l}
g(1,\lambda)+g(1,g+\lambda)+g(1,2q+\lambda)+g(1,3q+\lambda)+\dots \\
\qquad\quad \swarrow \qquad\qquad \nearrow \qquad\qquad \swarrow \\
+g(2,\lambda)+g(2,q+\lambda)+g(2,2q+\lambda)+g(2,3q+\lambda)+\dots \\
\qquad\quad \nearrow \qquad\qquad \swarrow \\
+g(3,\lambda)+g(3,q+\lambda)+g(3,2q+\lambda)+g(3,3q+\lambda)+\dots \\
\qquad\quad \swarrow \\
+g(4,\lambda)+g(4,q+\lambda)+g(4,2q+\lambda)+g(4,3q+\lambda)+\dots \\
+\dots\ .
\end{array}
$$

With this obervation, we have

Lemma 1. *For* $|t|>1$ *and* $\sigma>1$ *the following identity holds:*

$$(t^{qn}-1)^{-1} = \sum_{s=1}^{\infty}\sum_{\lambda=1}^{q} F^{*}_{(s-1)q+\lambda}(t,\sigma) \tag{3.2}$$

where

$$
\begin{aligned}
&F^{*}_{(s-1)q+\lambda}(t,\sigma) \\
&\quad = \sum_{j=1}^{s} \binom{(s-j)q^2+\lambda q+j-2}{j-1} \frac{(-\sigma^{-n})^{j-1}}{(t^n-\sigma^{-n})^{(s-j)q^2+\lambda q+j-1}}.
\end{aligned} \tag{3.3}
$$

Proof: It is easy to see the validity of the following expansion:

$$(t^{qn}-1)^{-1} = \sum_{s=1}^{\infty}\sum_{\lambda=1}^{\infty} \binom{\lambda q+s-2}{s-1} \frac{(-\sigma^{-n})^{s-1}}{(t^n-\sigma^{-n})^{\lambda q+s-1}}. \tag{3.4}$$

If we let $g(s,\lambda) = \binom{\lambda q+s-2}{s-1}\left((-\sigma^{-n})^{s-1}\right) \Big/ \left((t^n-\sigma^{-n})^{\lambda q+s-1}\right)$ in equation (3.1), then (3.2) follows immediately from (3.4) on observing that

$$g(j,(s-j)q+\lambda) = \binom{[(s-j)q+\lambda]q+j-2}{j-1} \frac{(-\sigma^{-n})^{j-1}}{(t^n-\sigma^{-n})^{[(s-j)q+\lambda]q+j-1}}.$$

$\square$

Now we define another sequence of help functions. Let

$$r^*_{n+m,n}(z,\,f,\,\nu) := \sum_{j=0}^{m+n} c^*_j(\nu) z^j/(z^n - \sigma^n), \qquad \nu = 1, 2, 3, \ldots \tag{3.5}$$

with

$$c^*_j(\nu) := \begin{cases} \frac{1}{2\pi i} \int_\Gamma \frac{K_1(t,\sigma)}{t^{j+1}} F^*_\nu(t,\,\sigma) f(t)dt, & 0 \leqslant j \leqslant m, \\ \frac{1}{2\pi i} \int_\Gamma \frac{K_2(t,\sigma)F^*_\nu(t,\sigma)}{t^{m-n+j+2}(t^n - \sigma^{-n})} f(t)dt, & m+1 \leqslant j \leqslant n-1, \\ \frac{1}{2\pi i} \int_\Gamma \frac{K_3(t,\sigma)F^*_\nu(t,\sigma)}{t^{j+1}(t^n - \sigma^{-n})} f(t)dt, & n \leqslant j \leqslant n+m\,. \end{cases} \tag{3.6}$$

For $\nu = 0$, we set

$$r^*_{n+m,n}(z,\,f,\,0) = r_{n+m,n}(z,\,f).$$

On using (1.3), an integral representation of $r^*_{n+m,n}(z,\,f,\,\nu)$, $\nu \geqslant 1$, is found to be

$$r^*_{n+m,n}(z,\,f,\,\nu) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)F^*_\nu(t,\,\sigma)}{(z^n - \sigma^n)(t-z)} \sum_{j=1}^{3} A_j(t,\,z)K_j(t,\,\sigma)dt. \tag{3.7}$$

For a fixed integer $\ell \geqslant 1$, we set

$$W^*_{n+m,n}(z,\,f,\,\ell) := \sum_{\nu=0}^{\ell-1} r^*_{n+m,n}(z,\,f,\,\nu). \tag{3.8}$$

With the above notation, we can now prove

Theorem 2. *Let $m \geqslant -1$, $q \geqslant 2$, and $\ell \geqslant 1$ be three fixed integers and $\sigma > 1$. If $f \in A_\rho$, $1 < \rho < \infty$, then*

$$\lim_{n\to\infty} \{R^*_{n+m,n}(z,\,f) - W^*_{n+m,n}(z,\,f,\,\ell)\} = 0, \quad \begin{cases} |z| < \rho^{\ell q+1} & \text{if } \sigma \geqslant \rho^{\ell q+1}, \\ |z| \neq \sigma & \text{if } \sigma < \rho^{\ell q+1}, \end{cases} \tag{3.9}$$

the convergence being uniform and geometric on any compact subset of the regions defined above. Moreover, the result is sharp.

Proof: As in (2.8), we use the relations (3.2), (3.7) and (3.8) to obtain

$$\begin{aligned} &R^*_{n+m,n}(z,\,f) - W^*_{n+m,n}(z,\,f,\,\ell) \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{\gamma_\ell(t,\,\sigma)f(t)}{(z^n - \sigma^n)(t-z)} \sum_{j=1}^{3} A_j(t,\,z)K_j(t,\,\sigma)dt \end{aligned} \tag{3.10}$$

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where $\gamma_\ell(t,\sigma) := \sum_{s=0}^{\infty}\sum_{\lambda=1}^{q} F^*_{sq+\lambda}(t,\sigma) - \sum_{\lambda=1}^{\ell-1} F^*_\lambda(t,\sigma)$ and Γ is the circle $|t| = \rho_1$ with $1 < \rho_1 < \rho$. If we write $\ell - 1 := aq + b$ with $a \geqslant 0$, $0 \leqslant b \leqslant q-1$, then we have

$$\begin{aligned}\gamma_\ell(t,\sigma) &= \sum_{s=0}^{\infty}\sum_{\lambda=1}^{q} F^*_{sq+\lambda}(t,\sigma) - \sum_{s=0}^{a-1}\sum_{\lambda=1}^{q} F^*_{sq+\lambda}(t,\sigma) - \sum_{\lambda=1}^{b} F_{aq+\lambda}(t,\sigma)\\ &= \sum_{s=a+1}^{\infty}\sum_{\lambda=1}^{q} F^*_{sq+\lambda}(t,\sigma) + \sum_{\lambda=b+1}^{q} F^*_{aq+\lambda}(t,\sigma).\end{aligned}$$

that is,

$$\gamma_\ell(t,\sigma) = \sum_{s=1}^{\infty}\sum_{\lambda=1}^{q} F^*_{(s+a)q+\lambda}(t,\sigma) + \sum_{\lambda=b+1}^{q} F^*_{aq+\lambda}(t,\sigma). \tag{3.11}$$

Substituting the value of $F^*_{aq+\lambda}(t,\sigma)$ from (3.3), we can write

$$\begin{aligned}&\sum_{\lambda=b+1}^{q} F^*_{aq+\lambda}(t,\sigma)\\ &\quad= \left(t^n - \sigma^{-n}\right)^{-aq^2} \sum_{\lambda=b+1}^{q} \left(t^n - \sigma^{-n}\right)^{-\lambda q}\\ &\qquad\times\left\{1 + \sum_{j=1}^{a}\binom{(a-j)q^2 + \lambda q + j - 1}{j}\frac{(-\sigma^{-n})^j}{(t^n - \sigma^{-n})^{(1-q^2)j}}\right\}.\end{aligned}$$

If $\sigma \geqslant \rho^{\ell q+1}$ and $|t| = \rho_1$, it is easy to see that

$$\begin{aligned}&\left|\sum_{\lambda=b+1}^{q} F^*_{aq+\lambda}(t,\sigma)\right|\\ &\quad\leqslant (q-b)\left(\rho_1^n - \sigma^{-n}\right)^{-(aq+b+1)q}\\ &\qquad\times\left\{1 + \frac{(\rho_1^n - \sigma^{-n})^{q^2-1}}{\rho^{\ell q n}}\sum_{j=1}^{a}\binom{(a-j)q^2 + \lambda q + j - 1}{j}\rho^{-jn}\right\},\end{aligned}$$

for all n sufficiently large. Since $aq + b + 1 =: \ell$, we obtain

$$\sum_{\lambda=b+1}^{q} F^*_{aq+\lambda}(t,\sigma) = O\left(\rho_1^{-\ell q n}\right). \tag{3.12}$$

It remains to estimate the double summation on the right side of (3.11). For this purpose, we set

$$h(\nu,\mu) = \binom{\mu q + (a+1)q^2 + \nu - 2}{\nu - 1}\frac{(-\sigma^{-n})^{\nu-1}}{(t^n - \sigma^{-n})^{\mu q + (a+1)q^2 + \nu - 1}}. \tag{3.13}$$

It remains to estimate the double summation on the right side of (3.11). For this purpose, we set

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Then using (3.3), we can rewrite

$$\sum_{s=1}^{\infty}\sum_{\lambda=1}^{q}F^*_{(s+a)q+\lambda}(t,\sigma)=I_1+I_2, \tag{3.14}$$

where

$$\begin{cases} I_1:=\displaystyle\sum_{s=1}^{\infty}\sum_{\lambda=1}^{q}\sum_{j=1}^{s}h(j,(s-j)q+\lambda),\\ I_2:=\displaystyle\sum_{s=1}^{\infty}\sum_{\lambda=1}^{\infty}\sum_{j=s+1}^{s+q+1}h(j,(s-j)q+\lambda).\end{cases}$$

Recalling the identity (3.1), we obtain

$$\begin{aligned}&\sum_{s=1}^{\infty}\sum_{\lambda=1}^{\infty}h(s,\lambda)\\ &=\frac{1}{(t^n-\sigma^{-n})^{(a+1)q^2}}\sum_{\lambda=1}^{\infty}\frac{1}{(t^n-\sigma^{-n})^{\lambda q}}\\ &\qquad\times\sum_{s=1}^{\infty}\binom{\lambda q+(a+1)q^2+s-2}{s-1}\left(\frac{-\sigma^{-n}}{t^n-\sigma^{-n}}\right)^{s-1}\\ &=(t^n-\sigma^{-n})^{-(a+1)q^2}\sum_{\lambda=1}^{\infty}(t^n-\sigma^{-n})^{-\lambda q}\left(1+\frac{\sigma^{-n}}{t^n-\sigma^{-n}}\right)^{-(\lambda q+(a+1)q^2)}\end{aligned}$$

so that

$$I_1=t^{-(a+1)nq^2}(t^{qn}-1)^{-1}=O\left(\rho_1^{-(a+1)nq^2-qn}\right). \tag{3.15}$$

Further, we notice that

$$I_2:=\sum_{s=1}^{\infty}\sum_{\lambda=1}^{q}\sum_{j=1}^{q+1}h(j+s,-jq+\lambda) \tag{3.16}$$

where in view of (3.13)

$$\begin{aligned}&h(j+s,-jq+\lambda)\\ &=\binom{(-jq+\lambda)q+(a+1)q^2+j+s-2}{(-jq+\lambda)q+(a+1)q^2-1}\frac{(-\sigma^{-n})^{j-1+s}}{(t^n-\sigma^{-n})^{(a+1-j)q^2+\lambda q+j+s-1}}.\end{aligned}$$

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Since $d(s) := \sum_{\lambda=1}^{q} \sum_{j=1}^{a+1} \binom{(\lambda-jq)q+(a+1)q^2+j+s-2}{(\lambda-jq)q+(a+1)q^2-1}$ is a polynomial in s of degree at most $(a+1)q^2 - 1$, it follows that for all n sufficiently large, the function $\sum_{s=1}^{\infty} d(s)(t^n - \sigma^{-n})^{-s}$ is analytic for $|t| > 1$ ([4], Lemma 2). Thus, there is a constant c_0 independent of n such that

$$\left|\sum_{s=1}^{\infty} d(s)\left(t^n - \sigma^{-n}\right)^{-s}\right| \leqslant c_0. \tag{3.17}$$

Since $\sigma \geqslant \rho^{\ell q+1}$ and $|t| = \rho_1$, it follows from (3.16) and (3.17) after some elementary algebra that for sufficiently large n

$$|I_2| \leqslant c_0 \rho^{-n(\ell q+1)} \left(\rho_1^n - \sigma^{-n}\right)^{-aq^2-q}. \tag{3.18}$$

Recall that $\ell q := (aq+b+1)q \leqslant (a+1)q^2$. Therefore, combining (3.11), (3.12), (3.14) and (3.18), we observe that

$$|\gamma_\ell(t, \sigma)| \leqslant \frac{c^*}{\rho_1^{n\ell q}}, \text{ for all sufficiently large } n, \tag{3.19}$$

where c^* is a constant independent of n. Using (3.10) and (3.19), an analysis of the kernels $A_j(t, z)\, K_j(t, \sigma)$, $j = 1, 2, 3$, shows that

$$\overline{\lim_{n\to\infty}} \{\max_{|z|=\tau} |R^*_{n+m,n}(z, f) - W^*_{n+m,n}(z, f, \ell)|\}^{1/n} \leqslant \frac{\tau}{\rho_1^{\ell q+1}}.$$

When $\sigma < \rho^{\ell q+1}$, a similar analysis of $\gamma_\ell(t, \sigma)$ and $A_j(t, z) \cdot K_j(t, \sigma)$ gives us

$$\lim_{n\to\infty} \{R^*_{n+m,n}(z, f) - W^*_{n+m,n}(z, f, \ell)\} = 0,$$

for all z with $|z| \neq \sigma$.

The sharpness of the result can be seen by considering

$$\widehat{f}(z) = \left(z - \rho e^{i\theta}\right)^{-1} \text{ where } 0 \leqslant \theta \leqslant 2\pi.$$

□

REMARK 2. Theorems 1 and 2 are also valid when $q = 1$ and $m = -1$ (see [2], Remark 3.1). Therefore, a result of Saff-Sharma ([6], Theorem 3.1), under the condition $m = -1$, is a special case of Theorem 1.

REMARK 3. If we fix $m = -1$ and let $\sigma \to \infty$ in either of Theorems 1 and 2, we get an extension of Rivlin's result given in (1.7). This follows from the fact that (see (2.1), (3.5)) for all integers $n \geqslant 1$, $\nu \geqslant 0$,

$$\lim_{\sigma\to\infty} \widetilde{r}_{n-1,n}(z, f, \nu) = \lim_{\sigma\to\infty} r^*_{n-1,n}(z, f, \nu) = S_{n-1,\nu}(z, f),$$

where $S_{n-1,\nu}(z, f)$ is described in (1.7).

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