Nagoya Math. J., **233** (2019), 155–169
DOI 10.1017/nmj.2017.31

CHARACTERIZATION OF PROJECTIVE SPACES AND $\mathbb{P}^r$-BUNDLES AS AMPLE DIVISORS

JIE LIU

Abstract. Let X be a projective manifold of dimension n. Suppose that T_X contains an ample subsheaf. We show that X is isomorphic to $\mathbb{P}^n$. As an application, we derive the classification of projective manifolds containing a $\mathbb{P}^r$-bundle as an ample divisor by the recent work of Litt.

§1. Introduction

Projective spaces are the simplest algebraic varieties. They can be characterized in many ways. A very famous one is given by the Hartshorne's conjecture, which was proved by Mori.

THEOREM A. [25, Theorem 8] *Let X be a projective manifold defined over an algebraically closed field k of characteristic $\geqslant 0$. Then X is a projective space if and only if T_X is ample.*

This result has been generalized, over the field of complex number, by several authors (see [1, 12, 27]).

THEOREM B. [1, Theorem] *Let X be a projective manifold of dimension n. If T_X contains an ample locally free subsheaf $\mathcal{E}$ of rank r, then $X \cong \mathbb{P}^n$ and $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ or $\mathcal{E} \cong T_{\mathbb{P}^n}$.*

This theorem was successively proved for $r = 1$ by Wahl [27] and later for $r \geqslant n-2$ by Campana and Peternell [12]. The proof was finally completed by Andreatta and Wiśniewski [1]. The main aim of the present article is to prove the following generalization.

THEOREM 1.1. *Let X be a projective manifold of dimension n. Suppose that T_X contains an ample subsheaf $\mathcal{F}$ of positive rank r, then $(X, \mathcal{F})$ is isomorphic to $(\mathbb{P}^n, T_{\mathbb{P}^n})$ or $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r})$.*

We refer to Section 2.1 for the basic definition and properties of ample sheaves. Comparing with Theorem B, we do not require *a priori* the

Received March 20, 2017. Revised August 10, 2017. Accepted August 13, 2017.
2010 Mathematics subject classification. Primary 14M20; Secondary 14C20, 37F75.

locally freeness of the subsheaf $\mathcal{F}$ in Theorem 1.1. In the case where the Picard number of X is one, Theorem 1.1 is proved in [2]. In fact, in [2], it was shown that the subsheaf $\mathcal{F}$ must be locally free under the additional assumption $\rho(X)=1$, and then Theorem B immediately implies Theorem 1.1. In particular, to prove Theorem 1.1, it suffices to show that X is isomorphic to some projective space if its tangent bundle contains an ample subsheaf $\mathcal{F}$; then, the locally freeness of $\mathcal{F}$ follows from [2]. An interesting and important special case of Theorem 1.1 is when the subsheaf $\mathcal{F}$ comes from the image of an ample vector bundle E over X. This confirms a conjecture of Litt [23, Conjecture 2].

COROLLARY 1.2. *Let X be a projective manifold of dimension n, and let E be an ample vector bundle on X. If there exists a nonzero map $E \to T_X$, then $X \cong \mathbb{P}^n$.*

As an application, we derive the classification of projective manifolds containing a $\mathbb{P}^r$-bundle as an ample divisor. This problem has attracted a great deal of interest over the past few decades (see [6–8, 13, 26], etc.). Recently, in [23, Corollary 7], Litt proved that it can be reduced to Corollary 1.2. To be more precise, we have the following classification theorem.

THEOREM 1.3. *Let X be a projective manifold of dimension $n \geqslant 3$, and let A be an ample divisor on X. Assume that A is a $\mathbb{P}^r$-bundle, $p\colon A \to B$, over a manifold B of dimension $b>0$. Then one of the following holds.*

(i) $(X,A)=(\mathbb{P}(E),H)$ *for some ample vector bundle E over B such that $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$. p is equal to the restriction to A of the induced projection $\mathbb{P}(E) \to B$.*

(ii) $(X,A)=(\mathbb{P}(E),H)$ *for some ample vector bundle E over $\mathbb{P}^1$ such that $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$. $H=\mathbb{P}^1 \times \mathbb{P}^{n-2}$, and p is the projection to the second factor.*

(iii) $(X,A)=(Q^3,H)$, *where Q^3 is a smooth quadric threefold and H is a smooth quadric surface with $H \in |\mathcal{O}_{Q^3}(1)|$. p is the projection to one of the factors of $H \cong \mathbb{P}^1 \times \mathbb{P}^1$.*

(iv) $(X,A)=(\mathbb{P}^3,H)$. *H is a smooth quadric surface and $H \in |\mathcal{O}_{\mathbb{P}^3}(2)|$, and p is again a projection to one of the factors of $H \cong \mathbb{P}^1 \times \mathbb{P}^1$.*

Convention. Throughout, we work over the field $\mathbb{C}$ of complex numbers unless otherwise stated. Varieties are always assumed to be integral separated schemes of finite type over $\mathbb{C}$. If D is a Weil divisor on a projective normal variety X, we denote by $\mathcal{O}_X(D)$ the reflexive sheaf associated to D. Given a coherent sheaf $\mathcal{F}$ on a variety X of generic rank r, then we denote by $\mathcal{F}^\vee$ the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, and by $\det(\mathcal{F})$ the sheaf $(\wedge^r \mathcal{F})^{\vee\vee}$. We denote by $\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ the fiber of $\mathcal{F}$ at $x \in X$. If $\mathcal{F}$ is a coherent sheaf on a variety X, we denote by $\mathbb{P}(\mathcal{F})$ the Grothendieck projectivization $\mathrm{Proj}(\bigoplus_{m \geqslant 0} \mathrm{Sym}^m \mathcal{F})$. If $f\colon X \to Y$ is a morphism between projective normal varieties, we denote by $\Omega^1_{X/Y}$ the relative differential sheaf. Moreover, if Y is smooth, we denote by $K_{X/Y}$ the relative canonical divisor $K_X - f^*K_Y$, and by $\omega_{X/Y}$ the reflexive sheaf $\omega_X \otimes f^*\omega_Y^\vee$.

§2. Ample sheaves and rational curves

Let X be a projective manifold. In this section, we gather some results about the behavior of an ample subsheaf $\mathcal{F} \subset T_X$ with respect to a family of minimal rational curves on X.

2.1 Ample sheaves

Recall that an invertible sheaf $\mathcal{L}$ on a quasi-projective variety X is said to be *ample* if for every coherent sheaf $\mathcal{G}$ on X, there is an integer $n_0 > 0$ such that for every $n \geqslant n_0$, the sheaf $\mathcal{G} \otimes \mathcal{L}^n$ is generated by its global sections (see [18, Section II.7]). In general, a coherent sheaf $\mathcal{F}$ on a quasi-projective variety X is said to be *ample* if the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is ample on $\mathbb{P}(\mathcal{F})$ [22].

Well-known properties of ampleness of locally free sheaves still hold in this general setting.

(i) A sheaf $\mathcal{F}$ on a quasi-projective variety X is ample if and only if, for any coherent sheaf $\mathcal{G}$ on X, $\mathcal{G} \otimes \mathrm{Sym}^m \mathcal{F}$ is globally generated for $m \gg 1$ (see [22, Theorem 1]).

(ii) If $i\colon Y \to X$ is an immersion, and $\mathcal{F}$ is an ample sheaf on X, then $i^*\mathcal{F}$ is an ample sheaf on Y (see [22, Proposition 6]).

(iii) If $\pi\colon Y \to X$ is a finite morphism with X and Y quasi-projective varieties, and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F}$ is ample if and only if $\pi^*\mathcal{F}$ is ample. Note that $\mathbb{P}(\pi^*\mathcal{F}) = \mathbb{P}(\mathcal{F}) \times_X Y$, and $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ pulls back, by a finite morphism, to $\mathcal{O}_{\mathbb{P}(\pi^*\mathcal{F})}(1)$.

(iv) Any quotient of an ample sheaf is ample (see [22, Proposition 1]). In particular, the image of an ample sheaf under a nonzero map is also ample.

(v) If $\mathcal{F}$ is a locally free ample sheaf of rank r, then the sth exterior power $\wedge^s \mathcal{F}$ is ample for any $1 \leqslant s \leqslant r$ (see [17, Corollary 5.3]).

(vi) If $\mathcal{L}$ is an ample invertible sheaf on a quasi-projective variety X, then $\mathcal{L}^m$ is very ample for some $m > 0$; that is, there is an immersion $i\colon X \to \mathbb{P}^n$ for some n such that $\mathcal{L}^m = i^*\mathcal{O}_{\mathbb{P}^n}(1)$ (see [18, II, Theorem 7.6]).

2.2 Minimal rational curves

Let X be a normal projective variety. By $\mathrm{Hom}(\mathbb{P}^1, X)$ we denote the open subscheme $\subset \mathrm{Hilb}(\mathbb{P}^1 \times X)$ of morphisms from $\mathbb{P}^1$ to X. Let $\mathrm{Hom}_1(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be the open subscheme corresponding to those morphisms $f\colon \mathbb{P}^1 \to X$ that are birational onto their image. The group $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathrm{Hom}_1(\mathbb{P}^1, X)$ and its quotient "really parametrizes" morphisms from $\mathbb{P}^1$ into X. It can be proved that the quotient exists, and its normalization is denoted $\mathrm{RatCurves}^n(X)$ and called the *space of rational curve* on X. For more details we refer to [19].

Let $\mathcal{V}$ be an irreducible component of $\mathrm{RatCurves}^n(X)$. $\mathcal{V}$ is said to be *a covering family of rational curves* on X if the corresponding universal family dominates X. A covering family $\mathcal{V}$ of rational curves on X is called *minimal* if its general members have minimal anticanonical degree. If X is a uniruled projective manifold, then X carries a minimal covering family of rational curves. We fix such a family $\mathcal{V}$, and let $[\ell] \in \mathcal{V}$ be a general point. Then, the tangent bundle T_X can be decomposed on the normalization of ℓ as $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$, where $d + 2 = \det(T_X) \cdot \ell \geqslant 2$ is the *anticanonical degree* of $\mathcal{V}$.

Let $\bar{\mathcal{V}}$ be the normalization of the closure of $\mathcal{V}$ in $\mathrm{Chow}(X)$. We define the following equivalence relation on X. Two points $x, y \in X$ are $\bar{\mathcal{V}}$-equivalent if they can be connected by a chain of 1-cycle from $\bar{\mathcal{V}}$. By [10] (see also [21]), there exists a proper surjective morphism $\varphi_0\colon X_0 \to T_0$ from an open subset of X onto a normal variety T_0 whose fibers are $\bar{\mathcal{V}}$-equivalence classes. We call this map the *$\bar{\mathcal{V}}$-rationally connected quotient* of X.

The first step toward Theorem 1.1 is the following result, which was essentially proved in [3].

THEOREM 2.1. [5, Proposition 2.7] *Let X be a projective uniruled manifold, and let $\mathcal{V}$ be a minimal covering family of rational curves on X. If T_X contains a subsheaf $\mathcal{F}$ of rank r such that $\mathcal{F}|_\ell$ is an ample vector bundle for a general member $[\ell] \in \mathcal{V}$, then there exists a dense open subset X_0 of X and a $\mathbb{P}^{d+1}$-bundle $\varphi_0\colon X_0 \to T_0$ such that any curve on X parametrized by $\mathcal{V}$ and meeting X_0 is a line on a fiber of φ_0. In particular, φ_0 is the $\bar{\mathcal{V}}$-rationally connected quotient of X.*

Recall that *the singular locus* $\operatorname{Sing}(\mathcal{S})$ of a coherent sheaf $\mathcal{S}$ over X is the set of all points of X where $\mathcal{S}$ is not locally free.

REMARK 2.2. The hypothesis in Theorem 2.1 that $\mathcal{F}$ is locally free over a general member of $\mathcal{V}$ is automatically satisfied. In fact, since $\mathcal{F}$ is torsion-free and X is smooth, $\mathcal{F}$ is locally free in codimension one. By [19, II, Proposition 3.7], a general member of $\mathcal{V}$ is disjoint from $\operatorname{Sing}(\mathcal{F})$; hence, $\mathcal{F}$ is locally free over a general member of $\mathcal{V}$.

As an immediate application of Theorem 2.1, we can derive a weak version of [2, Theorem 4.2].

COROLLARY 2.3. *Let X be a projective uniruled manifold with $\rho(X)=1$, and let $\mathcal{V}$ be a minimal covering family of rational curves on X. If T_X contains a subsheaf $\mathcal{F}$ of rank r such that $\mathcal{F}|_\ell$ is ample for a general member $[\ell]\in\mathcal{V}$, then $X\cong\mathbb{P}^n$.*

COROLLARY 2.4. [2, Corollary 4.3] *Let X be a projective manifold with $\rho(X)=1$. Assume that T_X contains an ample subsheaf, then $X\cong\mathbb{P}^n$.*

Proof. Since the tangent bundle T_X contains an ample subsheaf $\mathcal{F}$, X is uniruled (see [24, Corollary 8.6]), and it carries a minimal covering family $\mathcal{V}$ of rational curves. Note that the restriction $\mathcal{F}|_C$ is ample for any curve $C\subset X$; thus, we can deduce the result from Corollary 2.3. $\square$

REMARK 2.5. Our approach above is quite different from that in [2]. The proof in [2] is based on a careful analysis of the singular locus of $\mathcal{F}$, and the locally freeness of $\mathcal{F}$ has been proved. Even though our argument does not tell anything about the singular locus of $\mathcal{F}$, it has the advantage of giving a rough description of the geometric structure of projective manifolds whose tangent bundle contains a "positive" subsheaf.

§3. Foliations and Pfaff fields

Let $\mathcal{S}$ be a subsheaf of T_X on a quasi-projective manifold X. We denote by $\mathcal{S}^{\mathrm{reg}}$ the largest open subset of X such that $\mathcal{S}$ is a subbundle of T_X over $\mathcal{S}^{\mathrm{reg}}$. Note that, in general, $\operatorname{Sing}(\mathcal{S})$ is a proper subset of $X\setminus\mathcal{S}^{\mathrm{reg}}$.

DEFINITION 3.1. Let X be a quasi-projective manifold, and let $\mathcal{S}\subsetneq T_X$ be a coherent subsheaf of positive rank. $\mathcal{S}$ is called a foliation if it satisfies the following conditions.

(i) $\mathcal{S}$ is saturated in T_X; that is, $T_X/\mathcal{S}$ is torsion-free.
(ii) The sheaf $\mathcal{S}$ is closed under the Lie bracket.

In addition, $\mathcal{S}$ is called an algebraically integrable foliation if the following holds.

(iii) For a general point $x \in X$, there exists a projective subvariety F_x passing through x such that

$$\mathcal{S}|_{F_x \cap \mathcal{S}^{\mathrm{reg}}} = T_{F_x}|_{F_x \cap \mathcal{S}^{\mathrm{reg}}} \subset T_X|_{F_x \cap \mathcal{S}^{\mathrm{reg}}}.$$

We call F_x the $\mathcal{S}$-leaf through x.

REMARK 3.2. Let X be a projective manifold, and let $\mathcal{S}$ be a saturated subsheaf of T_X. To show that $\mathcal{S}$ is an algebraically integrable foliation, it is sufficient to show that it is an algebraically integrable foliation over a Zariski open subset of X.

EXAMPLE 3.3. Let $X \to Y$ be a fibration with X and Y projective manifolds. Then $T_{X/Y} \subset T_X$ defines an algebraically integrable foliation on X such that the general leaves are the fibers.

EXAMPLE 3.4. [4, 4.1] Let $\mathcal{F}$ be a subsheaf $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ of $T_{\mathbb{P}^n}$ on $\mathbb{P}^n$. Then $\mathcal{F}$ is an algebraically integrable foliation and it is defined by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r}$. The set of points of indeterminacy S of this rational map is an $(r-1)$-dimensional linear subspace. Let $x \notin S$ be a point. Then the leaf passing through x is the r-dimensional linear subspace L of $\mathbb{P}^n$ containing both x and S.

DEFINITION 3.5. Let X be a projective variety, and r a positive integer. A Pfaff field of rank r on X is a nonzero map $\partial \colon \Omega_X^r \to \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on X.

LEMMA 3.6. [5, Proposition 4.5] *Let X be a projective variety, and let $n \colon \widetilde{X} \to X$ be its normalization. Let $\mathcal{L}$ be an invertible sheaf on X, let r be a positive integer, and let $\partial \colon \Omega_X^r \to \mathcal{L}$ be a Pfaff field. Then ∂ can be extended uniquely to a Pfaff field $\widetilde{\partial} \colon \Omega_{\widetilde{X}}^r \to n^*\mathcal{L}$.*

Let X be a projective manifold, and let $\mathcal{S} \subset T_X$ be a subsheaf with positive rank r. We denote by $K_{\mathcal{S}}$ the canonical class $-c_1(\det(\mathcal{S}))$ of $\mathcal{S}$. Then there is a natural associated Pfaff field of rank r:

$$\Omega_X^r = \wedge^r(\Omega_X^1) = \wedge^r(T_X^\vee) = (\wedge^r T_X)^\vee \to \mathcal{O}_X(K_{\mathcal{S}}).$$

LEMMA 3.7. [4, Lemma 3.2] *Let X be a projective manifold, and let $\mathcal{S}$ be an algebraically integrable foliation on X. Then there is a unique irreducible projective subvariety W of* $\mathrm{Chow}(X)$ *whose general point parametrizes a general leaf of $\mathcal{S}$.*

REMARK 3.8. Let X be a projective manifold, and let $\mathcal{S}$ be an algebraically integrable foliation of rank r on X. Let W be the subvariety of $\mathrm{Chow}(X)$ provided in Lemma 3.7. Let $Z \subset W$ be a general closed subvariety of W, and let $U \subset Z \times X$ be the universal cycle over Z. Let $\widetilde{Z}$ and $\widetilde{U}$ be the normalizations of Z and U, respectively. We claim that the Pfaff field $\Omega^r_X \to \mathcal{O}_X(K_{\mathcal{S}})$ can be extended to a Pfaff field $\Omega^r_{\widetilde{U}/\widetilde{Z}} \to n^* p^* \mathcal{O}_X(K_{\mathcal{S}})$.

$$\begin{tikzcd}
\widetilde{U} \arrow[r, "n"] \arrow[d, "\widetilde{q}"'] & U \arrow[r, phantom, "\subset"] \arrow[d, "q"] & Z \times X \arrow[r, "p"] \arrow[d, "q"] & X \\
\widetilde{Z} \arrow[r] & Z \arrow[r, "="] & Z &
\end{tikzcd}$$

Let V be the universal cycle over W with $v \colon V \to X$. From the proof of [4, Lemma 3.2], we know that the Pfaff field $\Omega^r_X \to \mathcal{O}_X(K_{\mathcal{S}})$ extends to be a Pfaff field $\Omega^r_V \to v^* \mathcal{O}_X(K_{\mathcal{S}})$. It induces a Pfaff field $\Omega^r_U \to p^* \mathcal{O}_X(K_{\mathcal{S}})$. Note that U is irreducible since Z is a general subvariety. By Lemma 3.6, it can be uniquely extended to a Pfaff field $\Omega^r_{\widetilde{U}} \to n^* p^* \mathcal{O}_X(K_{\mathcal{S}})$.

Let $\mathcal{K}$ be the kernel of the morphism $\Omega^r_{\widetilde{U}} \twoheadrightarrow \Omega^r_{\widetilde{U}/\widetilde{Z}}$. Let F be a general fiber of $\widetilde{q}$ such that its image under $p \circ n$ is an $\mathcal{S}$-leaf and the morphism $p \circ n$ restricted on F is finite and birational. Let $x \in F$ be a point such that F is smooth at x and $p \circ n$ is an isomorphism at a neighborhood of x. Then the composite map $\Omega^r_{\widetilde{U}}|_F \twoheadrightarrow \Omega^r_{\widetilde{U}/\widetilde{Z}}|_F \twoheadrightarrow \Omega^r_F$ implies that the composite map

$$\mathcal{K} \to \Omega^r_{\widetilde{U}} \to n^* p^* \mathcal{O}_X(K_{\mathcal{S}})$$

vanishes in a neighborhood of x; hence, it vanishes generically over $\widetilde{U}$. Since the sheaf $n^* p^* \mathcal{O}_X(K_{\mathcal{S}})$ is torsion-free, it vanishes identically and finally yields a Pfaff field $\Omega^r_{\widetilde{U}/\widetilde{Z}} \to n^* p^* \mathcal{O}_X(K_{\mathcal{S}})$.

Let X be a projective manifold, and let $\mathcal{S} \subset T_X$ be a subsheaf. We define its saturation $\overline{\mathcal{S}}$ as the kernel of the natural surjection $T_X \twoheadrightarrow (T_X/\mathcal{S})/(torsion)$. Then $\overline{\mathcal{S}}$ is obviously saturated.

THEOREM 3.9. *Let X be a projective manifold. Assume that T_X contains an ample subsheaf $\mathcal{F}$ of rank $r < \dim(X)$. Then its saturation $\overline{\mathcal{F}}$ defines an algebraically integrable foliation on X, and the $\overline{\mathcal{F}}$-leaf passing through a general point is isomorphic to $\mathbb{P}^r$.*

Proof. Let $\varphi_0 \colon X_0 \to T_0$ be as the morphism provided in Theorem 2.1. Since $\mathcal{F}$ is locally free in codimension one, we may assume that no fiber of φ_0 is completely contained in $\mathrm{Sing}(\mathcal{F})$.

The first step is to show that $\mathcal{F}|_{X_0} \subset T_{X_0/T_0}$. Since $\varphi_0 \colon X_0 \to T_0$ is smooth, we get a short exact sequence of vector bundles,

$$0 \to T_{X_0/T_0} \to T_X|_{X_0} \to \varphi_0^* T_{T_0} \to 0.$$

The composite map $\mathcal{F}|_{X_0} \to T_X|_{X_0} \to \varphi_0^* T_{T_0}$ vanishes on a Zariski open subset of every fiber. Since $\varphi_0^* T_{T_0}$ is torsion-free, it vanishes identically, and it follows that $\mathcal{F}|_{X_0} \subset T_{X_0/T_0}$.

Next, we show that, after shrinking X_0 and T_0 if necessary, $\mathcal{F}$ is actually locally free over X_0. By the generic flatness theorem [15, Théorème 6.9.1], after shrinking T_0, we can suppose that $(T_X/\mathcal{F})|_{X_0}$ is flat over T_0. Let $F \cong \mathbb{P}^{d+1}$ be an arbitrary fiber of φ_0. The short exact sequence of sheaves

$$0 \to \mathcal{F}|_{X_0} \to T_X|_{X_0} \to (T_X/\mathcal{F})|_{X_0} \to 0$$

induces a long exact sequence of sheaves

$$\mathcal{T}or((T_X/\mathcal{F})|_{X_0}, \mathcal{O}_F) \to \mathcal{F}|_F \to T_X|_F \to (T_X/\mathcal{F})|_F \to 0.$$

Since $(T_X/\mathcal{F})|_{X_0}$ is flat over T_0, it follows that $\mathcal{F}|_F$ is a subsheaf of $T_X|_F$; in particular, $\mathcal{F}|_F$ is torsion-free. Without loss of generality, we may assume that the restrictions of $\mathcal{F}$ on all fibers of φ_0 are torsion-free. By Remark 2.5, the restrictions of $\mathcal{F}$ on all fibers of φ_0 are locally free. This yields, in particular, that the dimension of the fibers of $\mathcal{F}$ is constant on every fiber of φ_0 due to $\mathcal{F}(x) = (\mathcal{F}|_F)(x)$. Note that no fiber of φ_0 is contained in $\mathrm{Sing}(\mathcal{F})$. We conclude that the dimension of the fibers $\mathcal{F}(x)$ of $\mathcal{F}$ is constant over X_0. Hence, $\mathcal{F}$ is locally free over X_0.

Now, we claim that $\overline{\mathcal{F}}$ actually defines an algebraically integrable foliation on X_0. Let $F \cong \mathbb{P}^{d+1}$ be an arbitrary fiber of φ_0. We know that $(F, \mathcal{F}|_F)$ is isomorphic to $(\mathbb{P}^{d+1}, T_{\mathbb{P}^{d+1}})$ or $(\mathbb{P}^{d+1}, \mathcal{O}_{\mathbb{P}^{d+1}}(1)^{\oplus r})$ (cf. Theorem B); therefore, $\mathcal{F}$ defines an algebraically integrable foliation over X_0 (cf. Example 3.4). Note that we have $\mathcal{F}|_{X_0} = \overline{\mathcal{F}}|_{X_0}$, since $\mathcal{F}|_{X_0}$ is saturated in T_{X_0}. Hence, $\overline{\mathcal{F}}$ also defines an algebraically integrable foliation over X (cf. Remark 3.2). □

REMARK 3.10. Since $\mathcal{F}$ is locally free on X_0, it follows that $\mathcal{O}_X(-K_{\mathcal{F}})|_{X_0}$ is isomorphic to $\wedge^r(\mathcal{F}|_{X_0})$ and the invertible sheaf $\mathcal{O}_X(-K_{\mathcal{F}})$ is ample over X_0. Moreover, as $\mathcal{F}$ is locally free in codimension one, there exists an open subset $X' \subset X$ containing X_0 such that $\operatorname{codim}(X \setminus X') \geqslant 2$ and $\mathcal{O}_X(-K_{\mathcal{F}})$ is ample on X'.

§4. Proof of main theorem

The aim of this section is to prove Theorem 1.1. Let X be a normal projective variety, and let $X \to C$ be a surjective morphism with connected fibers onto a smooth curve. Let Δ be an effective Weil divisor on X such that (X, Δ) is log-canonical over the generic point of C. In [4, Theorem 5.1], it was proved that $-(K_{X/C} + \Delta)$ cannot be ample. In the next theorem, we give a variant of this result which is the key ingredient in our proof of Theorem 1.1.

THEOREM 4.1. *Let X be a normal projective variety, and let $f\colon X \to C$ be a surjective morphism with connected fibers onto a smooth curve. Let Δ be a Weil divisor on X such that $K_X + \Delta$ is Cartier and Δ^{hor} is reduced. Assume that there exists an open subset C_0 such that the pair $(X, \Delta^{\mathrm{hor}})$ is snc over $X_0 = f^{-1}(C_0)$. If $X' \subset X$ is an open subset such that no fiber of f is completely contained in $X \setminus X'$ and $X_0 \subset X'$, then the invertible sheaf $\mathcal{O}_X(-K_{X/C} - \Delta)$ is not ample over X'.*

Proof. To prove the theorem, we assume, to the contrary, that the invertible sheaf $\mathcal{O}_X(-K_{X/C} - \Delta)$ is ample over X'. Let A be an ample divisor supported on C_0. Then, for some $m \gg 1$, the sheaf $\mathcal{O}_X(-m(K_{X/C} + \Delta) - f^*A)$ is very ample over X' (see [14, Corollaire 4.5.11]). It follows that there exists a prime divisor D' on X' such that the pair $(X', \Delta^{\mathrm{hor}}|_{X'} + D')$ is snc over X_0 and

$$D' \sim (-m(K_{X/C} + \Delta) - f^*A)|_{X'}.$$

This implies that there exists a rational function $h \in K(X') = K(X)$ such that the restriction of the Cartier divisor $D = \operatorname{div}(h) - m(K_{X/C} + \Delta) - f^*A$ on X' is D', and D^{hor} is the closure of D' in X. Note that we can write $D = D_+ - D_-$ for some effective divisors D_+ and D_- with no common components. Then we have $\operatorname{Supp}(D_-) \subset X \setminus X'$. In particular, no fiber of f is supported on D_-. By [20, Theorem 4.15], there exists a log-resolution $\mu\colon \widetilde{X} \to X$ such that we have the following.

https://doi.org/10.1017/nmj.2017.31 Published online by Cambridge University Press

(i) The induced morphism $\widetilde{f} = f \circ \mu \colon \widetilde{X} \to C$ is prepared (cf. [11, Section 4.3]).
(ii) The birational morphism μ is an isomorphism over X_0.
(iii) $\mu_*^{-1}\Delta^{\mathrm{hor}} + \mu_*^{-1}D^{\mathrm{hor}}$ is a snc divisor.

Let E be the exceptional divisor of μ. Note that we have $\widetilde{f}_*(E) \neq C$. Moreover, we also have

$$K_{\widetilde{X}} + \mu_*^{-1}\Delta + \frac{1}{m}\mu_*^{-1}D_+ = \mu^*\left(K_X + \Delta + \frac{1}{m}D\right) + \frac{1}{m}\mu_*^{-1}D_- + E_+ - E_-,$$

where E_+ and E_- are effective μ-exceptional divisors with no common components.

Set $\widetilde{D} = m\mu_*^{-1}\Delta + \mu_*^{-1}D_+ + mE_-$. Then $\widetilde{D}^{\mathrm{hor}} = m\mu_*^{-1}\Delta^{\mathrm{hor}} + \mu_*^{-1}D^{\mathrm{hor}}$ is an snc effective divisor with coefficients $\leqslant m$. Since D is linearly equivalent to $-m(K_{X/C} + \Delta) - f^*A$, we can write

$$K_{\widetilde{X}/C} + \frac{1}{m}\widetilde{D} \sim_{\mathbb{Q}} -\frac{1}{m}\widetilde{f}^*A + \frac{1}{m}\mu_*^{-1}D_- + E_+.$$

After multiplying by some positive l divisible by the denominators of the coefficients of E_+ and E_-, we may assume that $\mathrm{lm}\, E_+$ and $\mathrm{lm}\, E_-$ are integer coefficients. By replacing $\widetilde{D}$ by $l\widetilde{D}$, the weak positivity theorem [11, Theorem 4.13] implies that the direct image sheaf

$$\begin{aligned}\widetilde{f}_*(\omega_{\widetilde{X}/C}^{\mathrm{lm}} \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{D})) &\simeq \widetilde{f}_*(\mathcal{O}_{\widetilde{X}}(-l\widetilde{f}^*A + \mathrm{lm}\, E_+ + l\mu_*^{-1}D_-)) \\ &\simeq \mathcal{O}_C(-lA) \otimes \widetilde{f}_*\mathcal{O}_{\widetilde{X}}(\mathrm{lm}\, E_+ + l\mu_*^{-1}D_-)\end{aligned}$$

is weakly positive.

Observe that $\widetilde{f}_*(\mathcal{O}_{\widetilde{X}}(\mathrm{lm}\, E_+ + l\mu_*^{-1}D_-)) = \mathcal{O}_C$. Indeed, E_+ is a μ-exceptional divisor. It follows that $\mu_*(\mathcal{O}_{\widetilde{X}}(\mathrm{lm}\, E_+ + l\mu_*^{-1}D_-)) = \mathcal{O}_X(lD_-)$. Note that we have $f_*(\mathcal{O}_X(lD_-)) = \mathcal{O}_C(P)$ for some effective divisor P on C such that $\mathrm{Supp}(P) \subset f(\mathrm{Supp}(D_-))$. Let V be an open subset of C, and let $\lambda \in H^0(V, \mathcal{O}_C(P))$. That is, λ is a rational function on C such that $\mathrm{div}(\lambda) + P \geqslant 0$ over V. It follows that $\mathrm{div}(\lambda \circ f) + lD_- \geqslant 0$ over $f^{-1}(V)$. Since there is no fiber of f completely supported on D_-, the rational function $\lambda \circ f$ is regular over $f^{-1}(V)$. Consequently, the rational function λ is regular over V. This implies that the natural inclusion $\mathcal{O}_C \to \mathcal{O}_C(P)$ is surjective, which yields $\widetilde{f}_*(\mathcal{O}_{\widetilde{X}}(\mathrm{lm}\, E_+ + l\mu_*^{-1}D_-)) = \mathcal{O}_C$. However, this shows that $\mathcal{O}_C(-lA)$ is weakly positive, which is a contradiction. Hence, $\mathcal{O}_X(-K_{X/C} - \Delta)$ is not ample over X'. $\square$

LEMMA 4.2. *Let X be a normal projective variety, and let $f\colon X \to C$ be a surjective morphism with reduced and connected fibers onto a smooth curve C. Let D be a Cartier divisor on X. If there exists a nonzero morphism $\Omega^r_{X/C} \to \mathcal{O}_X(D)$, where r is the relative dimension of f, then there exists an effective Weil divisor Δ on X such that $K_{X/C} + \Delta = D$.*

Proof. Since all of the fibers of f are reduced, the sheaf $\Omega^r_{X/C}$ is locally free in codimension one. Hence, the reflexive hull of $\Omega^r_{X/C}$ is $\omega_{X/C} \simeq \mathcal{O}_X(K_{X/C})$. Note that $\mathcal{O}_X(D)$ is reflexive; the nonzero morphism $\Omega^r_{X/C} \to \mathcal{O}_X(D)$ induces a nonzero morphism $\omega_{X/C} \to \mathcal{O}_X(D)$. This shows that there exists an effective divisor Δ on X such that $K_{X/C} + \Delta = D$. $\square$

As an application of Theorem 4.1, we derive a special property about foliations defined by an ample subsheaf of T_X. A similar result was established for Fano foliations with mild singularities in the work of Araujo and Druel (see [4, Proposition 5.3]), and we follow the same strategy.

PROPOSITION 4.3. *Let X be a projective manifold. If $\mathcal{F} \subset T_X$ is an ample subsheaf of rank $r < n = \dim(X)$, then there is a common point in the closure of general leaves of $\overline{\mathcal{F}}$.*

Proof. Since $\mathcal{F}$ is torsion-free and X is smooth, $\mathcal{F}$ is locally free over an open subset $X' \subset X$ such that $\operatorname{codim}(X \setminus X') \geqslant 2$. In particular, $\mathcal{O}_X(-K_{\mathcal{F}})$ is ample over X'. By Theorem 2.1, there exist an open subset $X_0 \subset X$ and a $\mathbb{P}^{d+1}$-bundle $\varphi_0\colon X_0 \to T_0$. Moreover, from the proof of Theorem 3.9, the saturation $\overline{\mathcal{F}}$ defines an algebraically integrable foliation on X, and we may assume that $\mathcal{F}$ is locally free over X_0. In particular, we have $X_0 \subset X'$. In view of Lemma 3.7, we denote by W the subvariety of $\mathrm{Chow}(X)$ parametrizing the general leaves of $\overline{\mathcal{F}}$, and by V the normalization of the universal cycle over W. Let $p\colon V \to X$ and $\pi\colon V \to W$ be the natural projections. Note that there exists an open subset W_0 of W such that $p(\pi^{-1}(W_0)) \subset X_0$.

To prove our proposition, we assume to the contrary that there is no common point in the general leaves of $\overline{\mathcal{F}}$.

First, we show that there exists a smooth curve C with a finite morphism $n\colon C \to n(C) \subset W$ such that we have the following.

(i) Let U be the normalization of the fiber product $V \times_W C$ with projection $\pi\colon U \to C$. Then the induced morphism $\widetilde{p}\colon U \to X$ is finite onto its image.
(ii) There exists an open subset C_0 of C such that the image of U_0 under p is contained in X_0. In particular, $U_0 = \pi^{-1}(C_0)$ is a $\mathbb{P}^r$-bundle over C_0.

https://doi.org/10.1017/nmj.2017.31 Published online by Cambridge University Press

(iii) For any point $c \in C$, the image of the fiber $\pi^{-1}(c)$ under $\widetilde{p}$ is not contained in $X \setminus X'$.
(iv) All of the fibers of π are reduced.

Note that we have $X \setminus X' = \mathrm{Sing}\,(\mathcal{F})$ and $\mathrm{codim}(\mathrm{Sing}\,(\mathcal{F})) \geqslant 2$. We consider the subset

$$Z = \{w \in W \mid \pi^{-1}(w) \subset p^{-1}(\mathrm{Sing}(\mathcal{F}))\}.$$

Since π is equidimensional, it is a surjective universally open morphism (see [16, Théorème 14.4.4]). Therefore, the subset Z is closed. Note that the general fiber of π is disjoint from $p^{-1}(\mathrm{Sing}(\mathcal{F}))$, so $\mathrm{codim}(Z) \geqslant 1$. Moreover, by the definition of Z, we have $p(\pi^{-1}(Z)) \subset \mathrm{Sing}(\mathcal{F})$ and $\mathrm{codim}(\mathrm{Sing}(\mathcal{F})) \geqslant 2$. Hence, we can choose some very ample divisors H_i $(1 \leqslant i \leqslant n)$ on X such that the curve B defined by the complete intersection $\widetilde{p}^*H_1 \cap \cdots \cap \widetilde{p}^*H_n$ satisfies the following conditions.

(i′) There is no common point in the closure of the general fibers of π over $\pi(B)$.
(ii′) $\pi(B) \cap W_0 \neq \emptyset$.
(iii′) $\pi(B) \subset W \setminus Z$.

Let $B' \to B$ be the normalization, and let $V_{B'}$ be the normalization of the fiber product $V \times_B B'$. The induced morphism $V_{B'} \to V$ is denoted by μ. Then it is easy to check that B' satisfies (i), (ii) and (iii). By [9, Theorem 2.1], there exists a finite morphism $C \to B'$ such that all of the fibers of $U \to C$ are reduced, where U is the normalization of $U \times_{B'} C$. Then we see at once that C is the desired curve.

The next step is to get a contradiction by applying Theorem 4.1. From Remark 3.8, we see that the Pfaff field $\Omega_X^r \to \mathcal{O}_X(K_{\overline{\mathcal{F}}})$ extends to a Pfaff field $\Omega^r_{V_{B'}/B'} \to \mu^*p^*\mathcal{O}_X(K_{\overline{\mathcal{F}}})$, and it induces a Pfaff field $\Omega^r_{U/C} \to \widetilde{p}^*\mathcal{O}_X(K_{\overline{\mathcal{F}}})$. The natural inclusion $\mathcal{F} \hookrightarrow \overline{\mathcal{F}}$ induces a morphism $\mathcal{O}_X(K_{\overline{\mathcal{F}}}) \to \mathcal{O}_X(K_{\mathcal{F}})$. This implies that we have a Pfaff field $\Omega^r_{U/C} \to \widetilde{p}^*\mathcal{O}_X(K_{\mathcal{F}})$. By Lemma 4.2, there exists an effective Weil divisor Δ on U such that $K_{U/C} + \Delta = \widetilde{p}^*K_{\mathcal{F}}$.

Let Δ^{hor} be the π-horizontal part of Δ. After shrinking C_0, we may assume that $\Delta|_{U_0} = \Delta^{\mathrm{hor}}|_{U_0}$. According to the proof of Theorem 3.9, for any fiber $F \cong \mathbb{P}^r$ over C_0, we have $(\widetilde{p}^*K_{\mathcal{F}})|_F - K_F = 0$ or H, where $H \in |\mathcal{O}_{\mathbb{P}^r}(1)|$. This shows that either Δ^{hor} is zero or Δ^{hor} is a prime divisor such that $\Delta|_{U_0} = \Delta^{\mathrm{hor}}|_{U_0} \in |\mathcal{O}_{U_0}(1)|$. In particular, the pair $(U, \Delta^{\mathrm{hor}})$ is snc over U_0

and Δ^{hor} is reduced. Note that $\widetilde{p}\colon U \to \widetilde{p}(U)$ is a finite morphism, so the invertible sheaf $\widetilde{p}^*\mathcal{O}_X(-K_{\mathcal{F}})$ is ample over $U' = U \cap \widetilde{p}^{-1}(X')$. That is, the sheaf $\mathcal{O}_U(-K_{U/C} - \Delta)$ is ample over U', which contradicts Theorem 4.1. □

Now, our main result immediately follows.

Proof of Theorem 1.1. Theorem 2.1 implies that there exist an open subset $X_0 \subset X$ and a normal variety T_0 such that $X_0 \to T_0$ is a $\mathbb{P}^{d+1}$-bundle and $d+1 \geqslant r$. Without loss of generality, we may assume that $r < \dim(X)$. By Theorem 3.9 followed by Proposition 4.3, $\overline{\mathcal{F}}$ defines an algebraically integrable foliation over X such that there is a common point in the closure of general leaves of $\overline{\mathcal{F}}$. However, this cannot happen if $\dim(T_0) \geqslant 1$. Hence, we have $\dim T_0 = 0$ and $X \cong \mathbb{P}^n$. □

§5. $\mathbb{P}^r$-bundles as ample divisors

As an application of Theorem 1.1, we classify projective manifolds X containing $\mathbb{P}^r$-bundles as ample divisors. This was originally conjectured by Beltrametti and Sommese (see [8, Conjecture 5.5.1]). In the remainder of this section, we follow the same notation and assumptions as in Theorem 1.3.

The case $r \geqslant 2$ follows from Sommese's extension theorem [26, Proposition III] (see also [8, Theorem 5.5.2]). For $r = 1$ and $b = 1$, it is due to Bădescu [6, Theorem D] (see also [8, Theorem 5.5.3]). For $r = 1$ and $b = 2$, it is due to the work of several authors (see [7, Theorem 7.4]). As mentioned in the introduction, Litt proved the following result, by which we can deduce Theorem 1.3 from Corollary 1.2.

PROPOSITION 5.1. [23, Lemma 4] *Let X be a projective manifold of dimension $\geqslant 3$, and let A be an ample divisor. Assume that $p\colon A \to B$ is a $\mathbb{P}^1$-bundle, then either p extends to a morphism $\widehat{p}\colon X \to B$, or there exists an ample vector bundle E on B and a nonzero map $E \to T_B$.*

For the reader's convenience, we outline the argument of Litt that reduces Theorem 1.3 to Corollary 1.2.

Proof of Theorem 1.3. Since the case $r \geqslant 2$ is already known, we can assume that $r = 1$; that is, $p\colon A \to B$ is a $\mathbb{P}^1$-bundle.

If p extends to a morphism $\widehat{p}\colon X \to B$, then the result follows from [7, Theorem 5.5] and we are in case (i) of the theorem.

If p does not extend to a morphism $X \to B$, by Proposition 5.1, there exists an ample vector bundle E over B with a nonzero map $E \to T_B$. Due to Corollary 1.2, we have $B \cong \mathbb{P}^b$. As the case $b \leqslant 2$ is also known, we may assume that $b \geqslant 3$. In this case, by [13, Theorem 2.1], we conclude that X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and we are in case (ii) of the theorem. $\square$

Acknowledgments. I would like to express my deep gratitude to my advisor A. Höring for suggesting to me to work on this question and also for his many valuable discussions, guidance and help during the preparation of this paper. I would like to express my sincere thanks to S. Druel for his careful reading of the first draft of this paper and for kindly pointing out several mistakes. Special gratitude is due to M. Beltrametti, D. Litt and C. Mourougane for their interest and helpful comments.

References

[1] M. Andreatta and J. A. Wiśniewski, *On manifolds whose tangent bundle contains an ample subbundle*, Invent. Math. **146**(1) (2001), 209–217.

[2] M. Aprodu, S. Kebekus and T. Peternell, *Galois coverings and endomorphisms of projective varieties*, Math. Z. **260**(2) (2008), 431–449.

[3] C. Araujo, *Rational curves of minimal degree and characterizations of projective spaces*, Math. Ann. **335**(4) (2006), 937–951.

[4] C. Araujo and S. Druel, *On Fano foliations*, Adv. Math. **238** (2013), 70–118.

[5] C. Araujo, S. Druel and S. J. Kovács, *Cohomological characterizations of projective spaces and hyperquadrics*, Invent. Math. **174**(2) (2008), 233–253.

[6] L. Bădescu, "*Hyperplane sections and deformations*", in *Algebraic Geometry, Bucharest 1982 (Bucharest, 1982)*, Lecture Notes in Mathematics **1056**, Springer, Berlin, 1984, 1–33.

[7] M. C. Beltrametti and P. Ionescu, "*A view on extending morphisms from ample divisors*", in *Interactions of Classical and Numerical Algebraic Geometry*, Contemporary Mathematics **496**, American Mathematical Society, Providence, RI, 2009, 71–110.

[8] M. C. Beltrametti and A. J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, de Gruyter Expositions in Mathematics **16**, Walter de Gruyter & Co., Berlin, 1995.

[9] S. Bosch, W. Lütkebohmert and M. Raynaud, *Formal and rigid geometry. IV. The reduced fibre theorem*, Invent. Math. **119**(2) (1995), 361–398.

[10] F. Campana, *Connexité rationnelle des variétés de Fano*, Ann. Sci. Éc. Norm. Supér. (4) **25**(5) (1992), 539–545.

[11] F. Campana, *Orbifolds, special varieties and classification theory*, Ann. Inst. Fourier (Grenoble) **54**(3) (2004), 499–630.

[12] F. Campana and T. Peternell, *Rational curves and ampleness properties of the tangent bundle of algebraic varieties*, Manuscripta Math. **97**(1) (1998), 59–74.

[13] M. L. Fania, E.-i. Sato and A. J. Sommese, *On the structure of 4-folds with a hyperplane section which is a* $\mathbf{P}^1$ *bundle over a surface that fibres over a curve*, Nagoya Math. J. **108** (1987), 1–14.

[14] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Publ. Math. Inst. Hautes Études Sci. **8** (1961), 222.

[15] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Publ. Math. Inst. Hautes Études Sci. **24** (1965), 231.

[16] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Publ. Math. Inst. Hautes Études Sci. **28** (1966), 255.
[17] R. Hartshorne, *Ample vector bundles*, Publ. Math. Inst. Hautes Études Sci. **29** (1966), 63–94.
[18] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977.
[19] J. Kollár, *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics] **32**, Springer, Berlin, 1996.
[20] J. Kollár, *Singularities of the Minimal Model Program*, Cambridge Tracts in Mathematics **200**, Cambridge University Press, Cambridge, 2013, With a collaboration of Sándor Kovács.
[21] J. Kollár, Y. Miyaoka and S. Mori., *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. **36**(3) (1992), 765–779.
[22] K. Kubota, *Ample sheaves*, J. Fac. Sci. Univ. Tokyo Sect. I A Math. **17** (1970), 421–430.
[23] D. Litt, *Manifolds containing an ample* $\mathbf{P}^1$*-bundle*, Manuscripta Math. **152**(3) (2017), 533–537.
[24] Y. Miyaoka, "*Deformations of a morphism along a foliation and applications*", in *Algebraic Geometry, Bowdoin 1985 (Brunswick, Maine, 1985)*, Proceedings of Symposia in Pure Mathematics **46**, American Mathematical Society, Providence, RI, 1987, 245–268.
[25] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110**(3) (1979), 593–606.
[26] A. J. Sommese, *On manifolds that cannot be ample divisors*, Math. Ann. **221**(1) (1976), 55–72.
[27] J. M. Wahl, *A cohomological characterization of* $\mathbf{P}^n$, Invent. Math. **72**(2) (1983), 315–322.

Laboratoire de Mathématiques J.A. Dieudonné
UMR 7351 du CNRS Université de Côte d'Azur
Parc Valrose, 06108 Nice Cedex 02
France

jliu@unice.fr