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SEMI-CLASSICAL SOLUTIONS FOR A NONLINEAR COUPLED ELLIPTIC-PARABOLIC PROBLEM

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We give an existence result for a fully nonlinear system consisting of a parabolic equation strongly coupled with an elliptic one. It models in particular miscible displacement in porous media. To this aim, we adapt the tools of Ladyzenskaja, Solonnikov and Uralćeva [**27, 28**] to the coupled nonlinear setting. Under some reasonable assumptions on the data, we state the existence of semi-classical solutions for the problem. We also give an existence result of weak solutions for a degenerate form of the problem.

1. INTRODUCTION

We consider a single-phase miscible displacement of an incompressible fluid by another in a porous medium. It is modeled by a fully nonlinear system of partial differential equations, consisting of an elliptic pressure equation strongly coupled with a parabolic one for the concentration. We assume that the displacement occurs during the time interval $(0,T)$, $T>0$, in a bounded $C^{2,\alpha}$ domain Ω of $\mathbb{R}^n$, for instance $n=2$ or 3, with $0<\alpha<1$. Its boundary is $\partial\Omega$. We denote by ν the exterior normal to $\partial\Omega$. Let also $\Omega_T=\Omega\times(0,T)$. We denote by c the concentration of mass of one of the two fluids of the mixture, and by p the pressure. The equations of the flow are given in Scheidegger [**37**], Peaceman [**34**], Douglas and Roberts [**16**]. The pressure $p(x,t)$ satisfies the incompressibility equation

$$\operatorname{div}(q)=f^+-f^- \quad \text{in } \Omega_T, \tag{1.1}$$

where the rate of flow $q(x,t)$ is given by the Darcy law

$$q=-\frac{k(x)}{\mu(c)}\nabla p \quad \text{in } \Omega_T. \tag{1.2}$$

In Equation (1.2), the function $k(x)$ is the rock permeability. The function $\mu(c)$ is the viscosity of the mixture, depending nonlinearly of the concentration. For instance in the Koval model [**25**], μ is defined on the interval $(0,1)$ by

$$\mu(u)=\mu(0)\left(1+(M^{1/4}-1)u\right)^{-4},$$

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where $M = \mu(0)/\mu(1)$ is the mobility ratio. For sake of simplicity, we neglect here the gravitational term. The concentration $c(x,t)$ is such that

$$\phi(x)\partial_t c + q\cdot\nabla c - \operatorname{div}\big(E(q)\nabla c\big) = f^+(1-c) \quad \text{in } \Omega_T. \tag{1.3}$$

We take into account the two main mechanisms of migration, the convection and the diffusion effects. The velocity-dependent dispersion is usually modeled by a nonlinear tensor which is of the form (see [**37**])

$$E(q) = \phi\left(d_m Id + |q|\left(d_L\mathcal{E}(q) + d_T\big(Id - \mathcal{E}(q)\big)\right)\right)$$

where $\mathcal{E}(q)_{ij} = q_i q_j/|q|^2$, the real d_m is the molecular diffusion, d_L and d_T are the longitudinal and transverse dispersion constants. However we emphasise that this work remains true for other settings: for more complex diffusion tensors containing effective drift correction (see for instance [**8, 33, 2**]), or in presence of turbulent diffusive effects (see [**10, 7**] for oceanic turbulent flows, [**30**] for atmospheric transport problems, [**41**] for bio-turbulent flows). See also [**40**] and the references therein for problems of dispersion in fixed beds. The only necessary assumptions are given in (1.9). In what follows, we assume that the porosity ϕ of the rock is $\phi = 1$ to make the computations clearer.

The equations (1.1)–(1.3) are provided with the initial and boundary conditions:

$$q\cdot\nu = 0,\ \ E(q)\nabla c\cdot\nu = 0 \quad \text{in } \partial\Omega\times(0,T); \tag{1.4}$$

$$c(x,0) = c_o(x) \quad \text{in } \Omega. \tag{1.5}$$

We shall also normalise the pressure by the following condition

$$\int_\Omega p(x,t)\,dx = 0, \quad t\in(0,T). \tag{1.6}$$

We now enumerate the assumptions used in this work. The rock permeability is such that

$$k \in \big(\mathcal{C}^{2,\alpha}(\Omega)\big)^{n\times n}, \quad k^-|\xi|^2 \leqslant k(x)\xi\cdot\xi, \quad \big|k(x)\xi\big| \leqslant k^+|\xi| \quad \text{in } \Omega,\ \forall\xi\in\mathbb{R}^n \tag{1.7}$$

where $0 < k^- \leqslant k^+$. We consider an extension of the viscosity μ to $\mathbb{R}$ such that

$$\mu\in W^{1,\infty}(\mathbb{R}), \qquad 0<\mu^-\leqslant\mu(u)\leqslant\mu^+ \quad \forall u\in\mathbb{R}. \tag{1.8}$$

We assume that the diffusion coefficients are such that $d_m > 0$, $d_T > 0$ and $d_L \geqslant 0$. Note that

$$\begin{cases} E(q)\xi\cdot\xi \geqslant \phi_-\big(d_m + d_T|q|\big)|\xi|^2, \quad \big|E(q)\xi\big| \leqslant \phi_+\big(d_m + d_L|q|\big)|\xi| \quad \forall\xi\in\mathbb{R}^n, \\ \Big|\partial_k\big(E(q)_{ij}\big)x\Big| \leqslant d_L\Big|\sum_{l=1}^n \partial_k q_l\, x\Big| \quad \forall x\in\mathbb{R},\ \forall(i,j,k)\in(1,n)^n. \end{cases} \tag{1.9}$$

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To ensure the existence of semi-classical solutions for the problem, we add an assumption between the viscosity function and the dispersive tensor, that is

$$\left\|\frac{\mu'}{\mu}\right\|_{L^\infty(0,1)} \ll d_T. \tag{1.10}$$

The exact meaning of the symbol $\ll$ is explained in Condition (4.21). Note that this latter assumption is consistent with the physical context of the problem. Indeed (1.10) does not limit the values but the variations of the viscosity compared with the dispersive effects modeled by the coefficient d_T. Thus it corresponds to the physics of flow in porous media where the convection effects are much greater than the diffusive ones (see also Remark 2 below). The injection and production source terms are f^+ and f^-, respectively. The functions f^+ and $-f^-$ are assumed nonnegative. They satisfy

$$(f^+, f^-) \in \left(L^\infty(0,T; W^{1,2s_o+2}(\Omega))\right)^2, \quad s_o > \frac{1}{2}(2n - 3 + \sqrt{4n^2 + 6n + 9}), \tag{1.11}$$

and the compatibility condition $\int_\Omega (f^+ - f^-)dx = 0$. The initial concentration is such that

$$c_0 \in C^{0,1}(\Omega), \quad 0 \leqslant c_0(x) \leqslant 1 \text{ in } \Omega. \tag{1.12}$$

2. Statement of the main results

We begin by defining the concept of weak solution for Pb. (1.1)–(1.5).

Definition 1: A pair (p, c) is a weak solution of Problem (1.1)–(1.5) if $p \in L^\infty(0,T; H^1(\Omega))$ and $c \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ satisfy the following identities

(i) $\displaystyle\int_0^T\int_\Omega \frac{k(x)}{\mu(c)} \nabla p \cdot \nabla\psi \, dxdt = \int_0^T\int_\Omega (f^+ - f^-)\psi \, dxdt$ for any $\psi \in C^\infty(\Omega_T)$;

(ii) $\displaystyle\int_\Omega p(x,t)\, dx = 0$ almost everywhere in $(0,T)$;

(iii) $\displaystyle\int_0^T\int_\Omega \left(-\phi c \partial_t\psi + (q \cdot \nabla c)\,\psi + E(q)\nabla c \cdot \nabla\psi\right) dxdt = \int_0^T\int_\Omega f^+(1-c)\psi \, dxdt$
$\displaystyle + \int_\Omega \phi(x) c_o(x) \psi(x,0)\, dx$ for any $\psi \in C^\infty(\Omega_T)$ with $\psi(x,T) = 0$.

We cite Fabrie and Langlais [**17**], Feng [**19**], Chen and Ewing [**11**] (and Choquet [**13**] for the compressible setting) for a proof of existence of weak solutions.

We then give a definition of semi-classical solution.

Definition 2: A pair (p, c) is a semi-classical solution of Problem (1.1)–(1.5) if it is a weak solution in the sense of Definition 1 with the following additional properties

$$(p, c) \in \left(L^\infty(0,T; C^{0,1}(\overline{\Omega}))\right)^2.$$

Our first goal is to prove the following result for the elliptic-parabolic problem.

THEOREM 1. *Suppose that assumptions* (1.7)–(1.12) *hold. There exists a unique semi-classical solution* (p, c) *of Problem* (1.1)–(1.5) *such that*

(i) *the pressure* p *is in* $L^\infty(0,T;H^{2,q}(\Omega))$ *for* $q > 3$ *and then in* $L^\infty(0,T;C^{2,\alpha}(\overline{\Omega}))$ *for* $\alpha \in (0,1)$;

(ii) *the concentration* c *is in* $H^{\alpha,\alpha/2}(\overline{\Omega}_T) \cap L^\infty(0,T;C^{2,\alpha}(\overline{\Omega}))$ *and* $0 \leqslant c(x,t) \leqslant 1$ *in* Ω_T.

REMARK 1. We do not prove the uniqueness part of Theorem 1 in this paper. This result is already shown in [**19**, Section 3] by Feng.

Let us mention some previous papers dealing with such a regularity analysis. Variants of the system have been analysed by series of authors. An elliptic-hyperbolic model without the couplings due to the dispersion term and the concentration-dependent viscosity was studied by Frid [**20**], Schroll and Tveito [**38**]. Amirat and Ziani considered in [**4**] an elliptic-parabolic model where the velocity is independent of the concentration. In the presence of capillary forces and without dispersion coupling term, Alt and DiBenedetto [**1**] and Kruzkov and Sukorjanskii [**26**] proved existence and uniqueness results for smooth solutions of the immiscible model. Frid and Shelukin [**21**] obtained similar results for a very particular triangular capillarity matrix, with periodic boundary conditions and without dependence on the velocity. We also can cite some studies of compressible models in the one-dimensional case. In the case $d_m > 0$ and $d_p = 0$, Feng [**18**] has proved local existence of strong solution and Choquet [**14**] has considered the question of global existence of weak and strong solutions. Amirat and Moussaoui [**3**] have studied a case with $d_m = d_p = 0$, but for a constant viscosity. But the study of the fully nonlinear and coupled problem is very seldom addressed. However the existence of classical "sufficiently smooth" solutions is for instance the first assumption in numerous numerical studies (see for instance [**23, 12**] and the references therein). Up to our knowledge the most complete analysis is performed by Mikelić in [**32**].

The last part of the paper is devoted to the study of a degenerate model. Actually the molecular diffusion d_m can be neglected in most of the porous media (see [**15**]). This motivates the study of the asymptotic behaviour of Model (1.1)–(1.5) as $d_m \to 0$. Amirat and Ziani obtained a first result in [**5**]. We complete their work with the following result.

THEOREM 2. *Assuming* $k \in \left(W^{1,\infty}(\Omega)\right)^n$, $(f^+, f^-) \in \left(L^\infty(0,T;H^1(\Omega))\right)^2$, $c_o \in H^1(\Omega)$ *and* $d_m = 0$, *there exists a weak solution* (p, c) *of the elliptic-degenerate parabolic problem without diffusion in any arbitrary bounded connected set* Ω *of* $\mathbb{R}^n$. *The solution satisfies*

$$p \in L^\infty(0,T;H^1(\Omega)) \cap L^\theta(0,T;W^{2,\theta}(\Omega)),\ \theta < 3/2,$$
$$c \in L^\infty(\Omega_T), \quad 0 \leqslant c(x,t) \leqslant 1 \text{ almost everywhere in } \Omega_T, \quad |q|^{1/2}\nabla c \in \left(L^2(\Omega_T)\right)^n.$$

If moreover $(f^+, f^-) \in \left(L^\infty(0,T; W^{1,4}(\Omega))\right)^2$ *and* $c_o \in W^{1,4}(\Omega)$, *then the results remains true for any* $\theta < 2$.

The paper is organised as follows. Sections 3 and 4 are devoted to the proof of Theorem 1. Our starting point is a regularised problem introduced by Feng in [**19**] to obtain an existence result of weak solutions for a similar problem. In Section 4 we obtain estimates in Hölder spaces for the solution $(p_\varepsilon, c_\varepsilon)$ of the regularised problem. The difficulty lies with the multiple couplings of the problem. On the first hand the viscosity in the pressure equation (1.1) is concentration dependent. It prevents the use of classical results for elliptic equations (of [**22**] for instance). Our pressure equation is more comparable to the ones studied in [**31, 9**]. But curiously the coupling with the concentration equation allows a weaker assumption on the data of our elliptic problem than the one used in these two latter works (see Remark 2 at the end of Section 4). On the other hand, the velocity in the dispersion tensor of the parabolic equation (1.3) does not allow a direct reference to the results of [**27**]. Furthermore, in view of Section 5, we have to obtain some estimates for the concentrations gradients which do not depend on the diffusion parameter d_m. We then let the regularisation parameter tend to let zero and we pass to the limit to get the result claimed in Theorem 1. Finally in Section 5, we briefly study the asymptotic behaviour of the model when the diffusion coefficient d_m tends to zero to justify the existence result of weak solutions for the degenerate problem announced in Theorem 2.

3. A REGULARISED PROBLEM

Our starting point is a regularised problem originally introduced by Feng [**19**] to state the existence of weak solutions for Pb. (1.1)–(1.5). Let $\varepsilon > 0$ be a given real. The pressure solution p_ε of the regularised problem satisfies

$$\operatorname{div} q_\varepsilon = f^+ - f^-, \quad q_\varepsilon = -\frac{k(x)}{\mu(c_\varepsilon)} \nabla p_\varepsilon \quad \text{in } \Omega_T, \tag{3.1}$$

while a truncated Darcy velocity Q_ε, defined by

$$Q_\varepsilon = \frac{q_\varepsilon}{1 + \varepsilon |q_\varepsilon|}, \tag{3.2}$$

is used to ensure the control of the dispersion effects in the following concentration equation

$$\partial_t c_\varepsilon + Q_\varepsilon \cdot \nabla c_\varepsilon - \operatorname{div}\big(E(Q_\varepsilon)\nabla c_\varepsilon\big) = f^+(1 - c_\varepsilon) \quad \text{in } \Omega_T. \tag{3.3}$$

Equations (3.1), (3.3) are provided with the initial and boundary conditions:

$$q_\varepsilon \cdot \nu = 0, \; E(Q_\varepsilon)\nabla c_\varepsilon \cdot \nu = 0, \qquad x \in \partial\Omega, \; t \in (0,T), \tag{3.4}$$

$$c_\varepsilon(x, 0) = c_o(x), \qquad x \in \Omega. \tag{3.5}$$

The pressure is normalised by the condition $\int_\Omega p_\varepsilon(x,t)\,dx = 0$ for $t \in (0,T)$.

Using a fixed point approach, Feng states in **[19]** an existence result for Problem (3.1)–(3.5). More precisely, the following result is established.

Theorem 3. *There exists a unique classical solution* $(p_\varepsilon, c_\varepsilon)$ *of Problem* (3.1)–(3.5) *with* $p_\varepsilon \in L^\infty(0,T;C^{2,\alpha}(\overline{\Omega}))$, $c_\varepsilon \in H^{\alpha,\alpha/2}(\overline{\Omega}_T) \cup L^\infty(0,T;C^{1,\alpha}(\overline{\Omega}))$, *for a real* $\alpha \in (0,1)$. *Moreover the following estimates are independent of* ε.

$$\|p_\varepsilon\|_{L^\infty(0,T;W^{1,r}(\Omega))} \leqslant C \text{ for some } r > 2\,;$$

$$\|c_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))\cap H^1(0,T;H^2(\Omega)^*)} \leqslant C, \ \left\|\left(d_m + d_T|Q_\varepsilon|^{1/2}\right)\nabla c_\varepsilon\right\|_{L^2(\Omega_T)^n} \leqslant C,$$

$$0 \leqslant c_\varepsilon(x,t) \leqslant 1 \text{ in } \Omega_T.$$

Our aim is now to describe the limit (p,q,c) of the sequence $(p_\varepsilon, q_\varepsilon, c_\varepsilon)$ as $\varepsilon \to 0$. The estimates listed in Theorem 3 lead to the existence of a weak solution for Pb. (1.1)–(1.5). But we claim that they can be considerably improved to get Theorem 1. The next section is dedicated to this work.

4. Hölder estimates

We begin by recalling the following classical regularity result for the elliptic pressure problem.

Lemma 1. *The following uniform estimate holds true.*

$$\|p_\varepsilon\|_{L^\infty(0,T;C^{0,\beta}(\overline{\Omega}))} \leqslant C \quad \text{for some } 0 < \beta < 1.$$

We refer to remarks in [**28**, p. 467] or to [**22**] for details, in particular for the oblique derivative problem of boundary conditions.

We then turn to the parabolic part of the problem. The pressure problem being strongly coupled with the concentration's one, we could compare in some sense the Darcy velocity q_ε with a nonlinear function of the concentration c_ε. Our idea is then to adapt the tools developed by [**27**] for the quasi-linear parabolic equations of divergence form to the concentration problem. The difficulty is of course the absence of an explicit relation between c_ε and q_ε. Hopefully, assumption (1.9) on the dispersion tensor gives a control of some energies weighted by this Darcy velocity q_ε.

For any $r \geqslant 0$, we denote by Ω_r the set $\Omega_r = \Omega \cap K_r$ where K_r is an arbitrary ball of radius r centred in a point $x_o \in \overline{\Omega}$. Let $\rho > 0$ a given real. Our first step is to estimate the integrals $\int_{\Omega_\rho} |\nabla c_\varepsilon|^{2s} dx$ for any $s \in \mathbb{R}_+$.

Lemma 2. *Let assumption* (4.21) *below be satisfied. There exists* $\rho_o > 0$ *depending only of the data of the problem such that for any* $\rho \leqslant \rho_o$ *the following uniform*

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estimate holds true.

$$\sup_{0\leqslant t\leqslant T}\int_{\Omega_\rho}\left|\nabla c_\varepsilon(x,t)\right|^{s+1}dx+\int_0^T\int_{\Omega_\rho}(d_m+d_T|Q_\varepsilon|^{1/\alpha})|\nabla c_\varepsilon|^{(2s+4)/\alpha}\,dxdt\leqslant C_s,$$

for any $0\leqslant s\leqslant s_o$ *and* $\alpha\geqslant 2$. *The constant* C_s *depends only on* s *and on the data of Problem* (1.1)–(1.5).

PROOF: Let $t_1\in(0,T)$. We aim to obtain an estimate for ∇c_ε as independent on d_m as possible. We thus adapt our proof to the variations of the velocity q_ε in the dispersion term. Then, for any $t\in(0,T)$ and $N\in\mathbb{N}^*$ with $N>1$, we consider the set $\Omega^N_{2\rho,t}$ which is the intersection of $\Omega_{2\rho}$ with the set of points

$$\left\{x\in\Omega;\ N-1<\left|q_\varepsilon(x,t)\right|\leqslant N\right\}.$$

In the same way, for any $N'\in\mathbb{N}^*$ such that $1\leqslant N'\leqslant 1/d_m$,

$$\Omega^{1/N'}_{2\rho,t}=\Omega_{2\rho}\cap\left\{x\in\Omega\,;\ \frac{1}{N'+1}<\left|q_\varepsilon(x,t)\right|\leqslant\frac{1}{N'}\right\}$$

and

$$\Omega^{d_m}_{2\rho,t}=\Omega_{2\rho}\cap\left\{x\in\Omega\,;\ 0\leqslant\left|q_\varepsilon(x,t)\right|\leqslant d_m\right\}.$$

We now work in each of these sets, bearing in mind that

$$\Omega_{2\rho}=\bigcup_N\Omega^N_{2\rho,t}\bigcup_{N'}\Omega^{1/N'}_{2\rho,t}\cup\Omega^{d_m}_{2\rho,t}.$$

We consider an arbitrary smooth function ξ of compact support in $\Omega_{2\rho}\times(0,t_1)$, with values between 0 and 1 and vanishing in the vicinity of the bounds of $\Omega_{2\rho}\times(0,t_1)$. We also consider for each $N\in\mathbb{N}^*$ (respectively $1\leqslant N'\leqslant 1/d_m$, d_m) a smooth function ξ_N (respectively $\xi_{1/N'}$, ξ_{d_m}) with values between 0 and 1, equal to zero in the vicinity of the lower base and lateral surface of $\bigcup\limits_{0\leqslant t\leqslant t_1}\Omega^N_{2\rho,t}$.

Let $N\in\mathbb{N}^*$ with $N>1$ and $s\geqslant 0$. In what follows, C denotes a generic constant independent of ε and N. Sometimes we emphasise the dependence with s using the notation C_s. Multiplying (3.3) by $\sum_{k=1}^n\partial_k\left(|\nabla c_\varepsilon|^{2s}\partial_k c_\varepsilon\xi\xi_N\right)$ and integrating over $\Omega_{2\rho}\times(0,t_1)$ we start from the following relation.

$$(4.1)\quad -\int_0^{t_1}\!\!\int_{\Omega_{2\rho}}\sum_{k=1}^n\Big(\partial_t c_\varepsilon+Q_\varepsilon\cdot\nabla c_\varepsilon-\operatorname{div}\big(E(Q_\varepsilon)\nabla c_\varepsilon\big)\Big)\partial_k\left(|\nabla c_\varepsilon|^{2s}\partial_k c_\varepsilon\,\xi\xi_N\right)dxdt$$
$$=-\sum_{k=1}^n\int_0^{t_1}\!\!\int_{\Omega_{2\rho}}f^+(1-c_\varepsilon)\,\partial_k\left(|\nabla c_\varepsilon|^{2s}\partial_k c_\varepsilon\,\xi\xi_N\right)dxdt.$$

We transform the first term in (4.1) by integrations by parts.

(4.2)
$$-\int_0^{t_1}\int_{\Omega_{2\rho}}\sum_{k=1}^n \partial_t c_\varepsilon \partial_k\big(|\nabla c_\varepsilon|^{2s}\partial_k c_\varepsilon\,\xi\xi_N\big)\,dxdt = \int_0^{t_1}\int_{\Omega_{2\rho}}\sum_{k=1}^n \partial_t(\partial_k c_\varepsilon)\,\partial_k c_\varepsilon\,|\nabla c_\varepsilon|^{2s}\xi\xi_N\,dxdt$$
$$= \frac{s}{s+2}\Big(\int_{\Omega_{2\rho}}|\nabla c_\varepsilon(x,t_1)|^{2s+2}(\xi\xi_N)(x,t_1)\,dx - \int_0^{t_1}\int_{\Omega_{2\rho}}|\nabla c_\varepsilon|^{2s+2}\partial_t(\xi\xi_N)\,dxdt\Big).$$

The convection term is

$$-\int_0^{t_1}\int_{\Omega_{2\rho}}\sum_{k=1}^n(Q_\varepsilon\cdot\nabla c_\varepsilon)\,\partial_k\big(|\nabla c_\varepsilon|^{2s}\partial_k c_\varepsilon\,\xi\xi_N\big)\,dxdt = -\sum_{k=1}^n\int_0^{t_1}\int_{\Omega_{2\rho}}(Q_\varepsilon\cdot\nabla c_\varepsilon)$$
$$\Big(|\nabla c_\varepsilon|^{2s}\partial_k c_\varepsilon\,\partial_k(\xi\xi_N) + |\nabla c_\varepsilon|^{2s}\partial^2_{kk}c_\varepsilon\,\xi\xi_N + 2s|\nabla c_\varepsilon|^{2s-2}\sum_{i=1}^n\partial_i c_\varepsilon\,\partial^2_{ik}c_\varepsilon\,\xi\xi_N\Big)\,dxdt.$$

Using the Cauchy-Schwarz and Young inequalities, we get

$$\Big|\int_0^{t_1}\int_{\Omega_{2\rho}}\sum_{k=1}^n(Q_\varepsilon\cdot\nabla c_\varepsilon)\,\partial_k\big(|\nabla c_\varepsilon|^{2s}\partial_k c_\varepsilon\,\xi\xi_N\big)\Big| \leqslant C\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}\,|\nabla(\xi\xi_N)|$$
$$+\delta_o\sum_{k=1}^n\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s}|\partial^2_{kk}c_\varepsilon|^2\xi\xi_N + \frac{C}{\delta_o}\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}\xi\xi_N$$
$$+\delta_1\sum_{i,k=1}^n\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s-2}\big|\partial_i c_\varepsilon\,\partial^2_{ik}c_\varepsilon\big|^2\xi\xi_N + \frac{C_s}{\delta_1}\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s}\xi\xi_N \tag{4.3}$$

for any δ_o, $\delta_1 > 0$. Each source term reads

$$\int_0^{t_1}\int_{\Omega_{2\rho}}f^+(1-c_\varepsilon)\,\partial_k\big(|\nabla c_\varepsilon|^{2s}\,\partial_k c_\varepsilon\,\xi\xi_N\big)\,dxdt = \int_0^{t_1}\int_{\Omega_{2\rho}}f^+|\nabla c_\varepsilon|^{2s}|\partial_k c_\varepsilon|^2\xi\xi_N\,dxdt$$
$$-\int_0^{t_1}\int_{\Omega_{2\rho}}\partial_k f^+(1-c_\varepsilon)\,|\nabla c_\varepsilon|^{2s}\,\partial_k c_\varepsilon\,\xi\xi_N\,dxdt.$$

The first term of the right hand-side is nonnegative. Since $f^+ \in L^{2s+2}(0,T;W^{1,2s+2}(\Omega))$ for any $s \leqslant s_o$, the second one is estimated in the following way.

$$\Big|\sum_{k=1}^n\int_0^{t_1}\int_{\Omega_{2\rho}}\partial_k f^+(1-c_\varepsilon)\,|\nabla c_\varepsilon|^{2s}\,\partial_k c_\varepsilon\,\xi\xi_N\,dxdt\Big|$$
$$\leqslant C\int_0^{t_1}\Big(\int_{\Omega_{2\rho}}|\nabla c_\varepsilon|^{2s+2}\,\xi\xi_N\,dx\Big)^{(2s+1)/(2s+2)}\Big(\int_{\Omega_{2\rho}}|\nabla f^+|^{2s+2}\,\xi\xi_N\,dx\Big)^{1/(2s+2)}dt$$
$$\leqslant C_s\int_0^{t_1}\int_{\Omega_{2\rho}}|\nabla c_\varepsilon|^{2s+2}\,\xi\xi_N\,dxdt + C_s\int_0^{t_1}\int_{\Omega_{2\rho}}|\nabla f^+|^{2s+2}\,\xi\xi_N\,dxdt. \tag{4.4}$$

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The diffusion part of (4.1) leads to

$$\int_0^{t_1}\int_{\Omega_{2\rho}} \sum_{k=1}^{n} \operatorname{div}\big(E(Q_\varepsilon)\nabla c_\varepsilon\big)\partial_k\big(|\nabla c_\varepsilon|^{2s}\partial_k c_\varepsilon\,\xi\xi_N\big)\,dxdt$$

$$= \int_0^{t_1}\int_{\Omega_{2\rho}} \sum_{k,j=1}^{n}\Big(\sum_{i=1}^{n}(\partial_k(E(Q_\varepsilon)_{ji})\partial_i c_\varepsilon + E(Q_\varepsilon)_{ji}\partial^2_{ki}c_\varepsilon)\Big)\Big(|\nabla c_\varepsilon|^{2s}\partial_k c_\varepsilon\partial_j(\xi\xi_N)$$

$$+|\nabla c_\varepsilon|^{2s}\partial^2_{jk}c_\varepsilon\,\xi\xi_N + 2s|\nabla c_\varepsilon|^{2s-2}\partial_k c_\varepsilon\big(\sum_{l=1}^{n}\partial_l c_\varepsilon\partial^2_{jl}c_\varepsilon\big)\,\xi\xi_N\Big)\,dxdt. \tag{4.5}$$

With Assumption (1.9), the terms containing $E(Q_\varepsilon)_{ji}\partial^2_{ki}c_\varepsilon$ are such that:

$$\int_0^{t_1}\int_{\Omega_{2\rho}} \sum_{i,j,k=1}^{n} E(Q_\varepsilon)_{ji}\partial^2_{ki}c_\varepsilon\,\Big(|\nabla c_\varepsilon|^{2s}\partial^2_{jk}c_\varepsilon + 2s|\nabla c_\varepsilon|^{2s-2}\partial_k c_\varepsilon\big(\sum_{l=1}^{n}\partial_l c_\varepsilon\partial^2_{jl}c_\varepsilon\big)\Big)\xi\xi_N\,dxdt$$

$$\geqslant d_T\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s}\xi\xi_N\sum_{i,j=1}^{n}(\partial^2_{ij}c_\varepsilon)^2\,dxdt$$

$$+2s\,d_T\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s-2}\xi\xi_N\Big(\sum_{i,j=1}^{n}\partial_i c_\varepsilon\partial^2_{ij}c_\varepsilon\Big)^2 dxdt \tag{4.6}$$

and

$$\Big|\int_0^{t_1}\int_{\Omega_{2\rho}} |\nabla c_\varepsilon|^{2s}\sum_{i,j,k=1}^{n} E(Q_\varepsilon)_{ji}\partial^2_{ki}c_\varepsilon\,\partial_k c_\varepsilon\,\partial_j(\xi\xi_N)\,dxdt\Big|$$

$$\leqslant 2d_L\int_0^{t_1}\int_{\Omega_{2\rho}} |\nabla c_\varepsilon|^{2s}|Q_\varepsilon|\Big|\sum_{i,j=1}^{n}\partial_i c_\varepsilon\partial^2_{ij}c_\varepsilon\Big|\,\big|(\xi\xi_N)^{1/2}\big|\,\big|\nabla(\xi\xi_N)^{1/2}\big|\,dxdt$$

$$\leqslant \delta_2\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s-2}\xi\xi_N\Big(\sum_{i,j=1}^{n}\partial_i c_\varepsilon\partial^2_{ij}c_\varepsilon\Big)^2 dxdt$$

$$+\frac{C}{\delta_2}\int_0^{t_1}\int_{\Omega^N_{2\rho,t}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}\big|\nabla(\xi\xi_N)^{1/2}\big|^2 dxdt, \tag{4.7}$$

for any $\delta_2 > 0$. The terms in (4.5) containing $\partial_k(E(Q_\varepsilon)_{ji})\partial_i c_\varepsilon$ are such that:

$$\int_0^{t_1}\int_{\Omega_{2\rho}} \sum_{j,k=1}^{n}\Big(\sum_{i=1}^{n}\partial_k\big(E(Q_\varepsilon)\big)_{ji}\partial_i c_\varepsilon\Big)\Big(\xi\xi_N\Big(|\nabla c_\varepsilon|^{2s}\partial^2_{jk}c_\varepsilon \tag{4.8}$$

$$+ 2s|\nabla c_\varepsilon|^{2s-2}\partial_k c_\varepsilon\Big(\sum_{l=1}^{n}\partial_l c_\varepsilon\partial^2_{jl}c_\varepsilon\Big)\Big) + |\nabla c_\varepsilon|^{2s}\partial_k c_\varepsilon\partial_j(\xi\xi_N)\Big)\,dxdt = I_1 + J_1 + K_1.$$

Let us begin with the estimate of I_1. We recall that $|Q_\varepsilon(x,t)| > 0$ for $(x,t) \in \bigcup_{0\leqslant t\leqslant T} \Omega^N_{2\rho,t}$. Thus, using Cauchy-Schwarz and Young inequalities and assumption (1.9), we can write

$$\begin{aligned}
|I_1| &\leqslant d_L \int_0^{t_1}\int_{\Omega_{2\rho}} \sum_{j,k=1}^{n} \Big| \Big(\sum_{i,l=1}^{n} \partial_k Q_{\varepsilon l} \partial_i c_\varepsilon\Big) |\nabla c_\varepsilon|^{2s} \partial^2_{jk} c_\varepsilon\, \xi\xi_N \Big| \,dxdt \\
&\leqslant d_L \int_0^{t_1}\int_{\Omega_{2\rho}} |\nabla c_\varepsilon|^{2s+1} \Big(\sum_{j,k=1}^{n} |\partial^2_{jk} c_\varepsilon|\, (\xi\xi_N)^{1/2}\Big) \Big|\sum_{k,l=1}^{n} \partial_k Q_{\varepsilon l}\, (\xi\xi_N)^{1/2}\Big| \,dxdt \\
&\leqslant \delta \int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\, |\nabla c_\varepsilon|^{2s} \sum_{j,k=1}^{n} |\partial^2_{jk} c_\varepsilon|^2 \xi\xi_N \,dxdt \\
&\quad + \sum_{k,l=1}^{n} \frac{C}{\delta} \int_0^{t_1}\int_{\Omega_{2\rho}} \frac{|\nabla c_\varepsilon|^{2s+2}}{|Q_\varepsilon|} \big|\partial_k Q_{\varepsilon l}\, (\xi\xi_N)^{1/2}\big|^2 \,dxdt,
\end{aligned} \tag{4.9}$$

for any $\delta > 0$. The Hölder inequality gives

$$\begin{aligned}
&\sum_{k,l=1}^{n} \frac{C}{\delta} \int_0^{t_1}\int_{\Omega_{2\rho}} \frac{|\nabla c_\varepsilon|^{2s+2}}{|Q_\varepsilon|} \big|\partial_k Q_{\varepsilon l}\, (\xi\xi_N)^{1/2}\big|^2 \,dxdt \\
&\qquad \leqslant \frac{C}{\delta} \int_0^{t_1} \Big(\int_{\Omega_{2\rho}} |Q_\varepsilon|\, |\nabla c_\varepsilon|^{2s+4} \xi\xi_N \,dx\Big)^{(s+1)/(s+2)} \\
&\qquad\qquad \times \Big(\sum_{k,l=1}^{n} \int_{\Omega_{2\rho}} \frac{|\partial_k Q_{\varepsilon l}\, (\xi\xi_N)^{1/2(s+2)}|^{2s+4}}{|Q_\varepsilon|^{2s+3}} \,dx\Big)^{1/(s+2)} dt.
\end{aligned} \tag{4.10}$$

Let us particularly consider the last term of this relation. In view of (3.2), it reads

$$\begin{aligned}
I_2 &= \sum_{k,l=1}^{n} \int_{\Omega_{2\rho}} \frac{|\partial_k Q_{\varepsilon l}\, (\xi\xi_N)^{1/2(s+2)}|^{2s+4}}{|Q_\varepsilon|^{2s+3}} \,dx \leqslant \sum_{k,l=1}^{n} \int_{\Omega_{2\rho}} \frac{|\partial_k q_{\varepsilon l}\, (\xi\xi_N)^{1/2(s+2)}|^{2s+4}}{(1+\varepsilon|q_\varepsilon|)^{2s+4} |Q_\varepsilon|^{2s+3}} \,dx \\
&\leqslant C_s \sum_{k,l=1}^{n} \Big(\int_{\Omega_{2\rho}} \frac{|\partial_k q_{\varepsilon l}\, (\xi\xi_N)^{1/2(s+2)} + q_{\varepsilon l}\, \partial_k (\xi\xi_N)^{1/2(s+2)}|^{2s+4}}{(1+\varepsilon|q_\varepsilon|)^{2s+4} |Q_\varepsilon|^{2s+3}} \,dx \\
&\qquad + \int_{\Omega_{2\rho}} \frac{|q_{\varepsilon l}\, \partial_k (\xi\xi_N)^{1/2(s+2)}|^{2s+4}}{(1+\varepsilon|q_\varepsilon|)^{2s+4} |Q_\varepsilon|^{2s+3}} \,dx \Big).
\end{aligned} \tag{4.11}$$

We denote by I_3 and I_4 the terms in the left-hand side of (4.11). Using the definition of $\Omega^N_{2\rho,t}$ and (3.2), we estimate I_3 by

$$\begin{aligned}
I_3 &= C_s \int_{\Omega_{2\rho}} \frac{|\partial_k (q_{\varepsilon l}\, (\xi\xi_N)^{1/2(s+2)})|^{2s+4}}{(1+\varepsilon|q_\varepsilon|)^{2s+4} |Q_\varepsilon|^{2s+3}} dx = C_s \int_{\Omega_{2\rho}} \frac{|Q_\varepsilon|}{|q_\varepsilon|^{2s+4}} |\partial_k (q_{\varepsilon l}\, (\xi\xi_N)^{1/2(s+2)})|^{2s+4} \,dx \\
&\leqslant \frac{C_s}{(1+\varepsilon(N-1))(N-1)^{2s+3}} \big\| q_\varepsilon\, (\xi\xi_N)^{1/2(s+2)} \big\|^{2s+4}_{W^{1,2s+4}(\Omega_{2\rho})}.
\end{aligned} \tag{4.12}$$

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The function $q_\varepsilon(\xi\xi_N)^{1/2(s+2)}$ is such that $q_\varepsilon(x,t)\,(\xi\xi_N)^{1/2(s+2)}(x,t) = 0$ for $x \in \partial\Omega_{2\rho}$ and $t \in (0,T)$. Moreover it satisfies

$$\Delta(q_\varepsilon\,(\xi\xi_N)^{1/2(s+2)}) = \nabla(q_\varepsilon \cdot \nabla(\xi\xi_N)^{1/2(s+2)}) + \nabla\big((f^+ - f^-)\,(\xi\xi_N)^{1/(2s+2)}\big) - \operatorname{curl}\Big(\nabla\Big(\frac{k(x)}{\mu(c_\varepsilon)}\,(\xi\xi_N)^{1/2(s+2)}\Big) \wedge \nabla p_\varepsilon\Big) = F$$

in $\Omega_{2\rho} \times (0,t_1)$. Thus we claim with **[24]** that

$$\|q_\varepsilon\,(\xi\xi_N)^{1/2(s+2)}\|^{2s+4}_{W^{1,2s+4}(\Omega_{2\rho})} \leqslant C\|F\|^{2s+4}_{L^{2s+4}_{-1}(\Omega_{2\rho})}. \tag{4.13}$$

Since $\nabla p_\varepsilon = -\mu(c_\varepsilon)k^{-1}(x)q_\varepsilon$, we get

$$\begin{aligned}\|q_\varepsilon(\xi\xi_N)^{1/(2(s+2))}\|^{2s+4}_{W^{1,2s+4}(\Omega_{2\rho})} &\leqslant C\int_{\Omega_{2\rho}} |q_\varepsilon|^{2s+4}\big|\nabla(\xi\xi_N)^{1/(2(s+2))}\big|^{2s+4}\\ &+\int_{\Omega_{2\rho}} |f^+ - f^-|^{2s+4}(\xi\xi_N)^{(s+2)/(s+1)} + \operatorname{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\int_{\Omega_{2\rho}}|q_\varepsilon|^{2s+4}\xi\xi_N\\ &+\Big\|\frac{\mu'_\varepsilon}{\mu_\varepsilon}\Big\|^{2s+4}_{\infty,\Omega^N_{2\rho,t}}\int_{\Omega_{2\rho}}|q_\varepsilon|^{2s+4}|\nabla c_\varepsilon|^{2s+4}\xi\xi_N.\end{aligned}$$

Here and below we denote by $\|\mu'_\varepsilon/\mu_\varepsilon\|_A$ the norm $\|\mu'(c_\varepsilon)/\mu(c_\varepsilon)\|_A$. Note that (1.7) ensures the existence of some constant $C > 0$ such that $\operatorname{osc}(k^{-1}\nabla k,\Omega_{2\rho}) \leqslant C(2\rho)^\beta$. Including the latter inequality in (4.12), we obtain

$$\begin{aligned}I_3 \leqslant \frac{C_s}{(1+\varepsilon(N-1))(N-1)^{2s+3}} \times \Big(&\Big\|\frac{\mu'_\varepsilon}{\mu_\varepsilon}\Big\|^{2s+4}_{\infty,\Omega^N_{2\rho,t}}\int_{\Omega_{2\rho}}(1+\varepsilon|q_\varepsilon|)\,|q_\varepsilon|^{2s+3}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\\ &+\int_k |f^+ - f^-|^{2s+4}(\xi\xi_N)^{(s+2)/(s+1)} + C\int_{\Omega_{2\rho}}|q_\varepsilon|^{2s+4}\big|\nabla(\xi\xi_N)^{1/2(s+2)}\big|^{2s+4}dx\\ &+\operatorname{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\int_{\Omega_{2\rho}}|q_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big).\end{aligned}$$

We note that $\int_k |f^+ - f^-|^{2s+4}(\xi\xi_N)^{(s+2)/(s+1)} \leqslant C$. Moreover, $|q_\varepsilon(x,t)|^{2s+3} \leqslant N^{2s+3}$ and $1+\varepsilon|q_\varepsilon(x,t)| \leqslant (1+\varepsilon N)$ in $\Omega^N_{2\rho,t}$. Then with (4.12) we get

$$\begin{aligned}\frac{1}{\delta^{s+2}}\Big(\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big)^{s+1} I_3 &\leqslant \frac{C_s}{\delta^{s+2}}\Big(\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big)^{s+1}\\ &\times\int_{\Omega_{2\rho}}|q_\varepsilon|\Big(|\nabla(\xi\xi_N)^{1/2(s+2)}|^{2s+4} + \operatorname{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\xi\xi_N\Big)dx\\ &+\frac{C_s}{\delta^{s+2}}\Big\|\frac{\mu'_\varepsilon}{\mu_\varepsilon}\Big\|^{2s+4}_{\infty,\Omega^N_{2\rho,t}}\Big(\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big)^{s+2} + \frac{C_s}{\delta^{s+2}}.\end{aligned} \tag{4.14}$$

On the other hand, the term I_4 of (4.11) is estimated by

$$I_4 = \sum_{k,l=1}^n\int_{\Omega_{2\rho}}\frac{|q_{\varepsilon l}\,\partial_k(\xi\xi_N)^{1/2(s+2)}|^{2s+4}}{(1+\varepsilon|q_\varepsilon|)^{2s+4}|Q_\varepsilon|^{2s+3}}\,dx \leqslant C\int_{\Omega_{2\rho}}|q_\varepsilon|\,\big|\nabla(\xi\xi_N)^{1/2(s+2)}\big|^{2s+4}dx.$$

Thus we have

$$\text{(4.15)}\quad \frac{1}{\delta^{s+2}}\Big(\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big)^{s+1}I_4 \leqslant \frac{C}{\delta^{s+2}}\Big(\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big)^{s+1} \times\Big(\int_{\Omega_{2\rho}}|q_\varepsilon|\,\big|\nabla(\xi\xi_N)^{1/2(s+2)}\big|^{2s+4}\,dx\Big).$$

With (4.11), and (4.14)–(4.15), we obtain

$$\begin{aligned}\frac{1}{\delta^{s+2}}\Big(\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big)^{s+1}I_2 &\leqslant \frac{C_s}{\delta^{s+2}}\Big(\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big)^{s+1}\\ &\times\int_{\Omega_{2\rho}}|q_\varepsilon|\Big(|\nabla(\xi\xi_N)^{1/2(s+2)}|^{2s+4}+\operatorname{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\xi\xi_N\Big)dx\\ &+\frac{C_s}{\delta^{s+2}}\Big\|\frac{\mu'_\varepsilon}{\mu_\varepsilon}\Big\|^{2s+4}_{\infty,\Omega^N_{2\rho,t}}\Big(\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big)^{s+2}+\frac{C_s}{\delta^{s+2}}\end{aligned}$$

and then, using in particular the Young inequality:

$$\begin{aligned}\frac{1}{\delta}\Big(\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big)^{(s+1)/(s+2)}I_2^{1/(s+2)} &\leqslant \frac{C_s}{\delta}\Big(\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\Big)^{(s+1)/(s+2)}\\ &\times\Big(\int_{\Omega_{2\rho}}|q_\varepsilon|\Big(|\nabla(\xi\xi_N)^{1/2(s+2)}|^{2s+4}+\operatorname{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\xi\xi_N\Big)dx\Big)^{1/(s+2)}\\ &+\frac{C_s}{\delta}\Big\|\frac{\mu'_\varepsilon}{\mu_\varepsilon}\Big\|^2_{\infty,\Omega^N_{2\rho,t}}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx+\frac{C_s}{\delta}\\ &\leqslant\Big(\frac{C_s\delta'}{\delta}+\frac{C_s}{\delta}\Big\|\frac{\mu'_\varepsilon}{\mu_\varepsilon}\Big\|^2_{\infty,\Omega^N_{2\rho,t}}\Big)\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dx\\ &+\frac{C_s}{\delta'}\int_{\Omega_{2\rho}}|q_\varepsilon|\Big(\big|\nabla(\xi\xi_N)^{1/2(s+2)}\big|^{2s+4}+\operatorname{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\xi\xi_N\Big)\,dx+\frac{C_s}{\delta'}.\end{aligned}\tag{4.16}$$

Combining (4.9), (4.10) and (4.16) integrated from 0 to t_1, we conclude that

$$\begin{aligned}|I_1| &= \Big|\sum_{j,k=1}^n\int_0^{t_1}\int_{\Omega_{2\rho}}\big(\sum_{i=1}^n\partial_k(E(Q_\varepsilon)_{ji})\partial_i c_\varepsilon\big)\,\big|\nabla c_\varepsilon|^{2s}\,\partial^2_{jk}c_\varepsilon\,\xi\xi_N\,dxdt\Big|\\ &\leqslant \delta_3\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s}\sum_{j,k=1}^n|\partial^2_{jk}c_\varepsilon|^2\,\xi\xi_N\,dxdt\\ &\quad+\frac{C_s}{\delta'_3}\int_0^{t_1}\int_{\Omega_{2\rho}}|q_\varepsilon|\Big(\big|\nabla(\xi\xi_N)^{1/2(s+2)}\big|^{2s+4}+\operatorname{osc}(k^{-1}\nabla k,k\Big)^{2s+4}\xi\xi_N)\,dxdt\\ &\quad+\Big(\frac{C_s\delta'_3}{\delta_3}+\frac{C_s}{\delta_3}\Big\|\frac{\mu'_\varepsilon}{\mu_\varepsilon}\Big\|^2_{\infty,\cup_{0\leqslant t\leqslant t_1}\Omega^N_{2\rho,t}}\Big)\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dxdt+\frac{C_s}{\delta_3},\end{aligned}\tag{4.17}$$

for any $\delta_3,\delta'_3>0$. One easily checks that the second integral J_1 in (4.8) which is exactly of the same order as I_1 also satisfies an estimate like (4.17). We then consider K_1. It

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satisfies for any $\delta > 0$

$$|K_1| = \Big|\sum_{j,k=1}^{n}\int_0^{t_1}\int_{\Omega_{2\rho}}(\sum_{i=1}^{n}\partial_k(E(Q_\varepsilon)_{ji})\partial_i c_\varepsilon)\,\big|\nabla c_\varepsilon\big|^{2s}\partial_k c_\varepsilon\,\partial_j(\xi\xi_N)\Big|$$
$$\leqslant d_L C\sum_{j,k,l=1}^{n}\int_0^{t_1}\int_{\Omega_{2\rho}}|\partial_k Q_{\varepsilon l}\,(\xi\xi_N)^{1/2}|\,|\nabla c_\varepsilon|^{2s+2}|\partial_j(\xi\xi_N)^{1/2}|$$
$$\leqslant \delta\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}\,|\nabla(\xi\xi_N)^{1/2}|^2 + \sum_{k=1}^{n}\frac{C}{\delta}\int_0^{t_1}\int_{\Omega_{2\rho}}\frac{|\partial_k Q_{\varepsilon l}(\xi\xi_N)^{1/2}|^2}{|Q_\varepsilon|}\,|\nabla c_\varepsilon|^{2s+2},$$

where the last integral is already estimated in the study of I_1. We conclude that for any $\delta_4, \delta_4' > 0$

$$|K_1| \leqslant \delta_4\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}\,|\nabla(\xi\xi_N)^{1/2}|^2\,dxdt$$
$$+\frac{C_s}{\delta_4'}\int_0^{t_1}\int_{\Omega_{2\rho}}|q_\varepsilon|\,(|\nabla(\xi\xi_N)^{1/2(s+2)}|^{2s+4} + \mathrm{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\xi\xi_N)\,dxdt$$
$$+\left(\frac{C_s\delta_4'}{\delta_4} + \frac{C_s}{\delta_4}\left\|\frac{\mu_\varepsilon'}{\mu_\varepsilon}\right\|^2_{\infty,\cup_{0\leqslant t\leqslant t_1}\Omega^N_{2\rho,t}}\right)\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dxdt. \tag{4.18}$$

Finally, with (4.1)–(4.8), (4.17) and (4.18), we have proven that

$$\frac{s}{s+2}\int_{\Omega_{2\rho}}|\nabla c_\varepsilon(x,t_1)|^{2s+2}(\xi\xi_N)(x,t_1) + \int_0^{t_1}\int_{\Omega_{2\rho}}f^+|\nabla c_\varepsilon|^{2s+2}\,\xi\xi_N\,dxdt$$
$$+\int_0^{t_1}\int_{\Omega_{2\rho}}(d_m Id + (d_T-\delta_o-\delta_3)|Q_\varepsilon|)\,|\nabla c_\varepsilon|^{2s}\xi\xi_N\sum_{i,j=1}^{n}(\partial^2_{ij}c_\varepsilon)^2$$
$$+2s\,(d_T-\delta_1-\delta_2)\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s-2}\xi\xi_N\Big(\sum_{i,j=1}^{n}\partial_i c_\varepsilon\partial^2_{ij}c_\varepsilon\Big)^2$$
$$\leqslant \int_0^{t_1}\int_{\Omega_{2\rho}}|\nabla c_\varepsilon|^{2s+2}\Big(\frac{s}{s+2}\partial_t(\xi\xi_N) + C_s\,\xi\xi_N + |Q_\varepsilon|\,\Big(\frac{C}{\delta_o}\,\xi\xi_N + \frac{C}{\delta_2}|\nabla(\xi\xi_N)^{1/2}|^2\Big)\Big)$$
$$+C_s\int_0^{t_1}\int_{\Omega_{2\rho}}|\nabla f|^{2s+2}\,\xi\xi_N + \frac{C}{\delta_1}\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s}\,\xi\xi_N + C_s\left(\frac{1}{\delta_3'}+\frac{1}{\delta_4'}\right)$$
$$+C_s\left(\frac{1}{\delta_3'}+\frac{1}{\delta_4'}\right)\int_0^{t_1}\int_{\Omega_{2\rho}}|q_\varepsilon|\,(|\nabla(\xi\xi_N)^{1/2(s+2)}|^{2s+4} + \mathrm{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\xi\xi_N)$$
$$+C_s\left(\frac{\delta_3'}{\delta_3}+\frac{\delta_4'}{\delta_4}+\left(\frac{1}{\delta_3}+\frac{1}{\delta_4}\right)\left\|\frac{\mu_\varepsilon'}{\mu_\varepsilon}\right\|^2_{\infty,\cup_{0\leqslant t\leqslant t_1}\Omega^N_{2\rho,t}}\right)\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N \tag{4.19}$$

for any $\delta_o, \delta_1, \delta_2, \delta_3, \delta_3', \delta_4, \delta_4' > 0$. We emphasise that this estimate is independent of N and ε. The last term of (4.19) is treated using Lemma 3 below. At this step one can only assert that $\omega \leqslant 1$. We denote by A the following quantity

$$A = \frac{C_s\big(\delta_3'/\delta_3 + \delta_4'/\delta_4 + (1/\delta_3 + 1/\delta_4)\,\|\mu_\varepsilon'/\mu_\varepsilon\|^2_{\infty,\cup_{0\leqslant t\leqslant t_1}\Omega^N_{2\rho,t}}\big)}{1-\gamma_o-\gamma_1-2(s+1)\gamma_2-\gamma_3-C_s\gamma_3'/\gamma_3-C_s/\gamma_3\|\mu_\varepsilon'/\mu_\varepsilon\|^2_{\infty,\cup_{0\leqslant t\leqslant t_1}\Omega^N_{2\rho,t}}}.$$

Lemma 3 then leads to

$$C_s\left(\frac{\delta_3'}{\delta_3}+\frac{\delta_4'}{\delta_4}+\left(\frac{1}{\delta_3}+\frac{1}{\delta_4}\right)\left\|\frac{\mu_\varepsilon'}{\mu_\varepsilon}\right\|^2_{\infty,\cup_{0\leqslant t\leqslant t_1}\Omega^N_{2\rho,t}}\right)\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N$$

$$\leqslant A\left(\frac{C}{\gamma_o}\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}\left|\nabla(\xi\xi_N)^{1/2}\right|^2+\frac{1}{\gamma_1}\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s}\sum_{i=1}^n|\partial^2_{ii}c_\varepsilon|^2\,\xi\xi_N\right.$$

$$+\frac{2(s+1)}{\gamma_2}\int_0^{t_1}\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s-2}\left(\sum_{i,j=1}^n\partial_jc_\varepsilon\partial^2_{ji}c_\varepsilon\right)^2\xi\xi_N$$

$$\left.+\frac{C_s}{\gamma_3'}\int_0^{t_1}\int_{\Omega_{2\rho}}|q_\varepsilon|\left(\mathrm{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\xi\xi_N+|\nabla(\xi\xi_N)^{1/2(s+2)}|^{2s+4}\right)\right)+\frac{C_s}{\gamma_3'}\tag{4.20}$$

for any $\gamma_o,\gamma_1,\gamma_2,\gamma_3,\gamma_3'$. In view of combining (4.19) and (4.20) we need to fulfill the following relation

$$A>0,\qquad \begin{cases} d_T-\delta_o-\delta_3-A\ \dfrac{C_s}{\gamma_1}>0,\\ d_T-\delta_1-\delta_2-A\ \dfrac{C_s}{\gamma_2}>0.\end{cases}\tag{4.21}$$

Since we can choose arbitrarly the numbers $\delta_i,\delta_i',\gamma_i,\gamma_i'$, (4.21) is translated by the condition $\|\mu'/\mu\|^2_{L^\infty(0,1)}\ll d_T$ in Assumption (1.10).

Now, combining (4.19) and (4.20) with Assumption (4.21), we obtain an estimate similar to the one get by [**27**, V.3.17] (but without any condition on ρ like (V.3.16) p. 435). The only major difference is that we used untill here the function $\xi\xi_N$ instead of the function ξ^2. We thus follow the lines of [**27**] adapting their tools to the term $|Q_\varepsilon||\nabla c_\varepsilon|^{2s+4}$ as in (4.9)–(4.14). In particular, $|Q_\varepsilon|^{1/2}|\nabla c_\varepsilon|$ and $|q_\varepsilon|$ are uniformly bounded in $L^2(\Omega_T)$ by the result of Theorem 3. Working successively with $s=0,\dots,s_o$, we then claim that for any $N>1$,

$$\int_0^{t_1}\int_{\Omega^N_{2\rho,t}}(d_m+|Q_\varepsilon|)\,|\nabla c_\varepsilon|^{2s+4}\,dxdt+\sup_{(0,t_1)}\int_{\Omega^N_{2\rho,t}}|\nabla c_\varepsilon|^{2s+2}\,dx\leqslant C_s,\tag{4.22}$$

for any $0\leqslant s\leqslant s_o$, s_o being defined in Assumption (1.11) on the source term. Using Theorem 3 and the Hölder inequality, we note that

$$|\Omega^N_{2\rho,t}|\leqslant\frac{C}{N}\left(\int_{\Omega^N_{2\rho,t}}|q_\varepsilon(x,t)|dx\right)\leqslant\frac{C}{N}\left(\int_{\Omega^N_{2\rho,t}}|q_\varepsilon(x,t)|^rdx\leqslant\right)^{1/r}|\Omega^N_{2\rho,t}|^{1-1/r}$$

and thus $|\Omega^N_{2\rho,t}|\leqslant C/N^r$ with $r>2$. The Hölder inequality then yields for all $\alpha>1$ to

$$\int_{\Omega^N_{2\rho,t}}|\nabla c_\varepsilon|^{(2s+2)/\alpha}\,dx\leqslant\left(\int_{\Omega^N_{2\rho,t}}|\nabla c_\varepsilon|^{2s+2}\,dx\right)^{1/\alpha}|\Omega^N_{2\rho,t}|^{(\alpha-1)/\alpha}\leqslant\frac{C}{N^{r(\alpha-1)/\alpha}}.$$

We choose $\alpha \geqslant 2$ and we sum up the latter results for $N > 1$. We conclude that for any $s \leqslant s_o$ and $\alpha \geqslant 2$, $\int_{\cup_{N>1}\Omega^N_{2\rho,t}} |\nabla c_\varepsilon|^{(2s+2)/\alpha}\, dx \leqslant C_s$ and then

$$\int_{\cup_{N>1}\Omega^N_{2\rho,t}} |\nabla c_\varepsilon|^{s+1}\, dx \leqslant C_s, \quad \forall t \in (0,T),\ \forall s \leqslant s_o. \tag{4.23}$$

The same tools give

$$\int_0^T \int_{\cup_{N>1}\Omega^N_{2\rho,t}} \left(d_m + |Q_\varepsilon|^{1/\alpha}\right) |\nabla c_\varepsilon|^{(2s+4)/\alpha}\, dxdt \leqslant C_s, \quad \forall \alpha \geqslant 2,\ \forall s \leqslant s_o. \tag{4.24}$$

We emphasise that we reach such an estimate in $\Omega^N_{2\rho,t}$ thanks to relation (4.20) which gives a result for the convective form $|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}$ and not only for $|\nabla c_\varepsilon|^{2s+4}$. We also note that the estimate in $\Omega^N_{2\rho,t}$ does not depend on d_m. Indeed, in this part of the proof we always use the control given by the dispersive term instead of the one given by the diffusion coefficient d_m. It is the gain in regard of [**27**]. The cost is the dividing factor $1/\alpha$ due to the renormalisation tools. Relation (4.20) allows us to exploit the control given by the dispersive part of the diffusion tensor.

We now consider the sets $\Omega^{1/N'}_{2\rho,t}$ and $\Omega^{d_m}_{2\rho,t}$. We note that all the estimates can also be carried out within $\Omega^{1/N'}_{2\rho,t}$ to get (4.19)–(4.20). Dividing by $(d_m + |Q_\varepsilon|)$ instead of $|Q_\varepsilon|$ in the estimates of I_1 (4.9), I_4 and K_1, we also obtain a relation of the form (4.19) in $\Omega^{d_m}_{2\rho,t}$. The analogous of (4.20) for $\int_0^{t_1}\int_{\Omega_{2\rho}} |\nabla c_\varepsilon|^{2s+4}\xi\xi_N$ has been proved by [**27**, II.5.8], assuming that $\rho \leqslant \rho_o$ where ρ_o is given by the data of the problem. Bearing in mind that $|q_\varepsilon(x,t)| \leqslant 1$ in all these sets, we sum up the relations (4.19)–(4.20) for $\Omega^{d_m}_{2\rho,t}$ and $\Omega^{1/N'}_{2\rho,t}$, $1 \leqslant N' \leqslant 1/d_m$, and we apply the Gronwall lemma to get directly

$$\int_{\Omega^{d_m}_{2\rho,t} \bigcup_{1\leqslant N'\leqslant 1/d_m} \Omega^{1/N'}_{2\rho,t}} |\nabla c_\varepsilon|^{2s+2}\, dx \leqslant C_s, \quad \forall t \in (0,T),\ \forall s \leqslant s_o. \tag{4.25}$$

With (4.23) and (4.25), we state that c_ε is uniformly bounded in $L^\infty\left(0,T; W^{1,s+1}(\Omega_\rho)\right)$, $0 \leqslant s \leqslant s_o$. With (4.24) and (4.25) and $0 \leqslant |q_\varepsilon(x,t)| \leqslant 1$ in $\Omega^{d_m}_{2\rho,t} \bigcup_{1\leqslant N'\leqslant 1/d_M} \Omega^{1/N'}_{2\rho,t}$, we conclude that

$$\int_0^T \int_{\Omega_\rho} (d_m + d_T|Q_\varepsilon|^{1/\alpha})|\nabla c_\varepsilon|^{(2s+4)\alpha}\, dxdt \leqslant C_s, \quad 0 \leqslant s \leqslant s_o,\ \alpha \geqslant 2.$$

This ends the proof of Lemma 2. ∎

We give here the following auxiliary result using the notations of the proof of Lemma 2.

Lemma 3. *Let* $\omega = \operatorname{osc}\Big(c_\varepsilon, \bigcup_{0\leqslant t\leqslant t_1} \Omega^N_{2\rho,t}\Big)$. *For any* $\gamma_o, \gamma_1, \gamma_2, \gamma_3, \gamma_3' > 0$, *the following estimate holds true.*

$$\left(1 - \gamma_o\omega - \gamma_1\omega - 2(s+1)\gamma_2\omega - \gamma_3\omega - \frac{C(s)\gamma_3'}{\gamma_3} - \frac{C(s)}{\gamma_3}\left\|\frac{\mu_\varepsilon'}{\mu_\varepsilon}\right\|^2_{\infty,\cup_{0\leqslant t\leqslant t_1}\Omega^N_{2\rho,t}}\right)$$

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$$\times \int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N \leqslant \frac{2(s+1)\omega}{\gamma_2}\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s-2}\Big(\sum_{i,j=1}^n \partial_j c_\varepsilon \partial^2_{ji} c_\varepsilon\Big)^2 \xi\xi_N$$
$$+\frac{C\omega}{\gamma_o}\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}\,|\nabla(\xi\xi_N)^{1/2}|^2 + \frac{\omega}{\gamma_1}\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s}\sum_{i=1}^n |\partial^2_{ii}c_\varepsilon|^2\,\xi\xi_N$$
$$+\frac{C_s}{\gamma_3'}\int_0^{t_1}\int_{\Omega_{2\rho}} |q_\varepsilon|\,\big(|\nabla(\xi\xi_N)^{1/2(s+2)}|^{2s+4} + \operatorname{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\xi\xi_N\big).$$

PROOF: We note that $|Q_\varepsilon||\nabla c_\varepsilon|^{2s+4}\xi\xi_N = \sum_{i=1}^n |Q_\varepsilon||\nabla c_\varepsilon|^{2s+2}(\partial_i c_\varepsilon)^2\xi\xi_N$. Let $x_o \in \Omega_{2\rho}$. We have by integration by parts

$$\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}(\partial_i c_\varepsilon)^2\xi\xi_N\,dxdt$$
$$= \int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}\partial_i c_\varepsilon\,\partial_i\big(c_\varepsilon(x,t)-c_\varepsilon(x_o,t)\big)\,\xi\xi_N\,dxdt = -(\mathcal{I}_{1i}+\mathcal{I}_{2i}+\mathcal{I}_{3i}+\mathcal{I}_{4i})$$
$$= -\int_0^{t_1}\int_{\Omega_{2\rho}} \big(c_\varepsilon(x,t)-c_\varepsilon(x_o,t)\big)\,|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}\partial_i c_\varepsilon\,\partial_i(\xi\xi_N)\,dxdt$$
$$-\int_0^{t_1}\int_{\Omega_{2\rho}} \big(c_\varepsilon(x,t)-c_\varepsilon(x_o,t)\big)\,|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}\partial^2_{ii} c_\varepsilon\,\xi\xi_N\,dxdt$$
$$-\int_0^{t_1}\int_{\Omega_{2\rho}} \big(c_\varepsilon(x,t)-c_\varepsilon(x_o,t)\big)\,|Q_\varepsilon|\,2(s+1)\,|\nabla c_\varepsilon|^{2s}\Big(\sum_{j=1}^n \partial_j c_\varepsilon\,\partial^2_{ji}c_\varepsilon\Big)\,\partial_i c_\varepsilon\,\xi\xi_N\,dxdt$$
$$-\int_0^{t_1}\int_{\Omega_{2\rho}} \big(c_\varepsilon(x,t)-c_\varepsilon(x_o,t)\big)\,\partial_i(|Q_\varepsilon|)\,|\nabla c_\varepsilon|^{2s+2}\partial_i c_\varepsilon\,\xi\xi_N\,dxdt. \tag{4.26}$$

We then estimate the four terms in (4.26). On the one hand, the Cauchy-Schwarz and Young inequalities give directly for any $\gamma_o,\gamma_1,\gamma_2,\gamma_3 > 0$

$$\Big|\sum_{i=1}^n \mathcal{I}_{1i}\Big| \leqslant \gamma_o\,\omega\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dxdt$$
$$+\frac{C\omega}{\gamma_o}\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+2}|\nabla(\xi\xi_N)^{1/2}|^2\,dxdt, \tag{4.27}$$

$$\Big|\sum_{i=1}^n \mathcal{I}_{2i}\Big| \leqslant \gamma_1\,\omega\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dxdt$$
$$+\frac{C\omega}{\gamma_1}\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s}\sum_{i=1}^n|\partial^2_{ii}c_\varepsilon|^2\,\xi\xi_N\,dxdt, \tag{4.28}$$

$$\Big|\sum_{i=1}^n \mathcal{I}_{3i}\Big| \leqslant 2(s+1)\gamma_2\,\omega\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dxdt$$
$$+\frac{2(s+1)\omega}{\gamma_2}\int_0^{t_1}\int_{\Omega_{2\rho}} |Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s-2}\big(\sum_{i,j=1}^n \partial_j c_\varepsilon\,\partial^2_{ji}c_\varepsilon\big)^2\,\xi\xi_N\,dxdt, \tag{4.29}$$

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$$\left|\sum_{i=1}^{n}\mathcal{I}_{4i}\right| \leqslant \gamma_3\,\omega\int_0^{t_1}\!\!\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N\,dxdt$$

$$(4.30)\qquad +\frac{\omega}{\gamma_3}\sum_{i,j=1}^{n}\int_0^{t_1}\!\!\int_{\Omega_{2\rho}}\frac{|\partial_i q_{\varepsilon j}\,(\xi\xi_N)^{1/2}|^2}{(1+\varepsilon|q_\varepsilon|)^2|Q_\varepsilon|}\,|\nabla c_\varepsilon|^{2s+2}\,dxdt.$$

The last term in (4.30) is already estimated in the proof of Lemma 2 (see (4.10)–(4.16)). It leads to

$$\frac{\omega}{\gamma_3}\sum_{i,j=1}^{n}\int_0^{t_1}\!\!\int_{\Omega_{2\rho}}\frac{|\partial_i q_{\varepsilon j}\,(\xi\xi_N)^{1/2}|^2}{(1+\varepsilon|q_\varepsilon|)^2|Q_\varepsilon|}\,|\nabla c_\varepsilon|^{2s+2}$$

$$\leqslant\left(\frac{C_s\gamma_3'}{\gamma_3}+\frac{C_s}{\gamma_3}\left\|\frac{\mu_\varepsilon'}{\mu_\varepsilon}\right\|^2_{\infty,\cup_{0\leqslant t\leqslant t_1}\Omega^N_{2\rho,t}}\right)\int_0^{t_1}\!\!\int_{\Omega_{2\rho}}|Q_\varepsilon|\,|\nabla c_\varepsilon|^{2s+4}\xi\xi_N$$

$$(4.31)\qquad +\frac{C_s}{\gamma_3'}\int_0^{t_1}\!\!\int_{\Omega_{2\rho}}|q_\varepsilon|\left(|\nabla(\xi\xi_N)^{1/2(s+2)}|^{2s+4}+\operatorname{osc}(k^{-1}\nabla k,\Omega_{2\rho})^{2s+4}\xi\xi_N\right)+\frac{C_s}{\gamma_3'},$$

for any $\gamma_3'>0$ The result of Lemma 3 follows from (4.26)–(4.31). ∎

Using Lemma 2, we can now state and prove the following result for the pressure function.

Lemma 4. *For any $t\in(0,T)$, the pressure p_ε satisfies*

$$\int_\Omega\left(|\nabla p_\varepsilon(x,t)|^{4+2s}+|\nabla p_\varepsilon(x,t)|^{2s}\sum_{i,j=1}^{n}|\partial^2_{ij}p_\varepsilon(x,t)|^2\right)dx\leqslant C_s,\quad \forall 0\leqslant s\leqslant\frac{s_o-1}{2}.$$

Proof: We multiply Eqation (3.1) by a test function of the form $\partial_r\eta(x)$, $1\leqslant r\leqslant n$, where η is an arbitrary sufficiently smooth function that is of compact support in Ω. With a double integration by parts, we get

$$\sum_{i=1}^{n}\int_\Omega\partial_r\Big(\frac{k}{\mu(c_\varepsilon)}\,\partial_i p_\varepsilon\Big)\,\partial_i\eta\,dx=\int_\Omega\partial_r\eta\,(f^+-f^-)\,dx.$$

Let $N>0$ be given. We set

$$b=\min(|\nabla p_\varepsilon|^2,N).$$

In the latter inequality, we choose $\eta=b^s\partial_r p_\varepsilon\xi^2$, where $s\geqslant 0$, ξ is a smooth function with compact support taking values between 0 and 1 in the sphere $\Omega_{2\rho}$. For sake of clarity we set $\kappa(c_\varepsilon)=k(x)/\mu(c_\varepsilon)$. For $1\leqslant r\leqslant n$, we obtain

$$\sum_{i=1}^{n}\int_{\Omega_{2\rho}}\left(\kappa'(c_\varepsilon)\,\partial_r c_\varepsilon\,\partial_i p_\varepsilon+\kappa(c_\varepsilon)\,\partial^2_{ri}p_\varepsilon\right)\left(b^s\partial^2_{ri}p_\varepsilon\,\xi^2\right.$$

$$\left.+\,s\,\partial_r p_\varepsilon\,b^{s-1}\,\partial_i b\,\xi^2+2\,b^s\,\partial_r p_\varepsilon\,\xi\,\partial_i\xi\right)dx=\int_{\Omega_{2\rho}}\partial_r(f^+-f^-)\,b^s\,\partial_r p_\varepsilon\,\xi^2\,dx.$$

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Summing up the latter result for r from 1 to n, we easily get

$$\sum_{i,r=1}^{n}\int_{\Omega_{2\rho}}\left(\kappa(c_\varepsilon)\,|\partial^2_{ri}p_\varepsilon|^2\,b^s\,\xi^2+\frac{s}{2}\,\kappa(c_\varepsilon)\,|\partial_i b|^2\,b^{s-1}\,\xi^2\right)dx$$
$$\leqslant C\int_{\Omega_{2\rho}}\Big(b^s\,|\nabla c_\varepsilon|\,|\nabla p_\varepsilon|\,\Big|\sum_{i,r=1}^{n}\partial^2_{ri}p_\varepsilon\Big|\,\xi^2+s\,b^{s-1}\,|\nabla c_\varepsilon|\,|\nabla p_\varepsilon|^2\,|\nabla b|\,\xi^2$$
$$+2\,b^s\,|\nabla c_\varepsilon|\,|\nabla p_\varepsilon|^2\,\xi\,|\nabla\xi|+2\,b^s\,|\nabla p_\varepsilon|\,\Big|\sum_{i,r=1}^{n}\partial^2_{ri}p_\varepsilon\Big|\,\xi\,|\nabla\xi|$$
$$+|\nabla(f^+-f^-)|b^s|\nabla p_\varepsilon|\,\xi^2\Big)\,dx. \tag{4.32}$$

Using the Cauchy–Schwarz and Young inequalities, we obtain the following estimates, for any $\delta>0$

$$\int_{\Omega_{2\rho}}b^s\,|\nabla p_\varepsilon|\,\Big|\sum_{i,r=1}^{n}\partial^2_{ri}p_\varepsilon\Big|\,\xi\,|\nabla\xi|\,dx\leqslant\delta\sum_{i,r=1}^{n}\int_{\Omega_{2\rho}}b^s\,|\partial^2_{ri}p_\varepsilon|^2\,\xi^2\,dx$$
$$+\frac{C}{\delta}\int_{\Omega_{2\rho}}b^s\,|\nabla p_\varepsilon|^2\,|\nabla\xi|^2\,dx, \tag{4.33}$$

$$\int_{\Omega_{2\rho}}b^s\,|\nabla c_\varepsilon|\,|\nabla p_\varepsilon|\,\Big|\sum_{i,r=1}^{n}\partial^2_{ri}p_\varepsilon\Big|\,\xi^2\,dx\leqslant\delta\sum_{i,r=1}^{n}\int_{\Omega_{2\rho}}b^s\,|\partial^2_{ri}p_\varepsilon|^2\,\xi^2\,dx$$
$$+\frac{C}{\delta}\int_{\Omega_{2\rho}}b^s\,|\nabla p_\varepsilon|^2\,|\nabla c_\varepsilon|^2\,\xi^2\,dx, \tag{4.34}$$

and

$$\int_{\Omega_{2\rho}}b^s\,|\nabla c_\varepsilon|\,|\nabla p_\varepsilon|^2\,\xi\,|\nabla\xi|\,dx\leqslant\int_{\Omega_{2\rho}}b^s\,|\nabla p_\varepsilon|^2\,|\nabla\xi|^2\,dx+\int_{\Omega_{2\rho}}b^s\,|\nabla p_\varepsilon|^2\,|\nabla c_\varepsilon|^2\,\xi^2\,dx. \tag{4.35}$$

The term

$$\int_{\Omega_{2\rho}}s\,b^{s-1}\,|\nabla c_\varepsilon|\,|\nabla p_\varepsilon|^2\,|\nabla b|\,\xi^2\,dx\leqslant\int_{\Omega_{2\rho}}s\,b^s\,|\nabla c_\varepsilon|\,|\nabla p_\varepsilon|\,\Big|\sum_{i,r=1}^{n}\partial^2_{ri}p_\varepsilon\Big|\,\xi^2\,dx \tag{4.36}$$

is treated as in (4.34). We now estimate the integral $\int_{\Omega_{2\rho}}b^s|\nabla c_\varepsilon|^2|\nabla p_\varepsilon|^2\xi^2dx$ appeared in (4.34)–(4.35). Using the Hölder inequality, we get for any $\gamma>0$

$$\int_{\Omega_{2\rho}}b^s\,|\nabla c_\varepsilon|^2\,|\nabla p_\varepsilon|^2\,\xi^2\leqslant\left(\int_{\Omega_{2\rho}}b^{s+\gamma}\,|\nabla p_\varepsilon|^{(2(s+\gamma))/s}\,\xi^2\right)^{s/(s+\gamma)}\left(\int_{\Omega_{2\rho}}|\nabla c_\varepsilon|^{(2(s+\gamma))/\gamma}\,\xi^2\right)^{\gamma/(s+\gamma)}.$$

We choose $\gamma=s/(1+s)>0$ such that $b^{s+\gamma}|\nabla p_\varepsilon|^{2(s+\gamma)/s}=b^s|\nabla p_\varepsilon|^4$. With the result of Lemma 2 we ensure that the last term of the latter inequality is uniformly bounded by a constant C if $s\leqslant(s_o-1)/2$. We then write with a Young inequality

$$\int_{\Omega_{2\rho}}b^s\,|\nabla c_\varepsilon|^2\,|\nabla p_\varepsilon|^2\,\xi^2\leqslant C_s\left(\int_{\Omega_{2\rho}}b^s\,|\nabla p_\varepsilon|^4\,\xi^2\right)^{(s+1)/(s+2)}\leqslant C_s+\int_{\Omega_{2\rho}}b^s\,|\nabla p_\varepsilon|^4\,\xi^2. \tag{4.37}$$

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The integral form containing the source terms f^+ and f^- is similarly estimated. We recall that by (1.7)–(1.8), $\kappa(c_\varepsilon) = k(x)/\mu(c_\varepsilon) \geqslant k^-/\mu^+ > 0$ in Ω_T. Thus, with (4.32)–(4.37), we can write

$$(4.38)\quad \Big(\frac{k^-}{\mu^+} - \delta\Big) \int_{\Omega_{2\rho}} b^s \sum_{i,r=1}^{n} |\partial^2_{ri} p_\varepsilon|^2 \, \xi^2 \, dx + \int_{\Omega_{2\rho}} s\, b^{s-1} \, |\nabla b|^2 \, \xi^2 \, dx$$
$$\leqslant \frac{C}{\delta} \int_{\Omega_{2\rho}} b^s \, |\nabla p_\varepsilon|^2 \, |\nabla \xi|^2 \, dx + \frac{C}{\delta} \int_{\Omega_{2\rho}} b^s \, |\nabla p_\varepsilon|^4 \, \xi^2 \, dx + \frac{C_s}{\delta}$$

for any $\delta > 0$. Choosing for instance $\delta = k^-/(2\mu^+)$, Relation (4.38) is completely similar with [**28**, 3.6] with $m = 2$. We know by Theorem 3 that $|\nabla p_\varepsilon|$ is uniformly bounded in $L^\infty\big(0,T;L^2(\Omega)\big)$. Moreover Lemma 1 ensures that $\operatorname{osc}(p_\varepsilon, \Omega_{2\rho}) \leqslant C\rho^\beta$. Following the lines of [**28**], we thus obtain the result announced in Lemma 4. ∎

We turn back to the concentration problem. We now have enough uniform estimates on the velocity q_ε to state the following result.

Lemma 5. *The concentration c_ε is uniformly bounded in $H^{\alpha,\alpha/2}(\Omega_T)$ for $\alpha \in (0,1)$.*

Proof: We already know that $\max_{\Omega_T} |c_\varepsilon(x,t)| = 1$. Let $k \in [-1,1]$ be an arbitrary number. We consider a smooth function ξ with compact support in Ω_ρ, such that ξ is equal to zero in the vincinity of the bounds of the cylinder $\Omega_\rho \times [t_1, T_1]$, with $0 \leqslant t_1 \leqslant T_1 \leqslant T$. We multiply Equation (3.3) by $\xi^2(x,t) c_\varepsilon^{(k)}(x,t) = \xi^2(x,t) \max(c_\varepsilon(x,t) - k, 0)$ and we integrate over $\Omega_{2\rho}$. We obtain

$$\frac{1}{2}\frac{d}{dt} \int_{\Omega_\rho} {c_\varepsilon^{(k)}}^2 \xi^2 \, dx + \int_{\Omega_\rho} E(Q_\varepsilon) \nabla c_\varepsilon^{(k)} \cdot \nabla c_\varepsilon^{(k)} \, \xi^2 \, dx + 2 \int_{\Omega_\rho} E(Q_\varepsilon) \nabla c_\varepsilon^{(k)} \cdot \nabla \xi \, c_\varepsilon^{(k)} \xi \, dx$$
$$(4.39)\qquad - \int_{\Omega_\rho} {c_\varepsilon^{(k)}}^2 \xi \, \partial_t \xi \, dx + \int_{\Omega_\rho} (Q_\varepsilon \cdot \nabla c_\varepsilon^{(k)}) \, c_\varepsilon^{(k)} \, \xi^2 \, dx - \int_{\Omega_\rho} f^+ (1 - c_\varepsilon) \, c_\varepsilon^{(k)} \, \xi^2 \, dx = 0.$$

We now perform classical estimates on the terms of (4.39). We have clearly

$$\Big| \int_{\Omega_\rho} {c_\varepsilon^{(k)}}^2 \xi \, \partial_t \xi \, dx \Big| \leqslant \int_{\Omega_\rho} {c_\varepsilon^{(k)}}^2 \xi \, |\partial_t \xi| \, dx.$$

Since $0 \leqslant c_\varepsilon(x,t) \leqslant 1$ and then $|c_\varepsilon^{(k)}(x,t)| \leqslant 2$ in Ω_T, we write

$$\Big| \int_{\Omega_\rho} f^+ (1 - c_\varepsilon) \, c_\varepsilon^{(k)} \, \xi^2 \, dx \Big| \leqslant 2 \int_{\Omega_\rho} |f^+| \, \xi^2 \, dx.$$

Moreover, the convective term is such that

$$\Big| \int_{\Omega_\rho} (Q_\varepsilon \cdot \nabla c_\varepsilon^{(k)}) \, c_\varepsilon^{(k)} \, \xi^2 \, dx \Big| \leqslant \delta_o \int_{\Omega_\rho} |Q_\varepsilon| \, |\nabla c_\varepsilon^{(k)}|^2 \, \xi^2 \, dx + \frac{C}{\delta_o} \int_{\Omega_\rho} |Q_\varepsilon| \, \xi^2 \, dx$$

and the diffusive ones satisfy

$$\Big| \int_{\Omega_\rho} E(Q_\varepsilon) \nabla c_\varepsilon^{(k)} \cdot \nabla \xi \, c_\varepsilon^{(k)} \, \xi \, dx \Big| \leqslant \delta_1 \int_{\Omega_\rho} (d_m + d_L |Q_\varepsilon|^2) \, |\nabla c_\varepsilon^{(k)}|^2 \, \xi^2 dx + \frac{C}{\delta_1} \int_{\Omega_\rho} {c_\varepsilon^{(k)}}^2 |\nabla \xi|^2 dx$$

for any $\delta_o, \delta_1 > 0$, while

$$\int_{\Omega_\rho} E(Q_\varepsilon)\nabla c_\varepsilon^{(k)} \cdot \nabla c_\varepsilon^{(k)}\, \xi^2\, dx \geqslant \int_{\Omega_\rho} (d_m + d_T|Q_\varepsilon|)\, |\nabla c_\varepsilon^{(k)}|^2\, \xi^2\, dx.$$

Equation (4.39) then gives

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\rho} c_\varepsilon^{(k)2}\xi^2 + \int_{\Omega_\rho} \big(d_m(1-\delta_1) + (d_T - \delta_o)|Q_\varepsilon|\big)\, |\nabla c_\varepsilon^{(k)}|^2\, \xi^2$$

$$(4.39) \quad \leqslant \int_{\Omega_\rho} \Big(\xi\,|\partial_t\xi| + \frac{C}{\delta_1}\,|\nabla\xi|^2\Big)\, c_\varepsilon^{(k)2} + C\int_{A_{k,\rho}(t)} \Big(|f^+| + \frac{2}{\delta_o}|Q_\varepsilon| + \delta_1|Q_\varepsilon|^2\, |\nabla c_\varepsilon^{(k)}|^2\Big)\, \xi^2.$$

The domain $A_{k,\rho}(t)$ is the set of points $x \in \Omega_\rho$ at which $c_\varepsilon(x,t) > k$. Let us estimate the last term of the latter relation. To this aim, we denote by $\mathcal{F}$ the function defined in Ω_T by $\mathcal{F}(x,t) = |f^+(x,t)| + |Q_\varepsilon(x,t)| + |Q_\varepsilon(x,t)|^2|\nabla c_\varepsilon^{(k)}(x,t)|^2$. In view of the regularity of $f^+ \in L^\infty\big(0,T;W^{1,2s_o+2}(\Omega)\big)$ and of Lemmas 2 and 4, we ensure that $|Q_\varepsilon|^2|\nabla c_\varepsilon|^2$ and thus $|Q_\varepsilon|^2|\nabla c_\varepsilon^{(k)}|^2$ are uniformly bounded in $L^r\big(0,T;L^q(\Omega)\big)$ with $r = (s_o+2)/2$ and $q = (s_o+2)(2s_o - 2 + 4\alpha)/\alpha(4s_o + 1 + 4\alpha)$ for any $\alpha \geqslant 2$, that is $r \leqslant (s_o+2)/2$ and $q \leqslant (s_o+2)(s_o+3)/(4s_o+9)$. The function $\mathcal{F}$ is thus also bounded in $L^r\big(0,T;L^q(\Omega)\big)$. Since we assume in (1.11) that $s_o > (2n - 3 + \sqrt{4n^2+6n+9})/2$, some computations show that r and q satisfy

$$\frac{1}{r} + \frac{n}{2q} = 1 - \chi_1,\ 0 < \chi_1 < 1, \quad r \in \Big(\frac{1}{1-\chi_1}, \infty\Big),\ q \in \Big(\frac{n}{2(1-\chi_1)}, \infty\Big).$$

Using a Hölder estimate, we thus can write

$$\int_{t_1}^{T_1}\!\!\int_{A_{k,\rho}(t)} \Big(|f^+| + \frac{2}{\delta_o}|Q_\varepsilon| + \delta_1|Q_\varepsilon|^2\, |\nabla c_\varepsilon^{(k)}|^2\Big)\, \xi^2\, dx \leqslant C\int_{t_1}^{T_1}\!\!\int_{A_{k,\rho}(t)} |\mathcal{F}|\, \xi^2\, dxdt$$

$$\leqslant C\|\mathcal{F}\|_{L^r(0,T;L^q(\Omega))} \Big(\int_{t_1}^{T_1} \big(\mathrm{mes} A_{k,\rho}(t)\big)^{\bar r/\bar q} dt\Big)^{1/\bar r},$$

with $\bar r = r/(r-1)$ and $\bar q = q/(q-1)$. Then c_ε belongs to the space $\mathcal{B}_2\big(\Omega_T, 1, C, (1+2\chi_1/n)2r/(r-1), \infty, 2\chi_1/n\big)$ in the sense of [**27**]. This implies that c_ε is uniformly bounded in $H^{\alpha,\alpha/2}(\Omega_T)$ for $\alpha \in (0,1)$. Lemma 5 is proven. ∎

Classical regularity results then imply the following uniform estimates.

Lemma 6.

(i) *The pressure p_ε is uniformly bounded in the space $L^\infty\big(0,T;H^{2,q}(\Omega)\big)$, for $q > 3$.*

(ii) *The concentration c_ε is uniformly bounded in $L^\infty\big(0,T;C^{2,\alpha}(\overline{\Omega})\big)$.*

Proof: Since c_ε is uniformly bounded in $H^{\alpha,\alpha/2}(\Omega_T)$ for $\alpha \in (0,1)$, the coefficients $k/\mu(c_\varepsilon)$ in the elliptic pressure equation (3.1) and their partial derivatives are also uniformly bounded in $L^\infty\big(0,T;L^q(\Omega)\big)^{(n\times n)}$, for $q > 3$. Thus, by [**28**, Theorem 15.1], p_ε

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belongs to $\mathcal{C}^{1,\alpha'}(\overline{\Omega})\cap H^{2,q}(\Omega)$, with $\alpha' = 1-n/q$. Furthermore the norms $\|p_\varepsilon\|_{\mathcal{C}^{1,\alpha'}(\overline{\Omega})\cap H^{2,q}(\Omega)}$ are bounded from above by a quantity depending only on n, q and $\|p_\varepsilon\|_{L^2(\Omega)} \leqslant C$. The functions q_ε and Q_ε are then uniformly bounded in $\mathcal{C}^{0,\alpha}(\overline{\Omega})$. This bound for the coefficients of Equation (3.3) let us claim in [**27**, Section III.11] that the concentration c_ε is uniformly bounded in $L^\infty\big(0,T;\mathcal{C}^{2,\alpha}(\overline{\Omega})\big)$. Note that, turning back to the pressure equation (3.1), we can ensure with [**28**, Theorem 3.3.2] that p_ε is uniformly bounded in $L^\infty\big(0,T;\mathcal{C}^{2,\alpha}(\overline{\Omega})\big)$. Lemma 6 is proven. ∎

We now pass to the limit $\varepsilon \to 0$. By Theorem 3 there exist subsequences of (p_ε) and (c_ε), not relabeled for convenience, such that

$$p_\varepsilon \rightharpoonup p \text{ weak } * \text{ in } L^\infty\big(0,T;W^{1,r}(\Omega)\big), \quad q_\varepsilon \rightharpoonup q \text{ weak } * \text{ in } \Big(L^\infty\big(0,T;L^r(\Omega)\big)\Big)^n,$$

for some limit functions $p \in L^\infty\big(0,T;W^{1,r}(\Omega)\big)$ and $q \in \Big(L^\infty\big(0,T;L^r(\Omega)\big)\Big)^n$. Using a classical compactness argument of Aubin's type [**39**], we can also ensure the existence of a function $c \in L^2\big(0,T;H^1(\Omega)\big)$, with $0 \leqslant c(x,t) \leqslant 1$ almost everywhere in Ω_T, and of a convenient subsequence of (c_ε) such that

$$c_\varepsilon \rightharpoonup c \text{ weakly in } L^2\big(0,T;H^1(\Omega)\big), \text{ strongly in } L^2(\Omega_T) \text{ and almost everywhere in } \Omega_T.$$

These results are sufficient to pass to the limit in the pressure problem (3.1)–$(3.4)_1$ to get

$$\operatorname{div}(q) = f^+ - f^-, \quad q = -\frac{k}{\mu(c)}\nabla p \text{ in } \Omega_T, \quad q\cdot\nu = 0 \text{ on } \partial\Omega.$$

We then multiply (1.1) by p, (3.1) by p_ε and we integrate over Ω. We conclude that

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{k(x)}{\mu(c_\varepsilon)}\nabla p_\varepsilon\cdot\nabla p_\varepsilon\,dx = \int_\Omega \frac{k(x)}{\mu(c)}\nabla p\cdot\nabla p\,dx = \lim_{\varepsilon\to 0}\int_\Omega \frac{k(x)}{\mu(c_\varepsilon)}\nabla p_\varepsilon\cdot\nabla p\,dx,$$

and then

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{k(x)}{\mu(c_\varepsilon)}|\nabla p_\varepsilon - \nabla p|^2\,dx = 0.$$

Since $k(x)/\mu(c_\varepsilon) \geqslant k^-/\mu^+ > 0$ in Ω_T, we claim that it follows

$$\nabla p_\varepsilon \longrightarrow \nabla p \text{ strongly in } \Big(L^\infty\big(0,T;L^2(\Omega)\big)\Big)^n$$

and then $q_\varepsilon \to q$ and $Q_\varepsilon \to q$ strongly in $\Big(L^\infty\big(0,T;L^2(\Omega)\big)\Big)^n$. Passing to the limit in problem (3.3)–$(3.4)_2$ we get

$$\begin{aligned}&\partial_t c + q\cdot\nabla c - \operatorname{div}\big(E(q)\nabla c\big) = f^+(1-c) \text{ in } \Omega_T,\\ &E(q)\nabla c\cdot\nu = 0 \text{ in } \partial\Omega\times(0,T), \quad c(x,0) = c_o(x) \text{ in } \Omega.\end{aligned}$$

Finally the uniform estimates listed in Lemma 6 give the additional regularity properties announced in Theorem 1.

Let us finish this section with some remarks.

REMARK 2. (i) Contrary to [21], we have here very weak assumptions on the diffusion tensor $E(q)$. In particular, assuming $d_T > 0$, we respect the physics of the problem. Indeed the dispersion effects are very superior to the molecular diffusion effects in most of the flows (see [6] or [35]). This justifies the study of the degenerate problem in Section 5 below.

(ii) Assumption (1.10) $\|\mu'/\mu\|_\infty \ll d_T$ also respects in some sense the physics of the problem. Indeed numerous studies have shown that it limits the digitation phenomenon, one of the major causes of instability in groundwater flows (see for instance [29] and the references therein). This assumption is quite similar to the one used by Mikelić in [32]. Note that Sammon [36] obtain C^∞ regularity assuming a constant viscosity, that is $\mu' = 0$. On the other hand, (1.10) is considerably weaker that the hypothesis used by the authors who study the regularity of a completely decoupled pressure equation. Indeed in [31] as in [9], the idea is to consider coefficients a_{ij} corresponding here to $k_{ij}/\mu(c)$ not very "different" from the identity function, in the sense where there exists some constant $\varepsilon > 0$ such that $\|1 - a_{ij}\|_\infty \leqslant \varepsilon$. In our work, assumption (1.10) is only a limitation of the variations of the elliptic coefficients. It is the complex coupling with the concentration equation which improves our estimates.

5. Existence of a weak solution for a degenerate problem

This section is devoted to the statement of Theorem 2 which gives an existence result of weak solutions for the elliptic-degenerate parabolic problem. To this aim, we set $d_m = \eta$, where $\eta > 0$ is a given real. In view of letting the diffusion coefficient d_m tend to zero, we start from the following problem in a bounded connected set Ω of $\mathbb{R}^n$.

$$\operatorname{div}(q_\eta) = f^+ - f^-, \quad q_\eta = -\frac{k(x)}{\mu(c_\eta)}\nabla p_\eta \qquad \text{in } \Omega_T, \tag{5.1}$$

$$\partial_t c_\eta + q_\eta \cdot \nabla c_\eta - \operatorname{div}\big(E(q_\eta)\nabla c_\eta\big) = f^+(1 - c_\eta) \qquad \text{in } \Omega_T, \tag{5.2}$$

$$q_\eta \cdot \nu = 0, \;\; E(q_\eta)\nabla c_\eta \cdot \nu = 0 \qquad \text{in } \partial\Omega \times (0,T), \tag{5.3}$$

$$c_\eta(x,0) = c_{o,\eta}(x) \qquad \text{in } \Omega. \tag{5.4}$$

The pressure is normalised by $\displaystyle\int_\Omega p_\eta(x,t)\,dx = 0$ in $(0,T)$. We assume now that $k \in \big(W^{1,\infty}(\Omega)\big)^n$, $(f^+, f^-) \in \Big(L^\infty\big(0,T;H^1(\Omega)\big)\Big)^2$, $c_{o,\eta} \in C^{0,1}(\Omega)$, $0 \leqslant c_{o,\eta}(x) \leqslant 1$ in Ω, with $c_{o,\eta} \to c_o$ strongly in $H^1(\Omega)$. All the other hypotheses listed in Section 1 remind valid here. Theorem 1 ensures for any $\eta > 0$ the existence of an unique semi-classical solution (p_η, c_η) for Pb. (5.1)–(5.4).

We aim now let η tend to zero. The first step is the statement of uniform estimates for the solutions of (5.1)–(5.4) independently of $d_m = \eta$. We begin by a straightforward estimate for the pressure p_η.

LEMMA 7. *The sequence (p_η) is uniformly bounded in $L^\infty\big(0,T;H^1(\Omega)\big)$.*

PROOF: Multiplying Equation (5.1) by p_η, integrating over Ω and using the Poincaré inequality, one easily get Lemma 7. ∎

Energy estimates for the concentration give the following result.

LEMMA 8. *The sequence (c_η) is uniformly bounded in the space $L^\infty(\Omega_T)$. Moreover the gradients satisfy the following uniform estimate.*

$$\| |q_\eta|^{1/2} \nabla c_\eta \|_{(L^2(\Omega_T))^n} \leqslant C.$$

PROOF: We begin by recalling that the concentration c_η is physically admissible in the sense that $0 \leqslant c_\eta(x,t) \leqslant 1$ almost everywhere in Ω_T. The sequence (c_η) is then uniformly bounded in $L^\infty(\Omega_T)$. We now multiply Equation (5.2) by c_η and we integrate over Ω. We obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|c_\eta(\cdot,t)\right|^2 dx + \int_\Omega E(q_\eta)\nabla c_\eta \cdot \nabla c_\eta \, dx + \int_\Omega (q_\eta \cdot \nabla c_\eta)\, c_\eta \, dx = \int_\Omega f^+(1-c_\eta)c_\eta \, dx. \tag{5.5}$$

Since c_η is uniformly bounded in $L^\infty(\Omega_T)$, we can write using the Cauchy-Schwarz and Young inequalities and Lemma 7

$$\left|\int_\Omega (q_\eta \cdot \nabla c_\eta)\, c_\eta \, dx\right| \leqslant C\int_\Omega |q_\eta|\, dx + \frac{d_T}{2}\int_\Omega |q_\eta|\, |\nabla c_\eta|^2 \, dx \leqslant C + \frac{d_T}{2}\int_\Omega |q_\eta|\, |\nabla c_\eta|^2 \, dx.$$

The source term in (5.5) brings no difficulty. Using assumption (1.9), relation (5.5) thus leads to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|c_\eta(\cdot,t)\right|^2 dx + \int_\Omega \left(\eta + \frac{d_T}{2}|q_\eta|\right) |\nabla c_\eta|^2 \, dx \leqslant C.$$

We then prove Lemma 8 using the Gronwall lemma. ∎

At this step, we have enough estimates to conclude that p_η (respectively q_η) is actually bounded in $L^2(0,T;W^{2,4/3}(\Omega))$ (respectively in $\left(L^2(0,T;W^{1,4/3}(\Omega))\right)^n$). Following the lines of [5], we can assert that, for extracted subsequences,

$$p_\eta \to p \text{ strongly in } L^2(0,T;H^1(\Omega)), \quad q_\eta \to q \quad \text{strongly in } \left(L^4(\Omega_T)\right)^n,$$
$$c_\eta \rightharpoonup c \text{ in } L^\infty(\Omega_T) \text{ weak} - *, \quad c_\eta q_\eta \to cq \quad \text{strongly in } \left(L^2(\Omega_T)\right)^n,$$

where (p,c) is a weak solution of (1.1)–(1.5) with the following degenerate dispersion tensor

$$E(q) = |q|\left(d_L\mathcal{E}(q) + d_T\big(Id - \mathcal{E}(q)\big)\right).$$

But one actually can improve the estimate for the pressure p_η. We claim and prove the following result.

LEMMA 9. *The pressure p_η is uniformly bounded in $L^\theta(0,T;W^{2,\theta}(\Omega))$ for any $\theta < 3/2$.*

PROOF: We introduce the same type of domain decomposition as in the proof of Lemma 2. For any $t \in (0,T)$, we define the sets

$$\Omega_t^N = \Big\{x \in \Omega;\ N \leqslant |q_\eta(x,t)| < N+1\Big\},$$

for any $N \in \mathbb{N}$, and

$$\Omega_t^{N,1/2} = \Big\{x \in \Omega;\ N - 1/2 \leqslant |q_\eta(x,t)| < N+3/2\Big\}$$

for any $N \in \mathbb{N}^*$. Let also $\xi_N \in C_c^\infty\Big(\bigcup_{t\in(0,T)} \Omega_t^{N,1/2}\Big)$ such that $\xi_N = 1$ in $\bigcup_{t\in(0,T)} \Omega_t^N$. Let $N \geqslant 1$. We note that for any $N \in \mathbb{N}$, p_η satisfies

$$-\Delta(p_\eta\,\xi_N) = \frac{\mu'(c_\eta)}{k}\,(q_\eta \cdot \nabla c_\eta)\,\xi_N + \frac{\mu(c_\eta)}{k}\,(f^+ - f^-)\,\xi_N - \frac{\mu(c_\eta)}{k^2}\,(q_\eta \cdot \nabla k)\,\xi_N - p_\eta\,\Delta\xi_N.$$

Since $k \in W^{1,\infty}(\Omega)$, the analogous of relation (4.13) gives for any $1 \leqslant i,j \leqslant n$

$$\begin{aligned}(5.6)\quad \int_{\Omega_t^N} |\partial^2_{ij} p_\eta|^2 &\leqslant \int_{\Omega_t^{N,1/2}} |\partial^2_{ij} p_\eta|^2 \xi_N^2 \leqslant C \int_{\Omega_t^{N,1/2}} |q_\eta|^2 |\nabla c_\eta|^2 \xi_N^2 + C \int_{\Omega_t^{N,1/2}} |q_\eta|^2 \xi_N^2 \\ &+ C \int_{\Omega_t^{N,1/2}} |f^+ - f^-|^2 \xi_N^2 \leqslant C\Big(\int_{\Omega_t^{N,1/2}} |q_\eta|^3\Big)^{1/2} \Big(\int_{\Omega_t^{N,1/2}} |q_\eta|\,|\nabla c_\eta|^4 \xi_N^2\Big)^{1/2} + C_N,\end{aligned}$$

where $C_N = \int_{\Omega_t^{N,1/2}} (|q_\eta|^2 + |f^+ - f^-|^2)\xi_N^2\,dx$. Using Estimate (4.22) for $s_o = 0$, we write

$$\int_{\Omega_t^N} |\partial^2_{ij} p_\eta|^2\,dx \leqslant C(t)\Big(\int_{\Omega_t^{N,1/2}} |q_\eta|^3\,dx\Big)^{1/2} + C_N,$$

where $C(t)$ is uniformly bounded in $L^2(0,T)$. Let $1 < \theta < 2$ and $s > 1$. Using the Hölder inequality and noting that $\Omega_t^N \subset \Omega_t^{N,1/2}$, we obtain

$$\begin{aligned}&\int_{\Omega_t^N} |\partial^2_{ij} p_\eta|^\theta\,dx \\ &\quad\leqslant \Big(\int_{\Omega_t^N} |\partial^2_{ij} p_\eta|^2 dx\Big)^{\theta/2} |\Omega_t^N|^{1-(\theta/2)} \leqslant \Big(\int_{\Omega_t^N} |\partial^2_{ij} p_\eta|^2\,dx\Big)^{\theta/2} \frac{1}{N^{s(1-\theta/2)}} \Big(\int_{\Omega_t^N} |q_\eta|^s\Big)^{1-(\theta/2)} \\ &\quad\leqslant \frac{C(t)^{\theta/2}}{N^{s(1-\theta/2)}(N+3/2)^{(s-3)\theta/4}} \Big(\int_{\Omega_t^{N,1/2}} |q_\eta|^s\Big)^{1-\theta/2+\theta/4} + \frac{C_N^{\theta/2}}{N^{s(1-\theta/2)}} \Big(\int_{\Omega_t^{N,1/2}} |q_\eta|^s\Big)^{1-\theta/2} \\ &\quad\leqslant \Big(\frac{C(t)^{\theta/2}}{N^{s(1-\theta/2)}(N+3/2)^{(s-3)\theta/4}} + \frac{C_N^{\theta/2}}{N^{s(1-\theta/2)}(N-1/2)^{s\theta/4}}\Big) \Big(\int_{\Omega_t^{N,1/2}} |q_\eta|^s\Big)^{1-\theta/2+\theta/4} \\ &\quad\leqslant \frac{C(t)^{\theta/2}}{N^{s(1-\theta/2)}(N-1/2)^{(s-3)\theta/4}} \Big(\int_{\Omega_t^{N,1/2}} |q_\eta|^s\Big)^{4-\theta/4},\end{aligned}$$

the latter inequality being written because $f^+ - f^- \in L^\infty\big(0,T;L^2(\Omega)\big)$ and $|q_\eta|$ is uniformly bounded in $L^\infty\big(0,T;L^2(\Omega)\big)$. Let $r > 1$ and r' such that $1/r + 1/r' = 1$. We now use the discrete Hölder inequality and get

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$$\int_{\cup_{N\geqslant 1}\Omega_t^N} |\partial_{ij}^2 p_\eta|^\theta \, dx \leqslant \left(C(t)^{r\theta/2} \sum_{N\geqslant 1} \frac{1}{N^{rs(2-\theta)/2}\left(N-\frac{1}{2}\right)^{r(s-3)\theta/4}} \right)^{1/r} \times \left(\sum_{N\geqslant 1} \left(\int_{\Omega_t^{N,1/2}} |q_\eta|^s \right)^{(4-\theta)r'/4} \right)^{1/r'}.$$

We choose $r' = 4/(4-\theta)$ that is $r = 4/\theta$. If moreover $r(4s - \theta s - 3\theta)/4 > 1$ that is $s > 4\theta/(4-\theta)$, we can write

$$\int_{\cup_{N\geqslant 1}\Omega_t^N} |\partial_{ij}^2 p_\eta|^\theta \, dx \leqslant C(t)^{\theta/2} \left(\int_{\cup_{N\geqslant 1}\Omega_t^{N,1/2}} |q_\eta|^s \right)^{(4-\theta)/4} \leqslant 2^{(4-\theta)/4} C(t)^{\theta/2} \left(\int_{\cup_{N\geqslant 1}\Omega_t^N} |q_\eta|^s \right)^{(4-\theta)/4}.$$

Since $|q_\eta|^{1/2}\nabla c_\eta$ is uniformly bounded in $\left(L^2(\Omega_T)\right)^n$, the analogous of (5.6) in Ω_t^0 is

$$\int_{\Omega_t^0} |\partial_{ij}^2 p_\eta|^2 dx \leqslant C_o(t)$$

where $C_o(t)$ is bounded in $L^1(0,T)$. The two latter relations give

$$\|p_\eta\|_{W^{2,\theta}(\Omega)}^\theta \leqslant C_o(t)^{\theta/2} + C(t)^{\theta/2} \left(\int_\Omega |q_\eta|^s \right)^{(4-\theta)/4}. \tag{5.7}$$

We recall the following Gagliardo-Nirenberg inequality for $s = 12\theta/(\theta+6)$.

$$\|q_\eta\|_{L^s(\Omega)} \leqslant C\|p_\eta\|_{W^{1,s}(\Omega)} \leqslant C\|p_\eta\|_{W^{2,\theta}(\Omega)}^{1/2} \|p_\eta\|_{L^6(\Omega)}^{1/2},$$

where the last term is uniformly bounded since p_η is uniformly bounded in $L^\infty\left(0,T;H^1(\Omega)\right)$. This gives in (5.7)

$$\|p_\eta\|_{W^{2,\theta}(\Omega)}^\theta \leqslant C_o(t)^{\theta/2} + C(t)^{\theta/2} \|p_\eta\|_{W^{2,\theta}(\Omega)}^{3\theta(4-\theta)/2(\theta+6)}$$

Note that the condition $s = 12\theta/(\theta+6) > 4\theta/(4-\theta)$ is satisfied provided that $\theta < 3/2$. We now integrate the latter relation from 0 to T and we use once again the Hölder inequality for some $r > 1$ to get

$$\|p_\eta\|_{L^\theta(0,T;W^{2,\theta}(\Omega))}^\theta \leqslant \int_0^T C_o(t)^{\theta/2}\, dt + \left(\int_0^T C(t)^{r\theta/2}\, dt \right)^{1/r} \left(\int_0^T \|p_\eta\|_{W^{2,\theta}(\Omega)}^{3r'\theta(4-\theta)/2(\theta+6)} \right)^{1/r'}.$$

The first term of the right hand-side is bounded by a constant since $C_o(t) \in L^1(0,T)$ and $\theta < 2$. For the second one we choose $r' = 2(\theta+6)/3(4-\theta)$ so that we can write

$$\|p_\eta\|_{L^\theta(0,T;W^{2,\theta}(\Omega))}^\theta \leqslant C + \left(\int_0^T C(t)^{r\theta/2}\, dt \right)^{1/r} \|p_\eta\|_{L^\theta(0,T;W^{2,\theta}(\Omega))}^{3\theta(4-\theta)/2(\theta+6)}.$$

Moreover, since $C(t)$ is uniformly bounded in $L^2(0,T)$ and $r\theta/2 = (\theta+6)/5 < 2$, we write

$$\|p_\eta\|_{L^\theta(0,T;W^{2,\theta}(\Omega))}^\theta \leqslant C + C\|p_\eta\|_{L^\theta(0,T;W^{2,\theta}(\Omega))}^{3\theta(4-\theta)/2(\theta+6)},$$

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where $3(4-\theta)/2(\theta+6) < 1$. This proves Lemma 9. ∎

The estimate of Lemma 9 fully justifies Theorem 2. Note that [5] obtained $p \in L^\theta\big(0,T;W^{2,\theta}(\Omega)\big)$ for $\theta < 4/3$. Moreover, following the lines of the proof of Lemma 9 using (4.22) with $s_o = 1$ instead of $s_o = 0$, we reach an uniform bound for p_η in $L^\theta\big(0,T;W^{2,\theta}(\Omega)\big)$ for any $\theta < 2$.

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