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ON THE REPRESENTATION OF AN IDEMPOTENT AS A SUM OF NILPOTENT ELEMENTS

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ABSTRACT. In this paper we study in which rings a non-zero idempotent element can be presented as a sum of two nilpotent elements.

In [4] it was proved that Koethe's problem is equivalent to the problem of whether a ring which is the sum of a nil subring and a nilpotent subring must be nil. This gives a strong motivation for studying rings which are sums of two nil subrings. Recently Kelarev [5] showed that such rings (even rings which are sums of two locally nilpotent subrings) need not be Jacobson radical. Hence the class of rings which are sums of two nil subrings is quite large. However it is unknown whether it can contain rings with non-zero idempotents. In this paper we raise a connected more general question of *whether in some ring a non-zero idempotent can be presented as a sum of two nilpotent elements*. The question has an easy positive answer in certain rings with torsion elements. It is much more interesting and challenging for torsion-free rings. Some particular cases considered in this paper suggest the answer for these rings might be negative.

Formally, this is a question in combinatorial ring theory concerning the nature of the finitely presented ring

$$Z(m,n) = \langle e, x, y \mid e^2 = e, x^m = 0, y^n = 0, e = x + y \rangle.$$

The Diamond Lemma ([1, 3]) suggests itself as a tool in investigating this ring. However, the analysis of ambiguities, especially when m, $n > 3$, becomes progressively more difficult.

In Section 1 we show that the question has a positive answer in the class of algebras of positive characteristic and it has a negative answer for some types of algebras of characteristic 0 (*e.g.* PI and group algebras) or when one of the nilpotents is of index 2.

In Section 2 we introduce the rings $Z(m, n)$ generated by elements e, x, y, as above. Our main result shows that $Z(m, n)$ has finite Z-rank for $m = 2$, n arbitrary, and for $m = 3$, $n = 4$ or 5. As a consequence, we conclude that the above question has a negative answer for torsion-free rings provided the indices of nilpotency of the nilpotent elements are ≤ 3 and ≤ 5, respectively. The next cases to be treated, which are open, are the rings $Z(3, 6)$ and $Z(4, 4)$.

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In the last section we provide some examples of torsion-free rings where an idempotent is a sum of more than two nilpotent elements.

Throughout the paper all rings are associative with an identity element. We denote the ring of integers by Z and the field of rational numbers by Q.

1. We start with an example showing that a non-zero idempotent can be a sum of two nilpotent elements.

EXAMPLE 1. Let $R = M_{p+1}(F)$ be the ring of $(p+1)\times(p+1)$-matrices over a field F of characteristic $p > 0$. Let a be the strictly upper triangular matrix in R whose all entries above the diagonal are equal to -1 and let b be the transpose of a. It is easy to check that a and b are nilpotent elements of R and $a+b$ is a non-zero idempotent of R. ∎

The next proposition shows in particular that in torsion-free rings one cannot construct non-zero idempotents which are sums of two nilpotent elements (if they exist at all) so easily.

Observe that studying the problem for a torsion-free ring R one can pass to the quotient ring $Z^{-1}R$ and thus assume that R is a Q-algebra. Further, every algebra over a field can be embedded into the algebra of linear transformations on a vector space over the field. Hence one can assume that R is the algebra of linear transformations on a vector space over Q.

PROPOSITION 1. *Let R be a ring and $x, y \in R$ nilpotent elements such that $e = x + y$ is an idempotent.*

i) If e belongs to the center of R, then $e = 0$.

ii) If R is a subring of the matrix ring $M_t(K)$, where K is a field, then either $e = 0$ or the characteristic of K is non-zero.

iii) If R is a PI *algebra over a field of characteristic zero, then $e=0$.*

iv) If R is a group algebra over a field of characteristic zero, then $e = 0$.

PROOF. i) Suppose that $x^n = y^m = 0$. Since $e = e^2$ commutes with x,

$$e(e-x)(e+x+x^2+\cdots+x^{n-1}) = e - ex^n = e$$

and

$$\begin{aligned} e = e^m &= \left(e(e-x)(e+x+x^2+\cdots+x^{n-1})\right)^m \\ &= e(e-x)^m(e+x+x^2+\cdots+x^{n-1})^m \\ &= e^m y^m (e+x+x^2+\cdots+x^{n-1})^m \\ &= 0. \end{aligned}$$

ii) It is well known that a matrix over a field is nilpotent if and only if all its eigenvalues are equal to 0 and that the eigenvalues of an idempotent matrix are equal 0 or 1. Hence assuming that $e \neq 0$ and applying the trace map to the equation $e = x + y$ we get $0 = \operatorname{tr}(x) + \operatorname{tr}(y) = \operatorname{tr}(e) = k1$, for a positive integer k. This implies that if $e \neq 0$, then the characteristic of K is different from zero.

iii) Since the Jacobson radical of a ring contains no non-zero idempotents, it is enough to prove the result for primitive algebras. So assume that R is a primitive algebra. Then by Kaplansky's theorem, R is a simple finite dimensional algebra over its center C and there exists a field extension $C \subseteq K$ such that $R \otimes_C K \simeq M_n(L)$ for some n. Applying (ii) we get that $e = 0$.

iv) Similarly as ii), this result is a direct consequence of the fact (*cf.* [6], Theorem 2.1.8 and Lemma 3.8) that in group algebras over fields of characteristic zero the trace of non-zero idempotent elements is non-zero and the trace of nilpotent elements is zero. ■

REMARK. Let F be a field of characteristic $p > 0$, with the prime subfield F_0, and let G be the linear group $\mathrm{GL}(p+1, F_0)$. Then the group algebra $A = F[G]$ contains a non-zero idempotent which is a sum of two nilpotent elements. Indeed, by Wedderburn's Theorem, the Jacobson radical J of A is a nilpotent ideal and A/J is a direct sum of matrix rings over division rings. Observe that there is a natural homomorphism of A onto $M_{p+1}(F)$. This implies that one of the summands in A/J is isomorphic to $M_{p+1}(F)$. Consequently, A contains a subalgebra B such that $J \subseteq B$ and $B/J \simeq M_{p+1}(F)$. By Example 1, B/J contains a non-zero idempotent which is a sum of two nilpotent elements. Since the ideal J is nilpotent, the idempotent of B/J can be lifted to an idempotent of B. By lifting nilpotent elements of B/J to B one gets nilpotent elements. These easily imply that B contains a non-zero idempotent which is a sum of two nilpotent elements. ■

COROLLARY 1. *If an idempotent e of a Q-algebra R is equal to the sum of nilpotent elements x and y of R, then $e = 0$ if and only if the Q-subalgebra of R generated by 1, x and y is finite dimensional.*

PROOF. If $e = 0$ then $x = -y$ and the subalgebra of R generated by $1, x$ and y is linearly spanned by $1, x, x^2, \ldots, x^{n-1}$, where $x^n = 0$. To get the converse implication it suffices to apply Proposition 1 iii), since every finite dimensional algebra is a PI algebra. ■

The nilpotent elements in Example 1 are of index > 2. The following shows that this is indeed a necessary condition.

PROPOSITION 2. *If x, y are elements of a ring R such that $x^2 = 0$ and $y^n = 0$, for a natural number n, and $e = x + y$ is an idempotent, then $e = 0$.*

PROOF. We can assume that R is generated by $1, x, y$. If e belongs to the nil radical $N(R)$ of R then $e = 0$. This allows us to assume that $N(R) = 0$. Let m be the smallest natural number such that $y^m = 0$. If $m = 1$ then the result is clear. Thus suppose $m > 1$. From the relation $x + y = (x + y)^2 = y^2 + xy + yx$ we get that $y^{m-1}x = y^{m-1}xy$. Thus $y^{m-1}x = y^{m-1}xy = y^{m-1}xy^2 = \cdots = y^{m-1}xy^m = 0$. It follows that $(y^{m-1}R)^2 = 0$. Therefore $y^{m-1}R = 0$ and we get $y^{m-1} = 0$, a contradiction. ■

2. As we have seen, in some rings non-zero idempotents are sums of two nilpotent elements. However the question of whether this can happen in torsion-free rings is open. Observe that if a, b are elements of a ring R such $a^n = b^m = 0$ and $(a + b)^2 = a + b$, then there is an epimorphism f of the free ring $Z\langle X, Y\rangle$ onto the subring of R generated

by 1, a, b such that $f(X) = a + b$ and $f(Y) = -a$. The kernel of f contains the ideal $I(m, n)$ generated by $X^2 - X$, Y^n and $(X + Y)^m$. Hence the subring of R generated by 1, a, b is a homomorphic image of the ring $Z(n, m) = Z\langle X, Y\rangle / I(m, n)$. If the ring R is torsion-free, then in order to obtain that $a + b = 0$ it suffices to prove that $Z(n, m)$ is a finitely generated Z-module (*cf.* Corollary 1). We prove in this section that the Z-rank of $Z(3, 5)$ is finite. As a consequence, we obtain that if e is an idempotent of a torsion-free ring R and $e = a + b$ for some $a, b \in R$ such that $a^3 = 0$, $b^5 = 0$, then $e = 0$.

Given a natural number m, denote by $Z(m)$ the ring $Z\langle X, Y\rangle / I(m)$, where $I(m)$ is the ideal of $Z\langle X, Y\rangle$ generated by $X^2 - X$ and $(X + Y)^m$. Obviously $Z(n, m)$ is a homomorphic image of $Z(m)$.

Observe that in $Z(2)$, $ea = -a^2 - ae - e$, where $e = X + I(2)$ and $a = Y + I(2)$. Hence, if V is the Z-submodule of $Z(2)$ generated by the set $S = \{a^i, a^je \mid i, j = 0, 1, \ldots\}$, then $Va \subseteq V$. This implies that V is a right ideal of $Z(2)$ and, since $1 \in S$, we get that $Z(2) = V$. Consequently S generates $Z(2)$ as a Z-module. Applying this we get that $Z(n, 2)$ is generated by the set $\{a^i, a^je \mid 0 \leq i, j \leq n - 1\}$ as a Z-module, where $e = X + I(n, 2)$ and $a = Y + I(n, 2)$. Observe now that Proposition 2 implies that $Z(n, 2)$ is generated as a Z-module by the set $\{a^i \mid 0 \leq i \leq n - 1\}$.

Now we shall study $Z(3)$. As above we write $e = X + I(3)$ and $a = Y + I(3)$. Given natural numbers $i_1, \ldots, i_k$, define $[i_1, \ldots, i_k] = ea^{i_1}ea^{i_2} \cdots ea^{i_k}e$. For $w \in Z(3)$, let $\lambda(w)$ denote the minimum of the X-degree of polynomials $f(X, Y)$ such that $w = f(X, Y) + I(3)$. We call $\lambda(w)$ the e-degree of w.

LEMMA 1. *(i)* $\lambda[1] \leq 1$ *and* $\lambda[2] \leq 1$;

(ii) $\lambda\big([4] + a[3] + [3](a + 3)\big) \leq 1$;

(iii) $\lambda\big([3, 4] + [4, 3] + [3, 3](a + 3)\big) \leq 2$,

$\lambda\big([3, 4] + [4, 3] + (a + 3)[3, 3]\big) \leq 2$

and, consequently,

$\lambda([3, 3]a - a[3, 3]) \leq 2.$

PROOF. Since $(a + e)^3 = 0$ and $e^2 = e$, we have

$$eae + aea + (a^2 + a + 1)e + e(a^2 + a) + a^3 = 0. \tag{1}$$

This equation gives $\lambda[1] \leq 1$. Now we compute $(1) \cdot e$:

$$eae + aeae + (a^2 + a + 1)e + e(a^2 + a)e + a^3e = 0 \tag{2}$$

This gives $\lambda[2] \leq 1$.

Computing e$\cdot$(1)$\cdot$e, $ea^2\cdot$(1) and (1)$\cdot a^2e$ we get, respectively

$$eaeae + [3] + 2[2] + 3[1] + e = 0 \tag{3}$$

$$ea^2eae + [4] + [3](a + 1) + [2](a^2 + a + 1) + ea^5 = 0 \tag{4}$$

$$eaea^2e + [4] + (a + 1)[3] + (a^2 + a + 1)[2] + a^5e = 0 \tag{5}$$

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From $e \cdot (1) \cdot ae$ we obtain

$$eaea^2e + ea^2eae + 2eaeae + [4] + [3] + [2] + [1] = 0.$$

Applying to this (i) and (3), (4) and (5) we get (ii).

Observe that $[3, 4] + [4, 3] + [3, 3](a + 3) = ea^3\big([4] + a[3] + [3](a + 3)\big)$ and $[4, 3] + [3, 4] + (a+3)[3, 3] = \big([4]+[3]a+(a+3)[3]\big)a^3e$. Now (iii) is a consequence of (ii). ∎

Now we shall prove that $Z(3, 5)$ is a finitely generated Z-module. As above, we put $e = X + I(3, 5)$, $a = Y + I(3, 5)$, $[i_1, \ldots i_n] = ea^{i_1}ea^{i_2}\cdots a^{i_n}e$ and define the e-degree λ of elements in $Z(3, 5)$. Since $I(3) \subseteq I(3, 5)$, the relations of Lemma 1 hold in $Z(3, 5)$. In addition, $a^5 = 0$.

THEOREM 1. *The ring $Z(3, 5)$ is generated as a Z-module by the set $S = \{a^k, a^iea^j, a^i[3]a^j, a^i[3, 3], a^i[3, 4] \mid 0 \le k, i, j \le 4\}$.*

PROOF. Let V be the Z-span of S. Notice first that from Lemma 1 (i) and (ii) it follows that all elements of e-degree ≤ 2 are in V. Also by Lemma 1 (i), $\lambda[i, j] \le 2$ whenever $i \le 2$ or $j \le 2$. This implies that all elements $a^k[i, j]a^l$, where $i \le 2$ or $j \le 2$, are in V.

From Lemma 1 (iii) it follows that $a^k[3, 3]a^l \in V$. Observe now that

$$[4, 4] + [3, 5] + (a+3)[3, 4] = \big([4] + [3]a + (a+3)[3]\big)a^4e,$$

so by Lemma 1 (ii)

$$(a) \qquad \lambda\big([4, 4] + (a+3)[3, 4]\big) \le 2.$$

Hence $[4, 4] \in V$. Similarly

$$(b) \qquad \lambda\big([4, 4] + [4, 3](a+3)\big) \le 2.$$

Hence $[4, 3](a + 3) \in V$. By Lemma 1 (iii) we also have $[4, 3] \in V$. Thus $[4, 3]$ and $[4, 3]a$ are in V. From (b) and Lemma 1 (iii) we get

$$\lambda\big([4, 4] - [3, 4](a+3) - (a+3)^2[3, 3]\big) \le 2.$$

Applying to this (a) we obtain,

$$(c) \qquad \lambda\big((a+6)[3, 4] + [3, 4]a + (a+3)^2[3, 3]\big) \le 2.$$

Consequently $[3, 4]a \in V$. The foregoing give that $Va \subseteq V$ and $[4, 3]$, $[4, 4]$ are in V. This together with the observation made in the first paragraph imply that all elements of e-degree ≤ 3 are in V.

Since $1 \in V$, in order to obtain that $V = Z(3, 5)$, it suffices to prove that V is a left ideal of $Z(3, 5)$. However $aV \subseteq V$, so it is enough to check that $eV \subseteq V$. Since we know that all the elements of e-degree ≤ 3 are in V, it remains to check that $w = [i, j, k] \in V$ for all $0 \le i, j, k \le 4$. If one of i, j, k is ≤ 2, we can use Lemma 1 (i) to reduce w to an element of e-degree ≤ 3. Hence $w \in V$ in this case.

From the third relation in Lemma 1 (iii) we get

$$\lambda([3,3,3]-a[3,3,2])=\lambda\big(([3,3]a-a[3,3])a^2e\big)\leq 3$$

and

$$\lambda([3,3,4]-a[3,3,3])=\lambda\big(([3,3]a-a[3,3])a^3e\big)\leq 3.$$

These give that $[3,3,3]\in V$ and $[3,3,4]\in V$.

Similarly, applying (c) we get

$$\lambda\big([3,4,3]+(a+6)[3,4,2]+(a+3)^2[3,3,2]\big)\leq 3$$

and

$$\lambda\big([3,4,4]+(a+6)[3,4,3]+(a+3)^2[3,3,3]\big)\leq 3.$$

Hence $[3,4,3]\in V$ and $[3,4,4]\in V$.

From (a), it follows that

$$\lambda\big([4,4,3]+(a+3)[3,4,3]\big)\leq 3$$

and

$$\lambda\big([4,4,4]+(a+3)[3,4,4]\big)\leq 3.$$

Consequently, $[4,4,3]\in V$ and $[4,4,4]\in V$.

Finally, from the second relation in Lemma 1 (iii) we get $\lambda\big([3,4,j]+[4,3,j]+(a+3)[3,3,j]\big)\leq 3$ for $j=3,4$. This and the foregoing imply that $[4,3,j]\in V$ for $j=3,4$. The proof now is complete. ∎

Corollary 1 and Theorem 1 immediately give

COROLLARY 2. *If a, b are elements of a torsion-free ring R, $a^3=b^5=0$ and $a+b$ is an idempotent, then $a+b=0$.* ∎

REMARK. Obviously as a consequence of Corollary 2 one gets that if a and b are elements of a torsion-free ring R, $a^3=b^3=0$ and $a+b$ is an idempotent, then $a+b=0$. Some more precise calculations show that to get this result it suffices to assume that R contains no 2-torsion elements.

3. The arguments applied in Proposition 1 ii) and iv) work for sums of more than two nilpotent elements. Hence in PI algebras of characteristic zero (resp. group algebras over fields of characteristic zero), no non-zero idempotent can be represented as a sum of nilpotent elements. This cannot be extended to all algebras of characteristic zero. In ([2], Theorem 3) Bokut' proved that each algebra can be embedded into a simple algebra which is a sum of three nilpotent algebras of index 3. This immediately implies that for each field F there exists an F-algebra A containing elements e,a,b,c such that $0\neq e^2=e=a+b+c$ and $a^3=b^3=c^3=0$. We give a direct example of the sort in which $a^2=b^2=c^2=0$.

EXAMPLE 2. Let V be the vector space over a field F with basis $\{e_1, e_2, \dots\}$. Define the linear endomorphisms a, b, c of V putting

$$a(e_{2n-1}) = a(e_{2n}) = n(e_{2n-1} - e_{2n}) \text{ for } n \geq 1;$$
$$b(e_1) = 0, b(e_{2n-1}) = (n-1)(e_{2n-2} - e_{2n-1}) \text{ for } n \geq 2$$

and

$$b(e_{2n}) = n(e_{2n} - e_{2n+1}) \text{ for } n \geq 1;$$
$$c(e_{2n-1}) = -(n-1)e_{2n-2} + ne_{2n}, c(e_{2n}) = 0 \text{ for } n \geq 1.$$

One easily checks that $a^2 = b^2 = c^2 = 0$ and $e = a + b + c$ is a non-zero idempotent. ∎

Observe that if $M_2(F)$ is the ring of 2×2-matrices over a field F of characteristic 2, then the identity is the sum of *three* elements with zero squares:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Examples of this type are not possible in rings without 2-torsion elements.

PROPOSITION 3. *If R is a ring without 2-torsion elements, $x, y, z \in R$ are such that $x^2 = y^2 = z^2 = 0$, then $x + y + z \neq 1$.*

PROOF. Suppose that $x + y + z = 1$. Then $xyz = xz$ and for any other permutation a, b, c of elements x, y, z, we have $abc = ac$. Hence $(xz)x = (xyz)x = x(yzx) = x(yx)$. Moreover, multiplying $x + y + z = 1$ from left and right by x, we get $xyx + xzx = 0$. Thus $2xyx = 0$ and $xyx = 0$ follows since R has no 2-torsion elements. Similarly, for any a, $b \in \{x, y, z\}$, $aba = 0$. This implies that if $a, b, c, d \in \{x, y, z\}$ and a, b, c or b, c, d is not a permutation of x, y, z, then $abcd = 0$. If both a, b, c and b, c, d are permutations of x, y, z, we have $a = d$, so $abcd = (abc)a = aca = 0$. Thus $(x + y + z)^4 = 0$, a contradiction. ∎

We conclude with an example showing that in some torsion-free rings the identity is a sum of *four* elements with zero squares.

EXAMPLE 3. Let V be the vector space over a field F with basis $\{e_1, e_2, \dots\}$. Define F-endomorphisms a, b, c, d of V putting, for $n \geq 1$,

$$a(e_{2n-1}) = a(e_{2n}) = (2n-1)(e_{2n-1} - e_{2n}),$$
$$b(e_{2n-1}) = 2(n-1)(e_{2n-2} - e_{2n-1}), b(e_{2n}) = 2n(e_{2n} - e_{2n+1}),$$
$$c(e_{2n-1}) = 0, c(e_{2n}) = -(2n-1)e_{2n-1} + 2ne_{2n+1},$$
$$d(e_{2n}) = 0, d(e_{2n-1}) = -(2n-2)e_{2n-2} + (2n-1)e_{2n}.$$

It is easy to check that $a^2 = b^2 = c^2 = d^2 = 0$ and $a + b + c + d = 1$. ∎

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