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DIFFUSION ON LIE GROUPS

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ABSTRACT. The heat kernel of an amenable Lie group satisfies either $p_t \sim \exp(-ct^{1/3})$ or $p_t \sim t^{-a}$ as $t \longrightarrow \infty$. We give a condition on the Lie algebra which characterizes the two cases.

RÉSUMÉ. Pour le noyau de la chaleur sur un groupe de Lie moyennable on a soit $p_t \sim \exp(-ct^{1/3})$, soit $p_t \sim t^{-a}$ (lorsque $t \longrightarrow \infty$). On donne une condition sur l'algèbre de Lie qui caracterise les deux cas.

Let G be a connected real Lie group which is not assumed to be unimodular. Let $X_1, \ldots, X_k$ be left invariant fields on G (*i.e.* $X(f_g) = (Xf)_g, f_g(x) = f(gx)$), and let $\Delta = -\sum X_j^2$ and $T_t = e^{-t\Delta}$ be the left laplacian and the left diffusion semigroup they generate. I shall assume throughout that Δ is subelliptic (*i.e.* the X_i's are generators of the Lie algebra, *cf.* [1], [2], [3]). I shall denote by $dg = d^l g$ (resp. $d^r g$) the left (resp. right) Haar measure on G and by $p_t(x,y) = \phi_t(y^{-1}x)$ $(x, y \in G)$ the corresponding "left" diffusion kernel:

$$T_t f(t) = \int_G p_t(x,y) f(y)\, dy, \quad f \in C_0^\infty(G).$$

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{n} \subset \mathfrak{q} \subset \mathfrak{g}$ denote the radical and the nil radical of $\mathfrak{g}$ (*cf.* [4]). Then $\mathfrak{q}/\mathfrak{n} = V$ and $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}] = W$ are abelian Lie algebras that can be identified to real vector spaces; furthermore the derivation ad on $\mathfrak{g}$ induces ad: $V \longrightarrow \mathfrak{gl}(W) = \mathrm{End}_{\mathbb{R}}(W)$. An $\mathbb{R}$-linear complex valued mapping $\lambda: V \longrightarrow \mathbb{C}$ is called a *root* if $\big(\mathrm{ad}\, v - \lambda(v)\big)w = 0$ for all $v \in V$, and some $0 \neq w \in W \otimes \mathbb{C}$ (*cf.* [4]). We shall then consider $L_1, \ldots, L_n \in V^*$ $(n \geq 0)$ the finitely many non- zero real parts of these roots and we shall say that G satisfies condition (C) if the above set $(L_1, \ldots, L_n) \subset V^*$ is non-empty and if there exist $\alpha_j \geq 0$, $1 \leq j \leq n$, such that:

$$\text{(C)} \qquad \sum_{j=1}^n \alpha_j L_j = 0; \quad \sum_{j=1}^n \alpha_j > 0.$$

In this paper I shall prove the following:

THEOREM. *Let G, ϕ_t be as above and let us assume that G satisfies condition (C). Then there exists $c > 0$ s.t.*

$$\text{(E)} \qquad \phi_t(g) = O\big(\exp(-ct^{1/3})\big); \quad t \geq 1,\ g \in G.$$

The following remarks will help to clarify this theorem:

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(i) We may as well assume, and we shall do so throughout this paper, that G is amenable, for otherwise we have the even stronger decay: $\phi_t(e) = O\big(\exp(-ct)\big)$ for some $c > 0$, (*cf.* [5]).

(ii) When G is amenable we always have $\phi_t(e) \geq \exp(-ct^{1/3})$ for some $c > 0$ where e denotes throughout the neutral element of G. (*cf.* [9], [10]).

(iii) When G is amenable and does *not* satisfy condition (C) there exists $c > 0$ s.t. $\phi_t(e) \geq t^{-c}$, $(t \geq c)$. This assertion is anything but obvious but will not be proved in this paper. We hope to give sharp results in this opposite direction in a second instalment of this work.

Observe finally that it is easy to see that unimodular Lie groups of exponential volume growth do satisfy the above (C)-conditions. The conclusion of our theorem for these groups is already known to hold (*cf.* [2], [11]). Observe also that for G *non*-unimodular the following estimate (*cf.* [6]) is known to hold:

$$\phi_t(e) = O(t^{-3/2}) \quad (t \longrightarrow \infty).$$

1. **General facts on the heat kernel.** In this section, we shall put together without proofs some known general (some are highly non-trivial) facts on the group G and the heat kernel ϕ_t of $e^{-t\Delta}$ as defined in the previous section. A general reference for all these facts is [1], [2].

I shall consider $H \subset G$ a closed normal subgroup and I shall assume that both H and G/H are unimodular. I shall further denote by $m(g)$ the modular function on G defined by $d^r g = m(g)\, dg$ and normalised by $m(e) = 1$. The Haar measure on G can then be disintegrated by:

$$\int f(g)\, dg = \int_{G/H}\Big(\int_H f(gh)\, dh\Big)\, d\dot{g}; \quad f \in \mathrm{C}_0^{(\infty)}(G)$$

where we shall use throughout the notation $\dot{g} = \pi(g)$ for the canonical projection $\pi = G \longrightarrow G/H$. We have of course an analogous formula for the right measure $d^r g$ and we also have:

$$T_t f(x) = f * \mu_t(x) = \int_G f(xy^{-1})\, d\mu_t(y) = \int_G \phi_t(y^{-1}x) f(y)\, dy = \int_G f(xy^{-1})\phi_t(y)\, d^r y;$$
$$d\mu_t(y) = \phi_t(y)\, d^r y = \phi_t(y) m(y)\, dy.$$

I shall use the notation $\check{\mu}_t = \check{\pi}(\mu_t)$ $(t > 0)$ for the measure induced by the projection on G/H and it is clear that $e^{-t\check{\Delta}} f = f * \check{\mu}_t$ on G/H where $\check{\Delta} = -\Sigma\big(d\pi(X_j)\big)^2$. We have therefore $d\check{\mu}_t(\dot{g}) = \check{\phi}_t(\dot{g})\, d\dot{g}$ for the corresponding heat diffusion kernel $\check{\phi}_t$ on G/H and

$$d\check{\mu}_t(\dot{g}) = \Big(\int_H \phi_t(hg)\, dh\Big)\, d\dot{g} = m(\dot{g})\Big(\int_H \phi_t(gh)\, dh\Big)\, d\dot{g}$$

(observe that $H \subset \operatorname{Ker} m$ by our hypothesis). It follows therefore that:

$$\check{\phi}_t(\dot{g}) = \int_H \phi_t(hg)\, dh = m(\dot{g}) \int_H \phi_t(gh)\, dh. \tag{1.1}$$

I shall now denote by $d(x,y)$ the left invariant canonical distance on G defined by the fields $X_1,\dots,X_k$ (*cf.* [1], [2]) and also by $|g|=|g^{-1}|=d(e,g)$. For all fixed $0<a<b$ there exist $C_1,C_2>0$ s.t. for $t\in[a,b]$ we have the Gaussian estimate:

$$C_1^{-1}\exp\left(-\frac{C_2|g|^2}{t}\right)\le\phi_t(g)\le C_1\exp\left(-\frac{|g|^2}{C_2t}\right);\quad g\in G.$$

To extract this estimate from [1], [2], one has to observe further that the modular function satisfies

$$(1.2)\qquad C^{-1}\exp(-C|g|)\le m(g)\le C\exp(C|g|);\quad g\in G.$$

In fact, in the Gaussian estimate, we can choose $C_2=4+\varepsilon$ ($\varepsilon>0$) with ε arbitrarily small provided that $C_1=C_1(\varepsilon)$ is appropriately chosen. For the upper estimate this refinement is contained in [1], [2]. For the lower estimate we have to use the results of [7] (indeed in [7] it was shown that $\phi_t(g)\ge C_\varepsilon\exp\left(\frac{-|g|^2}{(4-\varepsilon)t}\right)$ for $|g|\le 1, 0<t<1$. We can deduce the same estimate for arbitrary $g\in G$ by the argument of [7] §2.4 applied to $p_t(x,y)=\phi_t(y^{-1}x)$ and dg). This refinement of the Gaussian estimate, however, will not be essential for us, but if we are prepared to use it will clarify some of the proofs in §4 further on.

I shall now go back to the normal subgroup $H\subseteq G$ and I shall specialize the above situation and assume that $H\cong\mathbb{R}^n$ is a vector subgroup. We can then restrict ϕ_t to each coset $gH\subset G$ so that the restricted function can be identified (up to translation $\cdot\longmapsto\cdot+\alpha$, $\alpha\in\mathbb{R}$) with $\theta(\cdot)\in L^1(\mathbb{R}^n)$. We shall then normalize $\Phi(x)=\left(\int\theta(x)\,dx\right)^{-1}\theta(x)$ and take the Fourier transform $f(\xi)=\hat{\Phi}(\xi)$ ($\xi\in\hat{H}$). This Fourier transform $f(\xi)=f_{t,\dot g}(\xi)$ depends, of course, also on t and the coset $\dot g\in G/H$ and is only defined up to a multiplicative complex exponential $e^{i\alpha\xi}$. We have clearly $|f(\xi)|\le 1$ but we also have:

$$(1.3)\qquad |f_g(\xi)|\le Ce^{c|g|^2}|\xi|^{-n};\quad \xi\in\hat{H},\ g\in G/H.$$

In this estimate we assume $0<A\le t\le B$ and $n\ge 1$ is arbitrary. The $C,c>0$ depend on A,B and n but are independent of g and ξ. (Observe that here and in what follows, when confusion does not arise I shall drop the dots above the $\dot g\in G/H$). Furthermore, (but this is not essential for us) $c=\varepsilon>0$ can be chosen arbitrarily small provided that $C=C(n,A,B,\varepsilon)$ also depends on $\varepsilon>0$. The proof of (1.3) is easy:

Indeed the standard local Harnack estimates (*cf.* [1], [2]) applied to $u(t,g)=\phi_t(g)$ (which is a solution of the equation $\frac{\partial}{\partial t}-\sum X_j^2=0$ on G) give at once that:

$$(1.4)\qquad |X_{i_1}\cdots X_{i_\alpha}u(t,g)|\le Cu(t',g);\quad g\in G$$

for $t'>t>0$ fixed and C depending on t,t' and $i_1,\dots,i_\alpha$. For any multi-index β, by expressing the euclidean derivatives $D^\beta\theta$ on $H\cong\mathbb{R}^n$ as linear combinations of left invariant differential operators as in the left hand side of (1.4) (this is possible by the Hörmander condition on the fields), we deduce:

$$|D^\beta\theta_{t,g}(x)|\le C\theta_{t',g}(x);\quad x\in\mathbb{R}^n,\ g\in G/H$$

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where C is independent of g and x.

Integrating the above estimate over x we obtain:

$$\int_H |D^\beta \Phi_{t,g}(x)|\, dx \leq C\frac{\check{\phi}_{t'}(g)}{\check{\phi}_t(g)}e^{C|g|} \tag{1.5}$$

where we use our previous notations relative to $G \longrightarrow G/H$ and (1.1) (1.2). It follows that we can bound the right hand side of (1.5) by $Ce^{c|g|^2}$ because of our Gaussian estimate (applied to G/H). It is also clear that the $c = \varepsilon > 0$ can be made arbitrarily small by choosing t' very close to t and by using the sharp ($C_2 = 4 + \varepsilon$) Gaussian estimate. Our estimate (1.3) now follows by taking Fourier transforms.

It should be remarked that for the application of the estimate (1.3) that we have in mind the group G/H will be of the special form $G/H \cong V \times K$ where $V \cong \mathbb{R}^a$ is a vector group and K is compact. For these groups the estimate of the right hand side of (1.5) is elementary and can be done directly (and with $c = \varepsilon > 0$ arbitrarily small) without referring to the Gaussian estimate on G/H. Indeed if $K = \{e\}$ then $\check{\phi}$ is the classical Gaussian function on $V = \mathbb{R}^a$. If $K \neq \{e\}$ is not trivial we can use a local Harnack on G/H to compare $\check{\phi}_t(g)$ with $\int_K \check{\phi}_{t'}(gk)\, dk = \tilde{\phi}_{t'}(g)$, and this clearly can again be identified with the standard Gaussian on V.

Be that as it may, what we shall need to extract from the estimate (1.3) is the following:

LEMMA. *Let $\hat{H} = \hat{H}_1 \oplus \cdots \oplus \hat{H}_k$ be a direct decomposition of $\hat{H}$ and let $\hat{H} \ni \xi = (\xi_1, \ldots, \xi_k)$, $\xi_j \in \hat{H}_j$, $1 \leq j \leq k$, be the corresponding coordinates. Then we have the estimate*

$$|f_g(\xi)| \leq f_g^{(1)}(\xi_1) \cdots f_g^{(k)}(\xi_k); \quad \xi \in \hat{H}$$

where the functions $f_g^{(i)}(\cdot)$ satisfy:

$$0 \leq f_g^{(i)} \leq 1; \quad \int_{\hat{H}_i} f_g^{(i)}(\xi)\, d\xi \leq Ce^{c|g|^2}.$$

The proof is trivial. Indeed, because of (1.3), by a change of variables it suffices to show that for $n \geq 1$ *large enough* we have $\min\left[1, |\xi|^{-n}\right] \leq f^{(1)}(\xi_1) \cdots f^{(k)}(\xi_k)$ for appropriate $f^{(j)}$ that satisfy:

$$0 \leq f^{(j)}(\xi) \leq 1; \quad \int_{\hat{H}_i} f^{(j)}(\xi)\, d\xi < +\infty$$

and for that it is enough to set, for $\xi \in \hat{H}_i$, $f^{(j)}(\xi) = \min\left[1, |\xi|^{-n_j}\right]$, for appropriate n_j.

2. **The roots of the action.** In this section I shall collect together without proofs some well known facts from linear algebra and elementary representation theory.

Let $H \cong \mathbb{R}^n$ be a real vector space, let $G = V \times K$ where $V \cong \mathbb{R}^n$ is a vector group and K a compact group, and let $\pi\colon G \longrightarrow \mathrm{GL}(H)$ be a representation.

Let $\lambda_1,\dots,\lambda_k\colon V \longrightarrow \mathbb{C}$ be the ($\mathbb{R}$-linear complex valued) roots of $d\pi|_V$. These are defined by the fact that there exists $0 \neq h \in H \otimes \mathbb{C}$ such that:

$$\big(d\pi(v) - \lambda(v)\big) = 0 \quad \forall v \in V.$$

I can then define $L_1,\dots,L_p \in V^*$ as the distinct real parts of all the roots ($L = \operatorname{Re}\lambda$ is a real functional on V). We shall use in the next section the following fact:

There exists a π-stable direct decomposition of $H = H_1 \oplus \cdots \oplus H_p$ such that if we denote by d_j =dimension H_j and $\pi_j = \pi|_{H_j}$ then for each $1 \leq j \leq p$

$$C^{-1} \leq e^{-d_jL_j(v)}\Big|\det\big(\pi_j(g)\big)\Big| \leq C; \quad g = (v,k) \in G = V \times K$$

where $C > 0$ is independent of g.

3. **The disintegration of the kernel.** I shall consider $H \subset G$ as in §2 with $H \cong \mathbb{R}^n$ and shall assume that $G/H \cong V \times S$ with $V \cong \mathbb{R}^m$ a vector subgroup and S compact. I shall disintegrate μ_t for $t = 1$

$$\mu_1 = \mu = \int_{G/H} \nu_{\dot g}\, d\check\mu(\dot g)$$

where $\nu_{\dot g}$ are probability measures on the fibers $gH = \dot g \in G/H$ (all the other notations are as before). From this it clearly follows that:

$$\mu_n = \mu^{*n} = \int_{G/H}\cdots\int_{G/H} \nu_{g_1} * \cdots * \nu_{g_n}\, d\check\mu(g_1)\cdots d\check\mu(g_n) \tag{3.1}$$

where the $*$ indicates convolution in G. I shall now identify, as I may, ν_g with a measure on H (by $H \leftrightarrow gH$ up to translation on H), and, for any $\nu \in \mathbb{P}(H)$ and $g \in G/H$, I shall denote by $\nu^g \in \mathbb{P}(H)$ the image of ν by the action $\pi\colon G/H \longrightarrow \operatorname{Aut}(H)$ on H induced by inner automorphisms. It is clear then that the integrand of (3.1) which, up to translation, can be identified to a measure on the coset $\dot g_1 \cdots \dot g_n \subset G$, can also be identified with:

$$\nu(g_1,\dots,g_n) = \nu_{g_1}^{s_1} * \nu_{g_2}^{s_2} * \cdots * \nu_{g_n}^{s_n}$$

where $s_j = g_1 \cdots g_j \in G/H$ and where the convolution is now taken in H. (This identification is now done for the "right product" identification $H \leftrightarrow Hs_n$). Now the measures $\nu(g_1,\dots,g_n)$ can be identified as $L^\infty(H)$ functions of H and we clearly have, since convolution goes by Fourier transform to pointwise product:

$$\|\nu(g_1,\dots,g_n)\|_\infty \leq \int_{\hat H}\Big|f_{g_1}\big(\pi(s_1)^*\xi\big)\cdots f_{g_n}\big(\pi(s_n)^*\xi\big)\Big|\, d\xi. \tag{3.2}$$

I shall analyze closely the estimate (3.2).

I shall first use the decomposition $\hat H = \hat H_1 \oplus \cdots \oplus \hat H_p$ coming from $L_j = \operatorname{Re}\lambda$ ($j = 1,\dots,p$) the real parts of the roots of the representation $\pi\colon G/H \longrightarrow \operatorname{GL}(H)$ as in §2. For the above decomposition and with the obvious notation $\xi = (\xi_1,\dots,\xi_p) \in \hat H$ we shall apply the Lemma of §1 and estimate

$$|f_g(\xi)| \leq f_g^{(1)}(\xi_1)\cdots f_g^{(p)}(\xi_p); \quad g \in G/H.$$

This estimate will be inserted in the integrant of (3.2) and therefore the right hand side of (3.2) can be estimated by:

$$\inf \int_{\hat{H}} f^{(1)}_{g_{j_1}}\big(\pi(s_{j_1})^*\xi_1\big) f^{(2)}_{g_{j_2}}\big(\pi(s_{j_2})^*\xi_2\big)\cdots f^{(p)}_{g_{jp}}\big(\pi(s_{j_p})^*\xi_p\big)\, d\xi$$

where the inf is taken over all choices $1 \leq j_i \leq n$ $(i = 1, \ldots, p)$. The above integration splits in $\hat{H}_1 \oplus \cdots \oplus \hat{H}_p$ and each integral $\int_{\hat{H}_j}$ can be explicitly computed by a change of variable whose determinant is known by §2.

Let us introduce the following notation $s_j = (b_j, \sigma_j)$, $g_j = (X_j, \tilde{\sigma}_j) \in V \times S$, $(j = 1, \ldots, n)$ and for each $g = (u, \sigma) \in G/H$ let us observe that $|g|_{G/H} \approx |u|_V =$ the norm in V (provided that $|g| \gg 1$) let us further denote by:

$$A_n(L_i) = \inf_{1\leq j\leq n} \exp\big(c|X_j|^2 - d_i L_i(b_j)\big).$$

It follows therefore from the above that:

$$\|\nu(g_1, \ldots, g_n)\|_\infty \leq CA_n(L_1)\cdots A_n(L_p). \tag{3.3}$$

4. **The probabilistic estimate.** The notations of the previous section will be preserved but now $b_j = b(j) \in V$ will denote the values the n-dimensional standard Brownian motion (up to coordinate change) $b(t) \in V$ $(t > 0)$ takes at times $t = 1, 2, \ldots$. The $X_j = b(j) - b(j-1)$ are then independent equidistributed normal variables.

In this section I shall show that

$$\mathbf{E}\{A_n(L_1)\cdots A_n(L_p)\} = O\big(\exp(-cn^{1/3})\big). \tag{4.1.}$$

I shall base the proof on the following well known fact (*cf.* [8] for an elementary proof) on the one-dimensional brownian motion. $(\beta(s); s > 0)$:

$$\mathbf{E}(e^{-a\max_{0<s<t}|\beta(s)|}) = O(e^{-ct^{1/3}}) \tag{4.2}$$

valid for any $a > 0$ with $c = c(a) > 0$.

Let us denote by:

$$m_j(t) = \sup_{0<s<t} L_j\big(b(s)\big) = \sup_{0<s<t} L_j^+\big(b(s)\big); \quad t > 0,\ 1 \leq j \leq p.$$

(with $L^+ = \sup(L, 0)$)

I shall prove first that if the "roots" $L_1, \ldots, L_p$ satisfy the condition (C) of our theorem we have (again with $a > 0$ arbitrarily):

$$\mathbf{E}\Big\{\exp\Big[-a\big(m_1(t) + \cdots + m_p(t)\big)\Big]\Big\} = O(e^{-ct^{1/3}}) \tag{4.3}$$

for some $c > 0$. Indeed let $L_{i_1}, \ldots, L_{i_k}$ be a minimal set of *non zero* elements that verifies $\sum_{j=1}^k \gamma_j L_{i_j} = 0$ with $k \geq 1$, $\gamma_j > 0$, $1 \leq j \leq k$. The L_{i_s}'s thus form a non-trivial simplex in some subspace $E \subset V^*$ ($=$ the dual space of V) that contains 0_E in its interior. From this it follows that $\max_s L_{j_s}(x) \geq \varepsilon_0 > 0$ $(x \in E; |x| = 1)$ for otherwise all the L_{j_s}'s would be on one side of some hyperplane in E in contradiction with the above condition.

But then by the positive homogeneity of the $L_i^+(x)$'s we conclude that

$$|x| \leq C \sum_s L_{i_s}^+(x) \leq C \sum_{j=1}^{p} L_j^+(x); \quad x \in E.$$

The estimate (4.3) is therefore an immediate consequence of (4.2).

From (4.3) the estimate (4.1) can easily be deduced. Indeed we have with an appropriate $a > 0$:

$$A_n(L_1)\cdots A_n(L_p) \leq \exp\Big[-a\Big(m_1(n) + \cdots + m_p(n)\Big)\Big] \exp(U_n + V_n) \tag{4.4}$$

where the U_n's and the V_n's are the following "correcting" variables:

The V_n corrects the gap between the "sup" on continuous time $0 < t < n$ and the discrete sampling $j = 1, 2, \ldots, n$. This can be estimated by:

$$U_n = C \sup_{\substack{|s_1 - s_2| \leq 1 \\ 0 < s_1, s_2 < n}} \big|b(s_1) - b(s_2)\big|.$$

The V_n corrects the terms $c|X_j|^2$ that appear in the definition of $A_n(L)$ and can be taken

$$V_n = pc \sup_{1 \leq j \leq n} |X_j|^2 = k \sup_{1 \leq j \leq n} |X_j|^2$$

where the c is the same $c > 0$ that we had in the definition of $A_n(L)$. Observe that this $c > 0$ can be assumed arbitrarily small (*cf.* §1). This means that $k > 0$ can be made $\leq 10^{-10}$. By elementary considerations on brownian motion it follows therefore that:

$$\mathbf{E} \exp(2U_n + 2V_n) = O(n^C) \tag{4.5}$$

i.e. that it grows at most polynomially. But then, if we take expectations on (4.4) and apply Hölder's inequality on the right hand side we immediately deduce (4.1) from (4.3) and (4.5).

REMARK. In the above proof, essential use was made of the fact that, in the definition of $A_n(L)$, $c = \varepsilon > 0$ can be picked up arbitrarily small. In fact, the estimate (4.1) still holds without this provision. The proof is slightly more subtle then, but still remains only an exercise in elementary probability theory. A motivated reader can, I am sure, supply the details for himself.

5. **The Proof of the Theorem.** I shall preserve in this section all the notations introduced up to now and I shall first consider a special class of Lie groups namely the Lie groups G that admit an exact sequence as in §3:

$$\{e\} \longrightarrow H \longrightarrow G \xrightarrow{\pi} G/H \cong V \times S \longrightarrow \{e\}. \tag{5.1}$$

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For such a group the function $u(t,x) = \phi_t(x) > 0$ satisfies the parabolic equation $(\frac{\partial}{\partial t} - \Sigma X_j^2)u = 0$ and is therefore subject to the standard Harnack-Bony estimates at $e \in G$ (*cf.* [1], [2]). These imply that:

$$\phi_n(e) \leq C \sup_{h\in H} \phi_{n+1}(gh) = C \sup_{h\in H} \phi_{n+1}(hg); \quad n \geq 1$$

provided that $\pi(g) \in \Omega \times S = \tilde{\Omega}$ for some compact $\Omega \subset\subset V$ ($C = C_\Omega$ depends on Ω but is independent of n). This trivially gives:

$$\phi_n(e) \leq \frac{C}{\check{\mu}^n(\tilde{\Omega})} \int \cdots \int_{g=g_1\cdot g_2\cdot\cdots\cdot g_n\in\tilde{\Omega}} \sup_{h\in H} \phi_{n+1}(gh)\, d\check{\mu}(g_1)\cdots d\check{\mu}(g_n) \tag{5.2}$$

but since $\tilde{\Omega}$ is compact we can replace

$$\sup_{h\in H} \phi_n(gh) \quad \text{by} \quad \|\nu(g_1,\ldots,g_n)\|_\infty$$

in the above integral, and use (3.3) to estimate $\|(\nu(g_1\cdots)\|_\infty$. Let $p\colon G/H = V \times S \longrightarrow V$ be the canonical projection then $\mu^V = \check{p}(\check{\mu})$ is a non degenerate Gaussian, indeed $\mu^V = \mu_1^V$ for the heat diffusion semigroup μ_t^V on V induced by the projected Laplacian (that Laplacian being hypoelliptic on V is in fact elliptic!). The probabilistic estimate of §4 can therefore be applied to the new integral (5.2) (where we can integrate for *all* $g \in G$, and not only for $g \in \tilde{\Omega}$, and where we note that $\check{\mu}^n(\tilde{\Omega}) \geq cn^{-c}$) provided, of course, that the roots satisfy the condition (C). This proves the condition (E) of our theorem for these special groups and $g = e$. The condition (E) for a general $g \in G$ then follows by Harnack.

What will allow us to obtain our result for more general groups is the following:

LEMMA 5.1. *Let $G_1 \longrightarrow G_2$ be two connected real Lie groups and a surjective homomorphism, and assume that the kernel of that homomorphism is unimodular. Assume also that G_2 satisfies the condition (E) of our theorem (for every subelliptic Laplacian Δ_2 on G_2). Then the same conclusion holds for G_1.*

This is automatic from the same local Harnack estimate that was used before and the passage from the heat kernel $\phi_t^{(1)}$ on G_1 to the heat kernel $\phi_t^{(2)}$ as explained in §1. The details will be left to the reader.

To illustrate this lemma let $\mathfrak{g} \supseteq \mathfrak{q} \supseteq \mathfrak{n}$ be as in §0 for a general simply connected amenable Lie group G, and let $[N,N] \subset N \subset Q \subset G$ be the analytic subgroups that correspond to $[\mathfrak{n},\mathfrak{n}] \subseteq \mathfrak{n} \subseteq \mathfrak{q} \subseteq \mathfrak{g}$. These are known to be closed subgroups (*cf.* [4], Chapter 3) and we have $N/[N,N] = H$ $G/N = V \times S$ where H and V are vector groups and S is compact. (We need simple connectedness to be able to apply the structure theorems of [4], and we need the amenability to guarantee that the semisimple Levi subgroup S is compact. G/N is a product as above because it is reductive and simply connected.) From the special case of our theorem that we have just proved, it follows therefore that the condition (E) is verified for the group $G/[N,N]$ (provided, of course, that G satisfies

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the condition (C)). Our lemma therefore implies that (E) also holds for G. This completes the proof in the simply connected case.

To give the proof for the general (non simply connected) groups we need to use some more (global) structure theory. An excellent reference for this are the last few sections of [4], Chapter 3.

Let Q now be an arbitrary soluble connected Lie group, let $\tilde{Q} \xrightarrow{p} Q$ be its simply connected cover and the covering transformation. Let us denote by $\tilde{N} \subset \tilde{Q}$ and $N \subset Q$ the corresponding nil radicals (*cf.* [4]) that are closed subgroups. Furthermore, we shall make the additional assumption that:

$$(\operatorname{Ker} p) \cap \tilde{N} = \{e\}. \tag{5.3}$$

Further, let M be a compact semisimple group that acts on Q so that we can form the semidirect product $G = Q \ltimes M$. We shall further assume that G (or equivalently Q) satisfies the condition (C) of our theorem. The conclusion of all this will be that (E) holds for the group G.

It is clear from (5.3) that $N \cong \tilde{N}$ is simply connected, the analytic subgroup $N_2 \subset N$ that corresponds to $[\mathfrak{n}, \mathfrak{n}]$ is thus closed and we can form the quotient $G_2 = G/N_2$. The group G_2 admits then the exact sequence:

$$\{e\} \longrightarrow N/N_2 = H \longrightarrow G_2 \longrightarrow G/N \cong Q/N \times M \longrightarrow \{e\} \tag{5.4}$$

(for the fact that the last group is a product *cf.* [4], §3.14). But $Q/N = V \times T$ where V is a vector group and T is a torus because it is a quotient of the vector group $\tilde{Q}/\tilde{N}$. This implies that (5.4) satisfies the conditions of the exact sequence (5.1). The condition (E) therefore holds for G_2 and therefore also for G by Lemma 5.1.

We shall now show that the condition (5.3) was unnecessary and that the same conclusion actually holds without it. Indeed, let $Q, \tilde{Q}$ and M be as before but we do not assume (5.3) to hold, and let $(\operatorname{Ker} p) \cap \tilde{N} = Z$ which is a central subgroup. Observe also that M acts on $\tilde{Q}$ and that its action stabilizes each point of $\operatorname{Ker} p$. Let V_Z be the vector subspace of $Z(\tilde{N})$ (= the center of $\tilde{N}$) that is generated by Z. Then V_Z is a normal subgroup of $\tilde{G} = \tilde{Q} \ltimes M$ and we can form the canonical projections:

$$\tilde{G} \longrightarrow \tilde{G}_1 = (\tilde{Q}/V_Z) \ltimes M = \tilde{Q}_1 \ltimes M \longrightarrow \big(Q/p(V_Z)\big) \ltimes M = G_1$$

where clearly $\tilde{Q}_1$ is simply connected (observe also that $p(V_Z)$ is compact!). It follows that $\tilde{Q}_1$ is the universal cover of $Q/p(V_Z)$ and that therefore G_1 satisfies the condition (5.3). The conclusion (E) holds therefore for G_1 and therefore also by Lemma 5.1 for G.

The proof of our theorem for a general connected amenable Lie group G now simply follows from the fact that for every such group there exists Q and M as above and a *finite* cover $G_f = Q \ltimes M \xrightarrow{f} G$ above it (*cf.* [4] §3.18). Indeed, $\operatorname{Ker}(f)$ being finite, the conclusion (E) passes from G_f to G by an easy application of our previous Harnack principle.

6. **More general differential operators.** We can consider on G left invariant differential operators $\Delta = -\sum_{j=1}^{n} X_j^2 + X_0$, where X_i are all left fields, that are not "sum of squares". However, if we insist that the heat diffusion kernel, *i.e.* the kernel of the semigroup $T_t = e^{-t\Delta}$ should be C^∞ we must impose the Hörmander condition *already* on $X_1, \ldots, X_n$ which should therefore be generators of the Lie algebra. For the above more general operators the theory is very much the same and our theorem still holds. Indeed the local Harnack estimate for the heat kernel of these operators clearly still works. What needs a different proof is the global (*i.e.* large $|g|$) Gaussian estimate.

If we carefully analyze our previous proof we see that it does go through, provided that in the projection $G \longrightarrow G/H$ of §1 followed by the projection $G/H = V \times S \longrightarrow V$ projects Δ into a standard Laplacian (up to coordinate change) on $V \cong \mathbb{R}^n$. Indeed as we have seen we can make a proof without the Gaussian estimate.

If the projected operator Δ_V, which is always a constant coefficient elliptic second order operator on $V \cong \mathbb{R}^n$ is *not* a "standard Laplacian" then it contains non-trivial first order terms $\sum a_i \frac{\partial}{\partial x_i}$ and the corresponding diffusion kernel on V decays exponentially at 0, *i.e.* $\check{\phi}_t^V(0) = O(e^{-ct})$ for some $c > 0$. But then, by local Harnack again, our original diffusion kernel on G satisfies $\phi_t(e) = O(e^{-ct})$ and our theorem follows trivially.

7. **The spectral gap and the non-amenable groups.** Let G be a general (not necessarily amenable) connected real Lie group and let $\Delta = -\Sigma X_j^2$ as before, we can then define $\lambda_0(\Delta) = \lambda_0 = \inf \operatorname{sp}(\Delta) \geq 0$ where Δ is considered as a formally self adjoined operator on $L^2(G; d^r g)$. G is amenable if and only if $\lambda_0(\Delta) = 0$ for some (equivalently for all) subelliptic Laplacian as above. (Indeed, $e^{-\lambda_0}$ is the $L_2 \longrightarrow L_2$ norm of the symmetric operator $f \longmapsto f * \mu_1$.)

If we bring the spectral gap in, we can improve our theorem in §0 and we obtain

THEOREM. *Let G, Δ and λ_0 be as above and let ϕ_t be as in §0. Let us assume that G satisfies the condition (C) of §0. Then*

$$\phi_t(e) = O\big(\exp(-\lambda_0 t - ct^{1/3})\big); \quad t \longrightarrow \infty \tag{7.1}$$

for some $c > 0$ independent of t.

An even more general theorem can be proved if we consider

$$\Delta = \Delta_0 + X_0 = -\sum_{j=1}^{n} X_j^2 + X_0$$

as in §6. We then obtain a decay as in (7.1) but with a λ_0 that is now the sum of the spectral gap $\lambda_0(\Delta_0)$ and the contribution from the drift term X_0 (that can be made explicit in a very precise way).

The proof of these facts is but an elaboration of our previous proof. The details will be presented elsewhere.

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