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ORIGINAL RESEARCH PAPER

A stochastic model for capital requirement assessment for mortality and longevity risk, focusing on idiosyncratic and trend components

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Abstract

This paper provides a stochastic model, consistent with Solvency II and the Delegated Regulation, to quantify the capital requirement for demographic risk. In particular, we present a framework that models idiosyncratic and trend risks exploiting a risk theory approach in which results are obtained analytically. We apply the model to non-participating policies and quantify the Solvency Capital Requirement for the aforementioned risks in different time horizons.

Keywords: Life insurance; Mortality & longevity risk; Risk theory; Solvency II; Solvency Capital Requirement

1. Introduction

The introduction of Directive 138/2009/EC (Solvency II) has introduced two key innovations in the actuarial framework. On the one hand, the valuation of assets and liabilities has become market consistent and, on the other hand, the risk-based assessment of capital requirement has been provided (see European Parliament and Council, 2009).

In this context, in addition to an internal assessment of their own risk profile, insurance companies should quantify the risk using either the methodology proposed in the Standard Formula (see European Parliament and Council, 2014) or a partial or total internal model.

Therefore, we propose a methodological framework, consistent with the accounting principles of Solvency II, based on the cohort approach and aimed both at identifying the different sources of demographic risk and at quantifying the Solvency Capital Requirements (SCRs). With this purpose, we separately model, within demographic risk, an idiosyncratic (or unsystematic) component and a trend component. A unified model, focused on quantifying the SCR in a closed form, is provided for policies both with survival-linked benefits and for death-linked benefits.

The dynamic assessment of the risk, through the use of realistic and updated technical bases, must reflect the structural conditions of the demographic system. To this end, we focus on the effects on the liabilities related to mortality and longevity risks considering both the idiosyncratic (or accidental) and trend aspects. Indeed, longevity trends observed in the past 30 years and the recent experience of mortality increase due to the COVID-19 pandemic emphasised the usefulness of analysing both structural changes and extreme shocks. In particular, the COVID-19 pandemic structurally changed the entity of mortality shocks as well known (see Schnürch *et al.*, 2021); consequently, the assessment of the capital requirement must be based on actuarial models that are able to capture these aspects allowing insurance companies to cope with the unfavourable effects of new adverse demographic scenarios.

In the literature, many works deal with this topic. Savelli & Clemente (2013) developed a stochastic model based on the cohort approach in a Local Generally Accepted accounting Principles (Local GAAP) context. Although this paper is not related to a market-consistent valuation (MCV), it models the single generations of the insurance portfolio with an exact individual approach. Clemente *et al.* (2021) describe the bridge between Local GAAP and Solvency II contexts, highlighting the effects of the market-consistent assessment of liabilities on the random variable (r.v.) demographic profit. This work presents recursive relationships for the evaluation of the market consistent demographic profit, but it does not focus on identifying the sources of uncertainty related to systematic and unsystematic variations in mortality rates. In this field, Olivieri & Pitacco (2008) analyse longevity risk by referring to a portfolio of annuities. In particular, through risk-neutral approaches, the authors reconcile the traditional approach with the market-consistent one. Jarner & Møller (2015) propose a partial internal model for the longevity risk component, which incorporates an unsystematic element linked to the size of the portfolio. Similar to Jarner & Møller (2015), we overcome the methodology provided by the Standard Formula based on a longevity shock. However, our approach is different since it includes also mortality risk in the evaluation and considers the volatility of the sums insured within the portfolio, neglected in Jarner & Møller (2015).

As regards trend risk, alternative models have been provided in the literature. In Plat (2011), a mortality model is proposed for forecasting trends. In Börger (2010), the shock given by Solvency II is compared with the results of the forward models proposed by Bauer *et al.* (2010) and (2008). Börger *et al.* (2014) propose an ad-hoc mortality model that considers both longevity and mortality and the dependency structure between the different cohorts. Moreover, Gylys & Šiaulys (2019) compare the run-off and the 1-year approach fitting stochastic mortality data on different years than those used for Solvency II calibration by EIOPA. Zhou *et al.* (2014) model a multi-population mortality model overcoming the common assumptions of dominant population. To this end, they model the joint evolution of mortality using a multivariate stochastic process with a symmetric structure. Richards *et al.* (2014) consider a 1-year value-at-risk framework, investigating which is the share of longevity trend risk to be considered.

The theme, albeit in a different way, was also considered in other sub-fields: Dhaene *et al.* (2017) define a fair valuation of liabilities related to a portfolio in a single period framework, such that it is both mark-to-market for any hedgeable part of a claim and mark-to-model for any claim that is independent of financial market evolutions. Hari *et al.* (2008) focus on the relevance of longevity risk for the solvency position of annuity portfolios distinguishing between micro and macro-longevity risks. Stevens *et al.* (2010) quantify the longevity risk capital requirement by applying the classical Lee–Carter model to estimate the uncertainty of future survival probabilities. Bauer & Ha (2015) compute the required risk capital based on least-squares regression and Monte Carlo simulations. Dahl (2004) models mortality intensity as a stochastic process and quantifies mortality risk by capturing the importance of time dependency and uncertainty. Ngugnie Diffouo & Devolder (2020) assess the capital requirement for longevity risk related to a portfolio of annuity by means of the Hull–White model.

The model in this paper presents a unified accounting framework consistent with the MCV introduced by Solvency II to evaluate the capital requirement for both mortality and longevity risk. Through this approach, it is possible to measure both trend and idiosyncratic components of the demographic risk. The latter concerns exclusively the volatility linked to the random variable policyholder deaths (or survivals), otherwise the trend one is related to the structural changes in the cohort mortality (or longevity) curve. According to the trend component, with respect to the current literature, we consider both longevity and mortality risks, whose importance has emerged also during pandemic situation, and we take into account also the effect of the different structures of the insurance coverages, typically neglected in the literature. The approach depends on the distribution of death probability and, therefore, can be applied using the alternative forecasting models proposed in the literature (in addition to those defined above, see Lee & Carter, 1992; Renshaw & Haberman, 2006; Apicella *et al.*, 2019; Cairns *et al.*, 2006).

Regarding the idiosyncratic risk, the use of a cohort approach allows to reach closed formulas for the calculation of the SCR which clearly highlight not only the pooling effects due to the portfolio size but also the effects of the volatility of the insured sums within the cohort and of the type of coverage. Our proposal represents a possible undertaking approach in the Solvency II context and overcome some drawbacks of the proposal provided in Quantitative Impact Study n.2 (see Committee of European Insurance and Occupational Pensions Supervisors, 2006).

A case study has been developed to test the proposed approach considering different life insurance contracts. For the trend risk evaluation, the estimation of the distribution of future mortality rates is based on the Poisson log-bilinear projection method proposed in Brouhns *et al.* (2005) where a parametric bootstrap is included to catch the volatility.

The paper is organised as follows. Section 2 concerns the presentation of the stochastic framework and identifies the two aforementioned risk components. Section 3 analyses the two components from an analytical point of view, providing results in a closed formula and approximations for the calculation of the SCR. In section 4, an application of the model is provided for two typical policies of the life insurance market, showing how the basic results can be easily extended to more complex contract.

2. Preliminaries

In order to provide an approach for the computation of the SCR for demographic risk, we introduce some preliminary results that will be used in the next section.

We assume to be at time t and we consider traditional life insurance products. We recall the definition of the random variable (r.v.) demographic profit $\tilde{y}_{t+1}^{MCV}$[1] (see Clemente *et al.*, 2021) for the decomposition of the whole insurance profit in a market consistent context according to the framework defined by the Solvency II directive (see European Parliament and Council, 2009 and European Parliament and Council, 2014)

$$\begin{aligned}\tilde{y}_{t+1}^{MCV} = (w_t - \tilde{s}_{t+1}) \cdot \left[be_t^{Rf(t),q(t)} + \pi \right] \cdot (1+j^*) + \\ - \left[\tilde{w}_{t+1} \cdot \tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)} + \tilde{x}_{t+1} \right].\end{aligned} \tag{1}$$

where random variables are indicated with tilde.

We denote by w_t the insured sums at time t and by $\tilde{s}_{t+1}$ the insured sums eliminated due to lapses in the period $(t, t+1]$. We indicate with π the pure risk premium (i.e. excluding expenses) and with j^* the first-order financial technical basis. The r.v. $\tilde{x}_{t+1}$ represents the lump sum paid to a beneficiary when policyholder dies (in case of term insurance or pure endowment) or the sum insured paid in case of annuities.

It is noteworthy that in formula (1) the best estimate (BE) rates[2] calculated at time t and $t+1$ ($be_t^{Rf(t),q(t)}$ and $\tilde{be}_t^{Rf(t+1),\tilde{q}(t+1)}$, respectively) are evaluated using risk-free curves $Rf(t)$ and $Rf(t+1)$ assumed known at time t and second-order demographic life table q(t) and $\tilde{q}(t+1)$. Since the aim of the paper is to assess the capital requirement for demographic risk on a 1 year

[1] From now on, we indicate random variables with tilde.

[2] According to Art. 77 of Directive 138/2009/EC and Art. 22 of the Delegated Regulation, the best estimates rates (indicated with lowercase letters, to highlight that the insured sums are unitary) are calculated as the expected present value of the benefits net of expected present value of premiums. The technical bases used are "realistic" and "specific" to the company's business.

time horizon, we consider only the variability of mortality rates and we assume that the spot curve $Rf(t+1)$ at time $t+1$ can be inferred from the curve $Rf(t)$ available in t.

We rewrite formula (1) in order to isolate the effect of risk-free rates:

$$\begin{aligned}\tilde{y}_{t+1}^{MCV} &= (w_t - \tilde{s}_{t+1}) \cdot \left[be_t^{Rf(t),q(t)} + \pi \right] \cdot (1 + i_t(t, t+1)) + \\ &- \left[\tilde{w}_{t+1} \cdot \tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)} + \tilde{x}_{t+1} \right] + \\ &+ \left(j^* - i_t(t, t+1) \right) \cdot (w_t - \tilde{s}_{t+1}) \cdot \left[be_t^{Rf(t),q(t)} + \pi \right].\end{aligned} \tag{2}$$

where $i_t(t, t+1)$ is the 1-year spot rate at time t. In formula (2), it is noticeable that the last term, here defined as $\tilde{y}_{t+1}^{NDM} = \left(j^* - i_t(t, t+1) \right) \cdot (w_t - \tilde{s}_{t+1}) \cdot \left[be_t^{Rf(t),q(t)} + \pi \right]$, does not depend on future mortality rates.

Focusing now on the element $\left(\tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)} \right)_{t\epsilon I}$, it is a stochastic process adapted to filtration $(\mathcal{F}_t)_{t\epsilon I}$, therefore measurable with respect to the natural filtration $\mathcal{F}_t$[3]. As a consequence, $\tilde{y}_{t+1}^{MCV}$ is also a stochastic process. In the next sections, we will focus on quantifying the capital requirement over an annual time horizon as prescribed by Solvency II. Therefore, for $t > 1$, we will assume that the stochastic process has followed its expected trajectory until t and we model the distribution of $\tilde{y}_{t+1}^{MCV}$ in a 1-year view (i.e. between t and $t+1$).

Our aim is to split the effects on the demographic profit variable of the idiosyncratic mortality (longevity) risk related to the volatility of the random variables from the risk linked to structural changes of the second-order assumptions. Therefore, we rewrite previous formula adding and subtracting the following amount $\tilde{w}_{t+1} \cdot be_{t+1}^{Rf(t+1),q(t)}$ as follows:

$$\begin{aligned}\tilde{y}_{t+1}^{MCV} &= (w_t - \tilde{s}_{t+1}) \cdot \left[be_t^{Rf(t),q(t)} + \pi \right] \cdot (1 + i_t(t, t+1)) + \\ &- \left[\tilde{w}_{t+1} \cdot be_{t+1}^{Rf(t+1),q(t)} + \tilde{x}_{t+1} \right] + \tilde{w}_{t+1} \left[be_{t+1}^{Rf(t+1,q(t))} - \tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)} \right] \\ &+ \tilde{y}_{t+1}^{NDM}.\end{aligned} \tag{3}$$

Our purpose is indeed to analyse separately the three components in formula (3) in order to evaluate the SCRs related to the involved risks. Hence, we define

- The first component $\tilde{y}_{t+1}^{Id}$ that measures the r.v. demographic profit due only to idiosyncratic volatility of mortality and longevity.

$$\begin{aligned}\tilde{y}_{t+1}^{Id} &= (w_t - \tilde{s}_{t+1}) \cdot \left[be_t^{Rf(t),q(t)} + \pi \right] \cdot (1 + i_t(t, t+1)) + \\ &- \left[\tilde{w}_{t+1} \cdot be_{t+1}^{Rf(t+1),q(t)} + \tilde{x}_{t+1} \right].\end{aligned} \tag{4}$$

[3] I is an index set with a total order $\leq$.

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- A second component $\tilde{y}_{t+1}^{T}$ that measures the effects on demographic profit of both volatility of insured sums and the difference between second-order demographic assumptions included in the BE calculation made in t and ones made in $t+1$. This component quantifies the effects of a variation in the insurance company's realistic assumptions about the mortality curve. For instance, it is affected by the presence of mortality or longevity trends.

$$\tilde{y}_{t+1}^{T} = \tilde{w}_{t+1} \left[be_{t+1}^{Rf(t+1),q(t)} - \tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)} \right]. \tag{5}$$

- A third component, previously defined $\tilde{y}_{t+1}^{NDM}$, that depends exclusively on the variability of lapses.

It is noticeable how SCRs for idiosyncratic and trend risks can be assessed by exploiting formulas (3) and (4). Third component is instead not relevant for mortality/longevity risk assessment.

3. The Stochastic Framework

3.1. The cohort approach

In this section, we present the general aspects of the model, in particular we follow the so-called cohort approach. Let us assume that the portfolio is divided into sub-portfolios of homogeneous risks. In this case, each policyholder within the same cohort has the same age, the same gender, the same survival probability and so on: the only difference between policyholders within the same cohort concerns the sums insured, denoted with C_i for the policyholder i.

The use of this assumption has two important consequences. On the one hand, it implies that the aggregation and dependencies between the different cohorts must be specifically modelled. On the other hand, it is possible to describe the survival of each policyholder in a given time span with a dichotomous random variable as a Bernoulli distribution. Additionally, the cohort approach is consistent with the framework defined by Solvency II and IFRS17.

We denote with $w_0 = \sum_{i=0}^{l_0} C_i$ the total sums insured of a cohort with l_0 policyholders at the inception of the contract $t = 0$. We assume that the sums insured of a cohort follow the following rule over time:

$$\tilde{w}_t = \tilde{w}_{t-1} - \tilde{s}_t - \tilde{z}_t \tag{6}$$

where $\tilde{z}_t$ are the sums insured of occurred deaths between $t-1$ and t.

We follow an individual approach and we consider a 1-year time horizon. We describe the lapse of each individual policyholder as a Bernoulli distribution with a parameter δ_t that represents the expected annual lapse rate at time t. Lapses are here assumed independent and identically distributed random variables. Therefore, considering a cohort of policyholders it is possible to obtain the following relations (representing the mean, the variance and the skewness, respectively) related to the r.v. number of lapses $\tilde{r}_{t+1}$ during the period $(t, t+1]$ in the cohort:

$$\begin{aligned} \mathbb{E}\left[\tilde{r}_{t+1}\right] &= l_t \cdot \delta_t \\ \sigma^2[\tilde{r}_{t+1}] &= l_t \cdot \delta_t \cdot (1-\delta_t) \\ \gamma\left[\tilde{r}_{t+1}\right] &= \frac{(1 - 2 \cdot \delta_t)}{\sqrt{l_t \cdot \delta_t \cdot (1-\delta_t)}} \end{aligned} \tag{7}$$

In an analogous way, we define the random variable $\tilde{d}_{t+1}$ number of deaths during the period $(t, t+1]$ as the sum of Bernoulli random variables, each one with parameter $q_t \cdot (1-\delta_t)$:

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$$\begin{aligned}
&\mathbb{E}\left[\tilde{d}_{t+1}\right] = l_t \cdot q_t \cdot (1-\delta_t) \\
&\sigma^2[\tilde{d}_{t+1}] = l_t \cdot q_t \cdot (1-\delta_t) \cdot [1 - q_t \cdot (1-\delta_t)] \\
&\gamma[\tilde{d}_{t+1}] = \frac{[1 - 2 \cdot q_t \cdot (1-\delta_t)]}{\sqrt{l_t \cdot q_t \cdot (1-\delta_t) \cdot [1 - q_t \cdot (1-\delta_t)]}}
\end{aligned} \tag{8}$$

Considering a generic policyholder i and a specific time period (e.g. $(t, t+1]$), we define the moment generating function $M_{\tilde{s}_i}(s)$ of the r.v. $\tilde{s}_i$ that denotes the sum insured eliminated due to lapses of the i-th policyholder: in this context the insured sum is not unitary, but equal to C_i. Notice that for the sake of simplicity, we neglected the notation related to the time period. Hence:

$$M_{\tilde{s}_i}(s) = 1 - \delta + e^{s \cdot C_i} \cdot \delta \tag{9}$$

Considering now the whole cohort, under the assumption of i.i.d. of the policyholders, we obtain the cumulant generating function of the r.v. s:

$$\Psi_{\tilde{s}}(s) = \sum_{i=1}^{l} ln(1 - \delta + e^{s \cdot C_i} \cdot \delta) \tag{10}$$

The characteristics of the distribution of $\tilde{s}_{t+1}$ are easily obtained:

$$\begin{aligned}
&\mathbb{E}\,[\tilde{s}_{t+1}] = l_t \cdot \delta_t \cdot \bar{C}_{t+1} = \mathbb{E}\,[\tilde{r}_{t+1}] \cdot \bar{C}_{t+1} \\
&\sigma^2[\tilde{s}_{t+1}] = l_t \cdot \delta_t \cdot (1-\delta_t) \cdot \bar{C}^2_{t+1} = \sigma^2[\tilde{r}_{t+1}] \cdot (\bar{C}_{t+1})^2 \cdot r_{2,C_{t+1}} \\
&\gamma[\tilde{s}_{t+1}] = \frac{(1 - 2 \cdot \delta_t)}{\sqrt{l_t \cdot \delta_t \cdot (1-\delta_t)}} \cdot \frac{\bar{C}^3_{t+1}}{(\bar{C}^2_{t+1})^{3/2}} = \gamma[\tilde{r}_{t+1}] \cdot \frac{r_{3,C_{t+1}}}{r^{3/2}_{2,C_{t+1}}}
\end{aligned} \tag{11}$$

where $\bar{C}^n_{t+1}$ is the simple moment of order n and the risk indices of order n are defined as $r_{n,C_{t+1}} = \frac{\bar{C}^n_{t+1}}{(\bar{C}_{t+1})^n}$. Similarly, for the r.v. sums insured in case of death, we define the cumulant generating function of the random variable $\tilde{z}$. Notice that for the sake of simplicity, we neglected the notation related to the time period.

$$\Psi_{\tilde{z}}(s) = \sum_{i=1}^{l} ln\Big(1 - ((1-q) \cdot \delta) + e^{s \cdot C_i} \cdot q \cdot (1-\delta)\Big) \tag{12}$$

As for the r.v. sums insured of occurred lapse, the following cumulants are obtained for $\tilde{z}_{t+1}$:

$$\begin{aligned}
&\mathbb{E}\,[\tilde{z}_{t+1}] = \mathbb{E}\left[\tilde{d}_{t+1}\right] \cdot \bar{C}_{t+1} \\
&\sigma^2[\tilde{z}_{t+1}] = \sigma^2[\tilde{d}_{t+1}] \cdot \bar{C}^2_{t+1} \cdot r_{2,C_{t+1}} \\
&\gamma[\tilde{z}_{t+1}] = \gamma[\tilde{d}_{t+1}] \cdot \frac{r_{3,C_{t+1}}}{r^{3/2}_{2,C_{t+1}}}
\end{aligned} \tag{13}$$

We observe that the expected value of the sums insured eliminated due to deaths increases on average with both the number of deaths and the amounts of sums insured. Volatility increases

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both as a function of the variance of the number of deaths and as the relative volatility of the sums insured increases. The sign of the skewness index depends exclusively on the sign of the skewness of the number of deaths.

3.2. Idiosyncratic risk

In this section, we consider formula (4) and we study the exact moments of the distribution in order to identify a SCR linked to the risk that a change in mortality rates will bring unexpected losses to the insurance company.

In order to clearly expose the characteristics of the r.v. $\tilde{y}_{t+1}^{Id}$,we will introduce a lemma aimed at defining the expected value and then we will introduce a more compact formulation of $\tilde{y}_{t+1}^{Id}$ from which to easily get closed formulations of standard deviation and skewness.

Lemma 1. *Considering a generic risk-free rate curve Rf(t) and a generic second-order demographic assumption q(t), regardless of the first-order pricing bases (demographic assumption q^* and technical rate j^*), when $t = 0$ it is possible to define:*

$$\mathbb{E}\left[\tilde{y}_1^{Id}\right] = -be_{0+}^{Rf(0),q(0)} \cdot \Big(1 + i_0(0,1)\Big) \cdot (w_0 - \mathbb{E}\left[\tilde{s}_1\right]) \tag{14}$$

where $be_{0+}^{Rf(0),q(0)}$ is the expected present value of the benefits net of the expected present value of the premiums, calculated in $t = 0$ using as demographic base q(0) and Rf(0) as risk-free discount curve.

Whereas when $t > 0$:

$$\mathbb{E}\left[\tilde{y}_{t+1}^{Id}\right] = (be_t^{Rf(t),q(t)} + \pi) \cdot (1 + i_t(t, t+1)) - p_{x+t} \cdot be_{t+1}^{Rf(t),q(t)} - q_{x+t} = 0 \tag{15}$$

For $t > 0$, formula (15) is a recursive equation; therefore, if there are no changes in second-order demographic bases (i.e. the company does not change its expectations on future mortality), $\mathbb{E}\left[\tilde{y}_{t+1}^{Id}\right]$ is equal to 0.

When $t = 0$, formula (14) of Lemma 1 is proved with following simple algebra:

$$\begin{aligned}\mathbb{E}\left[\tilde{y}_1^{Id}\right] = (w_0 - \mathbb{E}\left[\tilde{s}_1\right]) \cdot (0 + \pi) \cdot \Big(1 + i_0(0,1)\Big) + \\ - \left[\mathbb{E}\left[\tilde{w}_1\right] \cdot be_1^{Rf(1),q(0)} + \mathbb{E}\left[\tilde{x}_1\right]\right].\end{aligned} \tag{16}$$

because $be_0^{Rf(0),q(0)} = 0$ in $t = 0$.

Since for an endowment $\tilde{x}_{t+1} = \tilde{z}_{t+1}$[4] and $\mathbb{E}\left[\tilde{z}_{t+1}\right] = q_{x+t} \cdot \mathbb{E}\left[\tilde{w}_t - \tilde{s}_t\right]$, we have

$$\begin{aligned}\mathbb{E}\left[\tilde{y}_1^{Id}\right] = (w_0 - \mathbb{E}\left[\tilde{s}_1\right]) \cdot \\ \left[(0 + \pi) \cdot \Big(1 + i_0(0,1)\Big) - \Big(p_x \cdot be_1^{Rf(1),q(0)} + q_x\Big)\right]\end{aligned} \tag{17}$$

[4]Using an endowment policy is a sufficient condition to verify the validity of any other policy: if we had considered a pure endowment (or an annuity) $\tilde{x}_{t+1} = 0$, if instead we had considered a term insurance $\tilde{x}_{t+1} = \tilde{z}_{t+1}$.

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For an endowment contract without lapses, we can rewrite (17) in the following way:

$$\begin{aligned}\mathbb{E}\left[\tilde{y}_1^{Id}\right] = (w_0 - \mathbb{E}\left[\tilde{s}_1\right]) \cdot \Bigg(\pi \cdot \Big(1 + i_0(0,1)\Big) - \Bigg[p_x \cdot \\ \Bigg({}_{n-1}p_{x+1}\left[\prod_{h=1}^{n-1}(1 + i_1(0,h,h+1))\right]^{-1} + \\ + \sum_{k=0}^{n-2}\left({}_{k/1}q_{x+1} \cdot \left[\prod_{h=1}^{k+1}(1 + i_1(0,h,h+1))\right]^{-1}\right) + \\ - \pi \cdot \ddot{a}_{(x+1):(n-1)}\Bigg) + q_x\Bigg]\Bigg)\end{aligned} \tag{18}$$

The BE at $t = 0^+$ can be defined as follows:

$$\begin{aligned}be_{0^+}^{Rf(0),q(0)} = {}_np_x\left[\prod_{h=0}^{n-1}(1 + i_0(0,h,h+1))\right]^{-1} +_{/1} q_x \cdot (1 + i_0(0,0,1))^{-1} + \\ \sum_{k=1}^{n-1}{}_{k/1}q_x\left[\prod_{h=0}^{k}(1 + i_0(0,h,h+1))\right]^{-1} - \pi \cdot \sum_{h=1}^{n-1}{}_hE_x - \pi\end{aligned} \tag{19}$$

Therefore, from (18) and (19), we have formula (14).

It is noteworthy that formula (14) defines the profit or loss that is released at the inception of the contract. Demographic profit or loss is indeed generated by the differences between the life table used for computing premiums and the assumptions used for the computation of the BE ($be_{0^+}^{Rf(0),q(0)}$ and $be_1^{Rf(1),q(0)}$). Obviously in case no differences are observed, also the expected value in formula (14) is zero since $be_{0^+}^{Rf(0),q(0)}$ is zero.

We focus now on the other characteristics of the distribution of the r.v. $\tilde{y}_{t+1}^{Id}$, especially standard deviation and skewness.

As proved in Appendix A, it is possible to rewrite the profit (loss) deriving from idiosyncratic risk only as:

$$\tilde{y}_{t+1}^{Id} = D_{t+1}^C\Big[q_t(w_t - \tilde{s}_{t+1}) - \tilde{z}_{t+1}\Big]. \tag{20}$$

where D_{t+1}^C is the complete expenses of sum-at-risk (SAR) rate. It has been introduced in order to consider both life and death insurance policies in the same formula. It is defined as:

$$D_{t+1}^C = \begin{cases}(1 - be_{t+1}), & \text{for term insurance and endowment policies;} \\ -be_{t+1}, & \text{for pure endowment policies;} \\ -be_{t+1}, & \text{for annuity in accumulation period (t} \leq \text{m) ;} \\ -(1 + be_{t+1}), & \text{for annuity in the payment period (t} \geq \text{m).}\end{cases} \tag{21}$$

From formula (20), it is possible to observe that the random variable $\tilde{y}_{t+1}^{Id}$ can be calculated as the difference of two r.v.s $D_{t+1}^C \cdot q_t \cdot \tilde{s}_{t+1}$ and $D_{t+1}^C \cdot \tilde{z}_{t+1}$. Since we want to focus on the assessment of the SCR related to changes in mortality rates, we neglect the volatility of lapses. Indeed, such volatility would include a portion of the capital related to the lapse risk. Treating lapses as deterministic, we can assess the standard deviation of the demographic profit linked to the idiosyncratic

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risk as follows:

$$\begin{aligned}\sigma(\tilde{y}^{Id}_{t+1}) &\approx D^C_{t+1}\cdot\sigma(\tilde{z}_{t+1})\\ &=|D^C_{t+1}|\cdot\sqrt{\sigma^2(\tilde{d}_{t+1})\cdot\bar{C}^2_{t+1}\cdot r_{2,C_{t+1}}}\\ &=\left(|D^C_{t+1}|\cdot\bar{C}_{t+1}\cdot l_t\right)\cdot\sqrt{\frac{q_t\cdot(1-q_t)}{l_t}\cdot r_{2,C_{t+1}}}\end{aligned}\tag{22}$$

Therefore, considering formula (22) it is observed that the variability is a function of the SAR of the policies in force at time $t+1$ (i.e. the term $|D^C_{t+1}|\cdot\bar{C}_{t+1}\cdot l_t$) of the probability of death of the policyholders within the cohort and to the variability of the insured sums.

In an analogous way, assuming that lapses are deterministic, we derive the skewness as follows:

$$\begin{aligned}\gamma(\tilde{y}^{Id}_{t+1}) &\approx -\frac{(D^C_{t+1})^3}{|D^C_{t+1}|^3}\cdot\gamma(\tilde{z}_{t+1})\\ &=-\frac{(D^C_{t+1})^3}{|D^C_{t+1}|^3}\cdot\gamma(\tilde{d}_{t+1})\cdot\frac{r_{3,C_{t+1}}}{(r_{2,C_{t+1}})^{3/2}}\\ &=-\frac{(D^C_{t+1})^3}{|D^C_{t+1}|^3}\cdot\frac{(1-2q_t)}{\sqrt{l_t\cdot q_t\cdot(1-q_t)}}\cdot\frac{r_{3,C_{t+1}}}{(r_{2,C_{t+1}})^{3/2}}\end{aligned}\tag{23}$$

By exploiting the assumptions made in section 3.1, it is possible to simulate the distribution of $\tilde{y}^{Id}_{t+1}$. In this case, the capital requirement for idiosyncratic risk SCR_{Id} is defined as:

$$\begin{aligned}SCR^{Id}_{t+1}=VaR_{0.5\%}\Bigg((w_t-\tilde{s}_{t+1})\cdot\left[be_t^{Rf(t),q(t)}+\pi\right]\cdot(1+i_t(t,t+1))+\\ -\left[\tilde{w}_{t+1}\cdot be_{t+1}^{Rf(t+1),q(t)}+\tilde{x}_{t+1}\right]\Bigg).\end{aligned}\tag{24}$$

where $VaR_{0.5\%}$ is the 0.5% quantile of the distribution based on value at risk at the 99.5% confidence level. It is also interesting to notice that an undertaking specific parameter approach (USP) can be provided by making use of the exact characteristics of the r.v. $\tilde{y}^{Id}_{t+1}$. In particular, we propose to evaluate the capital requirement for idiosyncratic risk by means of a USP approach as follows:

$$SCR^{Id,USP}_{t+1}=k[\gamma(\tilde{y}^{Id}_{t+1})]\cdot\sqrt{\frac{q_t\cdot(1-q_t)}{l_t}}\cdot r_{2,C_{t+1}}\cdot\left(|D^C_{t+1}|\cdot\bar{C}_{t+1}\cdot l_t\right)\tag{25}$$

where $k[\gamma(\tilde{y}^{Id}_{t+1})]$ is a proper multiplier of the standard deviation calibrated according to the skewness of the distribution.

It is noteworthy that formula (25) shows some similarities with the factor-based approach provided for the assessment of idiosyncratic component of longevity risk in quantitative impact study 2 (see Committee of European Insurance and Occupational Pensions Supervisors, 2006). It was defined as follows:

$$SCR^{Id,QIS2}=2.58\cdot\sqrt{\frac{q\cdot(1-q)}{l}}\cdot Sum-at-Risk.\tag{26}$$

By comparing formulas (26) and (25), it is worth pointing out that $SCR^{Id,USP}_{t+1}$ can be seen as an extension of the proposal of QIS2. In particular, $SCR^{Id,QIS2}$ is based on the assumption of

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a Gaussian distribution (given a multiplier equal to 2.58). According to formula (23), we have instead that the distribution is typically skewed, with the sign of the skewness related to the sign of the SAR. Therefore, for policies with a positive SAR we expect a value of $k[\gamma(\tilde{y}_{t+1}^{Id})]$ higher than 2.58. Vice versa for policies subject to longevity risk, the multiplier 2.58 represents an overestimation. Additionally, $SCR^{Id,QIS2}$ neglects the volatility of the sums insured $r_{2,C}$ that is instead considered in $SCR_{t+1}^{Id,USP}$. This volatility, often neglected in the literature, could have indeed a relevant impact on the capital requirement.

3.3. Trend risk

In this last subsection, our purpose is to calculate the SCR linked to the random variable demographic profit (loss) of formula (5). In particular, we want to quantify the amount of capital related to changes in mortality rates adopted for the computation of the BE. In this adverse scenario, the insurer will have to use the funds allocated at the beginning of the year to finance the reserve jump from $be_{t+1}^{Rf(t+1),q(t)}$ to $\tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)}$ that occurs at the end of the year, that is, when the new technical provisions are calculated with the new demographic technical bases $\tilde{q}(t+1)$. In this context, the crucial element is the estimation of the distribution of the probabilities $\tilde{q}(t+1)$ that can be done with a proper forecasting model. For instance, if we assume to use the well-known Lee–Carter model (see Lee & Carter, 1992), it is possible to estimate the value at risk of the mortality (or survival) rates at the chosen confidence level $\tilde{q}^{99.5\%}(t+1)$ (or $\tilde{p}^{99.5\%}(t+1)$).

Focusing now on trend risk, we can define the capital requirement as:

$$SCR_{t+1}^{T} = \mathbb{E}\left[\tilde{w}_{t+1}\right] \cdot \left[be_{t+1}^{Rf(t+1),q(t)} - be_{t+1}^{Rf(t+1),\hat{q}_{t+1}^{99.5\%}}\right]. \tag{27}$$

It should be noted that this component requires stressing only the BE because it aims at quantifying the risk associated with the variation of the insurance company's realistic assumptions regarding mortality rates. Formula (27) focuses on the effect on the BE of stressed mortality (or longevity) rates. SCR_{t+1}^{T} depends also on the expected value of the sums insured since the variability of sums insured is already included in the capital requirement for idiosyncratic risk.

As for the idiosyncratic risk, the evaluation of the SCR is to be carried out on a 1-year time horizon. Therefore, we define the natural filtration representing the information available at the valuation date. To calculate future BEs, the natural filtration is expanded including the average deaths calculated with the second-order demographic basis. The distribution of deaths at time $t+1$ involved in the capital requirement assessment has been obtained by applying the Bootstrap model (see Brouhns *et al.*, 2005). In this way, we are able to disentangle the separate effects of idiosyncratic and trend risk. This approach is indeed consistent with the assumption that any fluctuations in deaths from the expected value fall under the definition of idiosyncratic risk. It is also noteworthy that the same approach can be applied also considering alternative models for measuring the distribution of future mortality rates (e.g. Bauer & Ha, 2015).

4. Numerical Results

In this section, we present the results of the application of the described model to two life insurance traditional (i.e. without profit) products available on the market: pure endowment and term insurance. We also observe that other products on the market can be described as the linear combination of the aforementioned policies, for example, an endowment can be obtained as the combination of a pure endowment and a term insurance with the same maturity and the same sum insured.

In Table 1, we present the characteristics of the cohort of policyholders (gender, age, number, sums insured), of their policies (type, duration, duration of the premium payments) and of the

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Table 1. Model parameters of both pure endowment and term insurance.

Year of birth of male policyholders	1978
Instant of evaluation and country	Italy, 2018 ($t = 0$)
Policy duration	20
Premium type	Annual premiums (20)
Number of policyholders (l_0)	15,000
Expected value of the single insured sum	100,000
CoV of the insured sums	1.99
Risk-free rates	1% (flat)
First-order financial rate j^*	1%

market (risk-free rates). We assume a flat risk-free rates equal to the first-order financial rate. However, similar results can be obtained considering a risk-free rate curve. This assumption has been made to avoid affecting the demographic results with the effect of the financial rates.

This section is organised in two subsections relating, respectively, to the analysis of idiosyncratic and trend risk. In each subsection, the results of the model will be presented with reference to a pure endowment and a term insurance on three distinct time horizons: $t = 0$, the first year for which profits and losses are calculated, $t = 9$, at the half of the coverage period and in $t = 19$, i.e. in the last year.

4.1. Results on idiosyncratic risk

As mentioned, we start the discussion of a pure endowment policy in which parameters are described in Table 1. With reference to the demographic basis used in the pricing phase, the so-called first-order demographic basis, a mortality table with an implicit safety loading, was used.

In particular, as a second-order demographic base (i.e. as realistic assumptions of the company) regarding the mortality of the cohort born in 1978, the data from 1872 to 2018 of Italian mortality, contained in the Human Mortality Database, were processed. Using these data, a Lee–Carter model was fitted on and, after estimating the parameters of the model, the mortality rates were forecasted. Hence, in summary, the company's demographic second-order base coincides with the estimated death rates using the Lee–Carter model (see Lee & Carter, 1992).

In order to insert a safety loading, that under-expected terms bring profit to the company, we assume that the demographic base of the first order has been calculated by reducing the average death rates estimated by the Lee–Carter model by 20%.

For the convenience of the reader, Figure 1 shows the trend of the BE rate and the SAR rate for the pure endowment. With reference to the BE rate, it is observed that in $t = 0$ it (i.e. $be_{0^+}^{Rf(0),q(0)}$) assumes a negative value: at the subscription there is an expected profit since, although the technical rate of the policy coincides with the risk-free rates, the use of prudential demographic basis implies an expected profit. For this reason, the first value assumed by the SAR rate is positive, while all the others are negative and converge to -1 at the end of the contract.

In Table 2, we present the results of the stochastic model described in the previous section. First of all, we notice that the number of simulations assured a good convergence of simulated values to exact ones. It is also interesting to observe the expected value of the demographic profit (loss) linked to idiosyncratic risk. Since prudential first-order demographic bases were used in the pricing phase, the BE rate at subscription is negative and the expected profit in $t = 0$ is positive, exactly as described by formula (14). In case of a first-order technical rate different from the risk-free rate (or different from the rates of a generic risk-free rate curve), the expected value of the idiosyncratic demographic profit in $t = 0$ would have also been affected by the spread between the two rates, while the other expected values for the different time periods would remain equal to 0.

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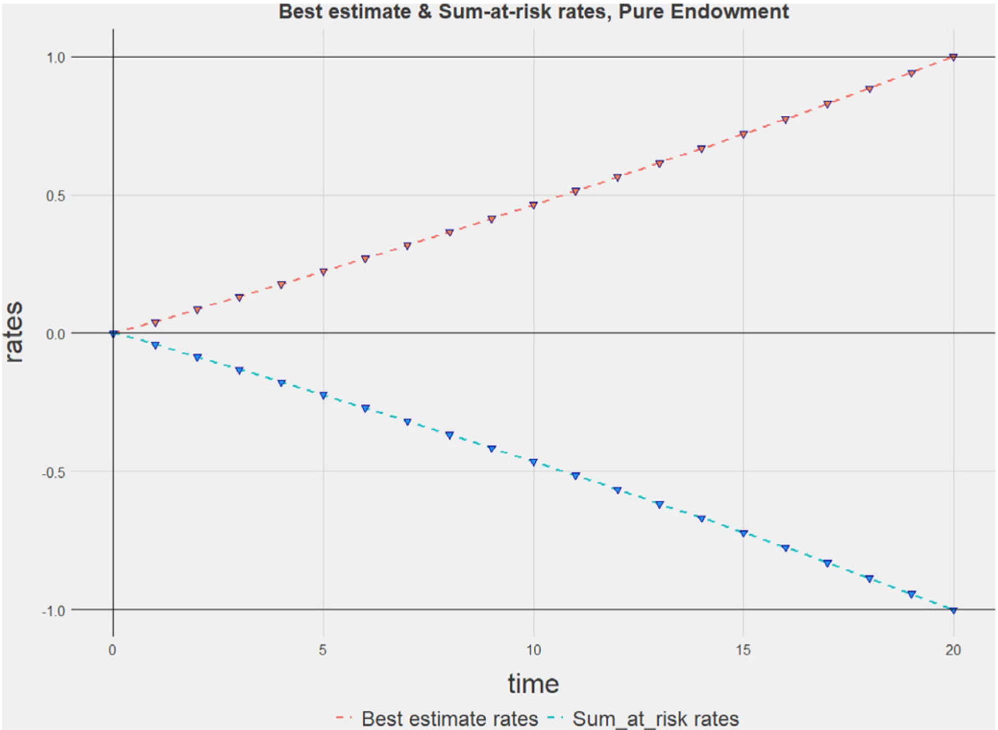

Figure 1. Best estimate and sum-at-risk rates.

Indeed, in a market consistent perspective profits are recognised when the differences between the bases of the first and second order are observed, differently from the Local GAAP context where profits are recognised only when actually realised.

With reference to the standard deviation, a strongly increasing trend is observed between the various time instants. By comparing the numerical results with formula (22), we can observe that the absolute value of D^b_{t+1} increases over time and, at the same time, we have an increasing volatility of the number of deaths. Additionally, we have a positive skewness, motivated by the negative SAR. As q increases, that is, as the cohort is getting older, the distribution becomes more asymmetric.

Figure 2 displays the behaviour of the idiosyncratic profit (loss) distribution during the coverage period. It is observed that, with the exception of the first distribution where the profits deriving from the implicit safety loading are recognised, all the distributions have average values equal to 0 and, as time increases, they become more volatile and skewed. In conclusion, it is noted that the SCR, defined as the loss that the company suffers in the worst-case scenario with a confidence level of 99.5% in the following year, is related both to the standard deviation and to the skewness. In particular, a negative capital requirement is initially derived because of a low volatility and a very large positive expected profit. Over time, the increasing volatility and the decreasing asymmetry of the left tail lead to an increase of the capital requirement related to idiosyncratic longevity risk.

Moreover, we observe that, with reference to formula (25), the multiplier $k[\gamma(\tilde{y}^{Id}_{t+1})]$ assumes values close to 1.5 in $t = 0$ and grows to 1.8 during the last year of coverage. This value is very far from the value assumed by QIS n.2 – Solvency II, equal to 2.58 because skewness of profit r.v. is rather positive and the tail is located on the right-hand side of profit distribution.

With reference to term insurance, first of all we specify that also in this context the second-order demographic basis coincides with the results obtained by the application of the Lee–Carter model.

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Table 2. Simulated and theoretical characteristics of idiosyncratic profit and loss distribution for a pure endowment contract for three different time periods. Last two rows summarise SCR and SCR ratio with respect to sums insured.

Pure endowment	$t=0$	$t=9$	$t=19$
Theoretical expected value	5,807,038	0	0
Simulated mean	5,807,024	−122	124
Theoretical standard deviation	23,115	464,869	1,657,030
Simulated standard deviation	23,091	463,478	1,632,732
Theoretical skewness	3.26	1.84	1.10
Simulated skewness	3.25	1.84	1.09
Solvency Capital Requirement	−5, 784, 999	704,780	2,985,434
SCR/Sums insured	−0.38%	0.05%	0.21%

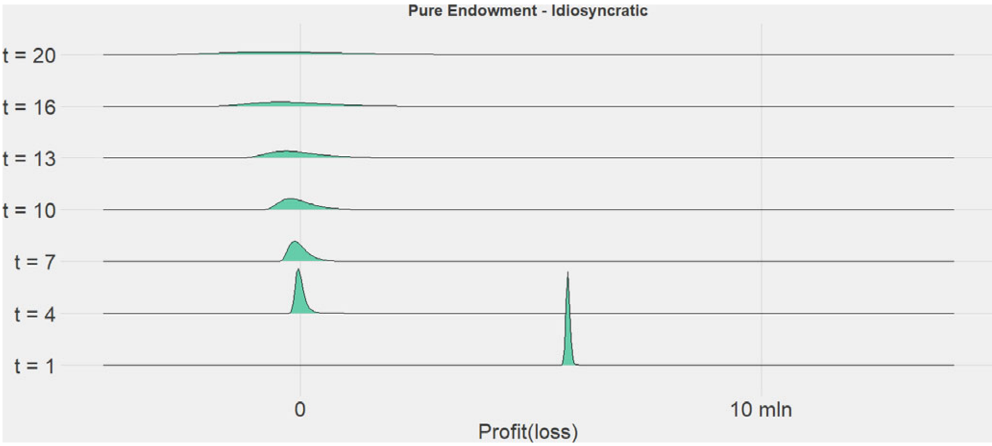

Figure 2. Distributions of idiosyncratic profit and loss for a pure endowment.

The only difference compared to pure endowment lies in the fact that for term insurance, the first-order demographic basis assumes an increase in mortality rates compared to the second-order basis, equal to 20%. Figure 3 shows the trend of the BE rate and the SAR rate for the term insurance, whose parameters are the same as for the pure endowment. On the left-hand side, the BE rate is negative at the time of subscription due to implicit safety loading and grows up to $t=14$ (where reaches the value of 0.025) and subsequently decreases to 0. On the right-hand side, we have instead the opposite behaviour of the SAR being quite always around 1 (sum insured). As well known, these patterns, very different from the pure endowment, have a crucial role in the assessment of idiosyncratic profit distribution.

Table 3 reports theoretical and simulated values of the idiosyncratic demographic profit distribution for the terms insurance. We observe a stronger effect of the safety loading on the expected value at time $t=0$ given by the higher SAR. For the same reason, we also notice a higher volatility in all the analysed periods. According to the skewness, the value is exactly the opposite of the skewness of the pure endowment. This behaviour is explained by the positive sign of the SAR and by the fact that number of policyholders and second-order bases are assumed to be the same for the two contracts.

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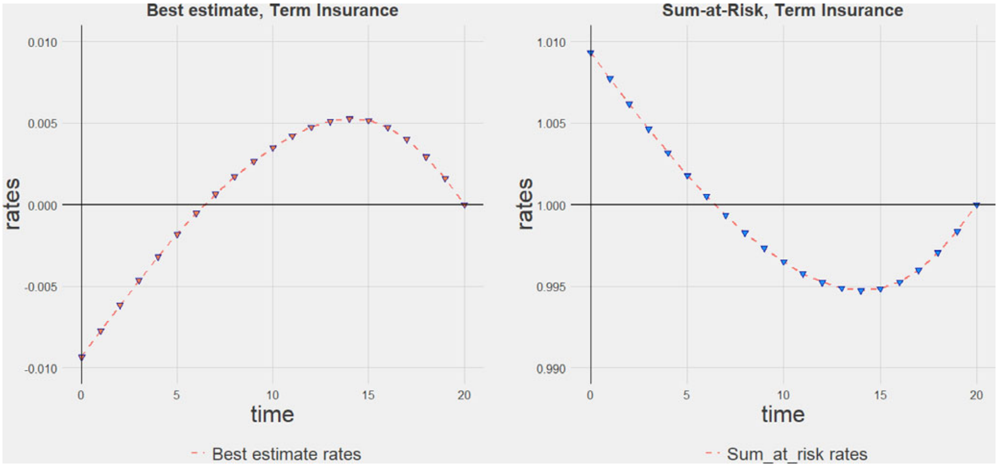

Figure 3. Best estimate and sum-at-risk rates.

Figure 4 shows the distributions of the demographic profit linked to the idiosyncratic risk for different time periods. It is interesting to compare these distributions with the pure endowment (Figure 2). It is observed that these distributions have different shapes, as the volatility of the term insurance is greater and the skewness is negative. This latter element also greatly influences the 0.5% percentile of the distribution necessary to quantify the SCR. Indeed, we obtain a multiplier $k[\gamma(\tilde{y}^{Id}_{t+1})]$ that varies between 3.7 and 4.5. Hence, we observe that the solution adopted by QIS n.2 for measuring the capital requirement of this risk provides in this case a significant underestimation of capital neglecting the variability of sums insured and assuming a Gaussian distribution for $\tilde{y}^{Id}_{t+1}$.

4.2. Results on trend risk

In this section, we focus on formula (5) and we aim at estimating the BE rate at the end of the year in the worst-case scenario at a confidence level of 99.5%.

Firstly, we specify that the estimate of the stressed BE rate is not bound to the use of a specific model. We assume here to forecast the distribution of future mortality rates following the approach provided in Brouhns *et al.* (see Brouhns *et al.*, 2005). This approach allows to consider the uncertainty in the estimates related to both trend changes and accidental deviations.

We briefly recall the methodology used to estimate the distribution of future mortality force $\mu_x(t)$:

- We assume that the number of deaths follows a Poisson distribution based on the following assumptions:

$$D_{x,t} \sim Poisson\left(E_{x,t}\mu_x(t)\right) \qquad (28)$$
$$\mu_x(t) = exp(\alpha_x + \beta_x\kappa_t)$$

  where $E_{x,t}$ is used to indicate the exposure-to-risk at age x during calendar year t, that is, the total time lived by people aged x in year t and where $\mu_x(t)$ is described by the Lee–Carter model (see Lee & Carter, 1992). α_x describes the general shape of mortality according to different ages. Furthermore, κ_t reproduces the underlying time trend, while the term β_x is considered in order to take into account the different effect of time t at each age.

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Table 3. Simulated and theoretical characteristics of idiosyncratic profit and loss distribution for a term insurance contract for three different time periods. Last two rows summarise SCR and SCR ratio with respect to sums insured.

Term insurance	$t = 0$	$t = 9$	$t = 19$
Theoretical expected value	8,696,520	0	0
Simulated mean	8,696,373	−49	331
Theoretical standard deviation	569,948	995,523	1,657,030
Simulated standard deviation	570,607	991,701	1,632,169
Theoretical skewness	−3.26	−1.84	−1.10
Simulated skewness	−3.25	−1.83	−1.09
Solvency Capital Requirement	−5, 826, 888	4,444,835	6,057,733
SCR/Sums insured	−0.38%	0.29%	0.41%

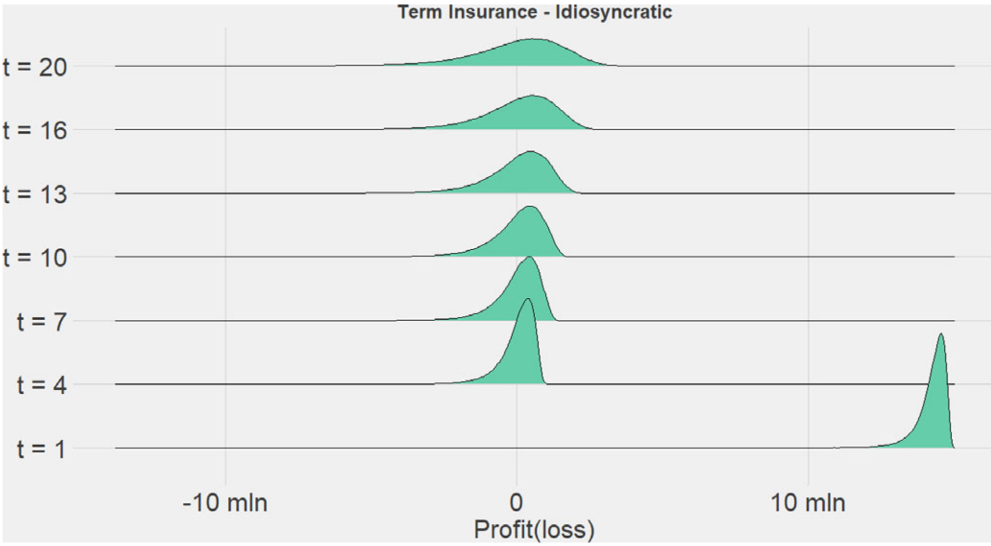

Figure 4. Distributions of idiosyncratic profit and loss for a term insurance.

- Instead of resorting to singular value decomposition, we estimate the model parameters by maximising the log-likelihood function defined as:

$$L(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}) = \sum_t \sum_x \left(D_{x,t}(\alpha_x + \beta_x \kappa_t) - E_{x,t} e^{(\alpha_x + \beta_x \kappa_t)} \right) + constant \tag{29}$$

  Vector of parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\kappa}$ is obtained by an iterative algorithm and under properly constraints to assure a unique solution: $\sum_t k_t = 0$ and $\sum_x \beta_x = 1$.
- Finally, Box–Jenkins methodology is used to generate the appropriate ARIMA time series model for forecasting future values of k_t. In particular, the parameter κ_t is considered as a discrete stochastic process of the ARIMA type (ARIMA (0,1,0) for men and ARIMA (0,1,1) for women, respectively). The methodology allows to obtain an estimate $\hat{\mu}_x(t)$ for each age x and time t.

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Table 4. Simulated and theoretical characteristics of trend profit and loss distribution for a pure endowment contract for three different time periods. Last two rows summarise SCR and SCR ratio with respect to sums insured.

Pure endowment	$t=0$	$t=9$
Be rate	0.04	0.46
Simulated mean	0.04	0.46
Simulated standard deviation	0.25%	0.14%
Simulated skewness	−0.43	−0.29
Solvency Capital Requirement	7,843,214	5,064,797
SCR/Sums insured	0.52%	0.33%

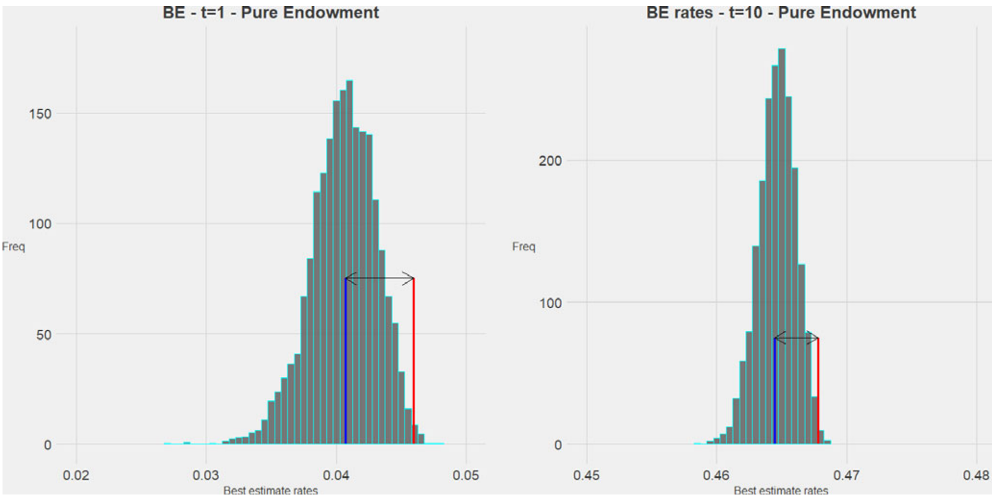

Figure 5. Distribution of BE rates at time 1 and time 10 for a pure endowment. Blue lines represent the expected values of the distributions, therefore $be_1^{Rf(1),q(0)}$ for the left figure and $be_{10}^{Rf(10),q(9)}$ for the right figure. The red lines indicate the 99.5% percentile of the distributions, and then the black lines indicate the spreads between the expected values and the stressed values indicate the SCRs.

- The procedure is iterated n times, following the parametric bootstrap proposed in Brouhns *et al.* (2005). Bootstrap samples have been derived by simulating $D^n_{x,t}$ from a Poisson:

$$D^n_{x,t} = E_{x,t} \cdot \mu_x(t) \tag{30}$$

  Therefore, we add a heteroskedastic noise that will consider greater volatility for the most extreme ages. Given the new set of deaths $D^n_{x,t}$ obtained in each iteration, the Lee–Carter methodology is applied in order to estimate $\hat{\mu}^n_x(t)$. The procedure involves the estimation of n Poisson Lee–Carter models and as many forecasts. It is time-consuming but allows to quantify the two volatilities previously mentioned in the calculation of future BE rates.

We now show the results obtained considering the two policies described in previous subsection at the time horizons $t=0$ and $t=9$. It is important to underline that our aim is to include in the valuation only the 1-year volatility to be consistent with the Solvency II framework. To this end, we applied the Lee–Carter model using deaths and exposures for ages 0–100 and calendar years 1872–2018 and we forecast the distribution of mortality force for next 20 years. A higher time span is not needed given the duration of the contracts. Therefore, the Lee–Carter model has

Table 5. Simulated and theoretical characteristics of trend profit and loss distribution for a term insurance contract for three different time periods. Last two rows summarise SCR and SCR ratio with respect to sums insured.

Term insurance	$t = 0$	$t = 9$
Be rate	−0.4%	0.5%
Simulated mean	−0.4%	0.5%
Simulated standard deviation	0.34%	0.17%
Simulated skewness	0.33	0.19
Solvency Capital Requirement	15,382,318	6,618,003
SCR/Sums insured	1.02%	0.44%

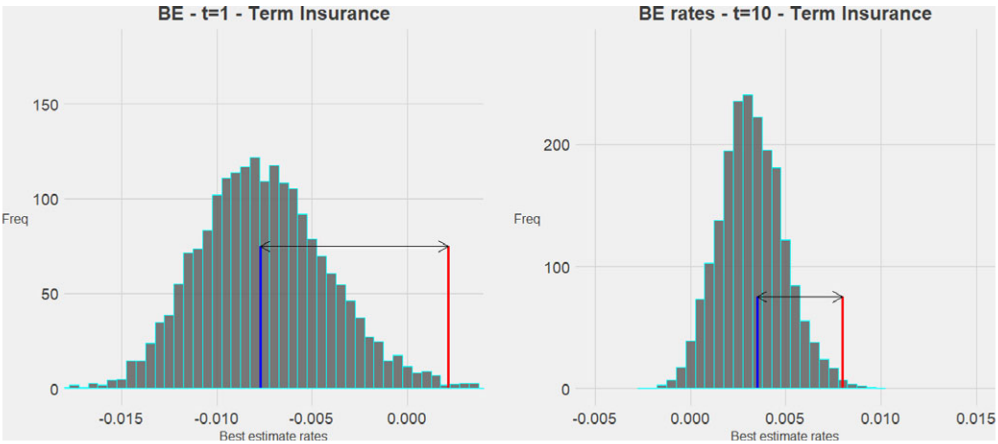

Figure 6. Distribution of BE rates at time 1 and time 10 for a term insurance. Blue lines represent the expected values of the distributions, therefore $be_1^{Rf(1),q(0)}$ for the left figure and $be_{10}^{Rf(10),q(9)}$ for the right figure. The red lines indicate the 99.5% percentile of the distributions, and then the black lines indicate the spreads between the expected values and the stressed values indicate the SCRs.

been used to obtain the expected value $\hat{\mu}_x^n(t)$ for the coverage period of the contract (i.e., the time period 2019–2028). Moreover, to catch the 1-year distribution of $\hat{\mu}_x^n(t)$ for the different times considered (i.e., $t = 0$ and $t = 9$), we applied the Bootstrap methodology with 50,000 iterations to a specific training dataset built assuming that the deaths follow the expected predicted behaviour up to the time t. In this way, we catch only the volatility in the period $t, t + 1$.

Table 4 reports exact and simulated characteristics of the profit and loss distribution with regard to the trend component. We observe that, given the high number of simulations, a good convergence of the simulated distribution is assured. Moreover, although the SAR rate is close to 0 when $t = 0$, a change in the expected mortality curve systematically affects all future expected cash flows, in fact in the worst-case scenario there is a SCR equal to almost 8 million. Over the time, we have a reduction of the volatility of the BE (see Figure 5) with a reduction of the SCR.

Table 5 and Figure 6 present the results deriving from the application of the model to a term insurance and, subsequently, the shapes of the distributions of the BE rates at the two times $t = 0$ and $t = 9$. Comparing the results of a term insurance with those obtained for the pure endowment, we observe that also in this case the volatility of the BE is particularly high in the first periods and then decreases until it reaches zero at maturity. Nevertheless, we observe that the ratio between the standard deviation and the expected BE rate is higher in the case of term insurance for the whole period. Furthermore the skewness is positive. These effects imply higher amount of capital

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and higher SCR ratios. The justification for these results lies in the fact that the annual survival probabilities of the policyholders are close to one, therefore the "best case scenario" that concerns the case in which the entire cohort survives, truncates the distribution on the left, while the right tail concerns the less probable cases in which more deaths occur.

5. Conclusions

This paper proposes a cohort approach for the assessment of capital requirement of traditional life insurance policies in a market consistent framework, as requested by Solvency II. We provide a unified stochastic framework for quantifying the capital related to mortality and longevity risk, taking into account separately idiosyncratic and trend risk.

In both cases, Monte Carlo approach can be used to evaluate the characteristics of the profit and loss distribution and to assess the capital requirement. For idiosyncratic risk, we propose a factor-based formula that could be used to quantify the capital requirement. We show how this formula overcomes some pitfalls of the closed formula given by QIS n.2 – Solvency II. A specific case study has been developed in the numerical section providing insights about the behaviour of the models for two specific life insurance contracts.

The model assumes to split the whole portfolio into cohorts of homogeneous risks. From a practical point of view, this is certainly complex since the cohorts can be hundreds (thousands in the most large cases). Despite this, the model adapts perfectly to the operational reality. On the one hand, it is observed that the increased computational capacity allows to operate even at the level of a single policy. On the other hand, it is possible to divide the undertaking portfolio in order to build clusters capable of identifying model points with a certain degree of approximation. They are defined as policies representative of the single clusters and the model is suitable for assessing their risk. Using this approach, it is therefore possible to assess the risk of the overall portfolio as a function of the risk of each model point. Further research will regard the development of the model considering the effects related to the aggregation and the management of heterogeneous portfolios, also connected with the natural hedging coming from the mixture of death-linked and survival-linked insurance products.

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Appendix A. Idiosyncratic Profit: A More Compact Version

Let us consider the formulation of the demographic profit arising from idiosyncratic risk as formulated in formula (4).

We break down the best estimate rate into a pure component be_t^P (expected present value of benefits net of expected present value of premiums) and a component relating to expenses be_t^E (expected present value of expenses net of expected present value of loadings per expense). In order to make the notation more usable, we do not report the subscripts related to the demographic and financial bases.

$$\begin{aligned}\tilde{y}_{t+1}^{Id} =&(w_t - \tilde{s}_{t+1}) \cdot \left[be_t^P + be_t^E + \pi \right] \cdot (1 + i_t(t, t+1)) + \\ &- \left[\tilde{w}_{t+1} \cdot (be_{t+1}^P + be_{t+1}^E) + \tilde{x}_{t+1} \right]\end{aligned} \tag{A.1}$$

Considering relation (6), we obtain

$$\begin{aligned}\tilde{y}_{t+1}^{Id} =&(w_t - \tilde{s}_{t+1}) \cdot \left[be_t^P + be_t^E + \pi \right] \cdot (1 + i_t(t, t+1)) + \\ &- \left[(w_t - \tilde{s}_{t+1} - \tilde{z}_{t+1}) \cdot (be_{t+1}^P + be_{t+1}^E) + \tilde{x}_{t+1} \right]\end{aligned} \tag{A.2}$$

By simple steps, we have

$$\begin{aligned}\tilde{y}^{Id}_{t+1} =& \left[(be^P_t + \pi)(1 + i_t(t, t+1)) - be^P_{t+1}\right](w_t - \tilde{s}_{t+1}) - \tilde{x}_{t+1} + be^P_{t+1}\tilde{z}_{t+1} \\ & - be^E_t\left[w_t - \tilde{s}_{t+1}(1 + i_t(t, t+1))\right] - be^E_{t+1}(w_t - \tilde{s}_{t+1}) + be^E_{t+1}\tilde{z}_{t+1}\end{aligned} \tag{A.3}$$

In order to further break down formula (A.3), we now present a decomposition of the pure premium rate π into the risk premium rate π^r and the rate of savings premium π^s. This is possible because formula (15) cited in Lemma 1 is a recursive equation.

They are defined as:

$$\begin{aligned}\pi^r &= q_t(I - be^P_{t+1})(1 + i_t(t, t+1))^{-1} \\ \pi^s &= be^P_{t+1}(1 + i_t(t, t+1))^{-1} - be^P_t\end{aligned} \tag{A.4}$$

where I is an indicator function equal to 1 in case of term insurance and endowment policies and equal to 0 in case of Pure endowment policies.

It follows that the π^r risk premium rate can be written as:

$$\pi^r = (\pi + be_t) - be_{t+1}(1 + i_t(t, t+1))^{-1} \tag{A.5}$$

In a completely analogous way, it is possible to define for the component relating to expenses:

$$\pi^{er} = (\pi^e + be^E_t) - be^E_{t+1}(1 + i_t(t, t+1))^{-1} \tag{A.6}$$

By formula (A.3), we can write

$$\begin{aligned}\tilde{y}^{Id}_{t+1} =& \left[\pi^r(1 + i_t(t, t+1))(w_t - \tilde{s}_{t+1})\right] - \tilde{x}_{t+1} + be^P_{t+1} + \\ & - \pi^{er}_{t+1}(1 + i_t(t, t+1))(w_t - \tilde{s}_{t+1}) + be^E_{t+1}\tilde{z}_{t+1}\end{aligned} \tag{A.7}$$

and

$$\begin{aligned}\tilde{y}^{Id}_{t+1} =& q_t(I - be^P_{t+1})(w_t - \tilde{s}_{t+1}) - \tilde{x}_{t+1} + be^P_{t+1}\tilde{z}_{t+1} \\ & - be^E_{t+1}(q_t(w_t - \tilde{s}_{t+1}) - \tilde{z}_{t+1})\end{aligned} \tag{A.8}$$

On the one hand, we now consider the case of term insurance and endowment policies, where $\tilde{x}_t = \tilde{z}_t$ and $I = 1$, and on the other hand the case of pure endowment and annuities where $\tilde{x}_t = 0$ and $I = 0$. In both cases, it is possible to rewrite the demographic profit linked to the idiosyncratic risk as:

$$\tilde{y}^{Id}_{t+1} = D^C_{t+1}\left[q_t(w_t - \tilde{s}_{t+1}) - \tilde{z}_{t+1}\right] \tag{A.9}$$

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