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SOME REMARKS ON THE Q CURVATURE TYPE PROBLEM ON $\mathbb{S}^N$

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Abstract

In this paper, we prove the existence, uniqueness and multiplicity of positive solutions of a nonlinear perturbed fourth-order problem related to the Q curvature.

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1. Introduction

In recent years, there has been an intensive study of the relationship between conformally covariant operators and partial differential equations. See some recent survey papers by Chang [8] and Chang and Yang [10]. Given a smooth four-dimensional compact Riemannian manifold (M, g), let R_g and Ric_g be the scalar curvature and the Ricci curvature of g, respectively, div_g the divergence operator and d the de Rham differential; then the Paneitz operator is defined in the following way:

$$P_g\psi = \Delta_g^2\psi - div_g(\tfrac{2}{3}R_g - 2Ric_g)\, d\psi;$$

see Paneitz [22]. For the case $N \geq 5$, the Paneitz operator P_g is defined by

$$P_g = \Delta_g^2 - div_g[a_N R_g g + b_N Ric_g] + \frac{N-4}{2}Q_g.$$

Here

$$Q_g = \frac{1}{2(N-1)}\Delta R_g + \frac{N^3 - 4N^2 + 16N - 16}{8(N-1)^2(N-2)^2}R_g^2 - \frac{2}{(N-2)^2}|Ric|^2$$

and

$$a_N = \frac{(N-2)^2 + 4}{2(N-1)(N-2)},$$

$$b_N = -\frac{4}{N-2}.$$

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When $N \geq 5$, the operator P_g has the following property: if $\overline{g} = u^{4/(N-4)}g$ is a conformal metric of g, then for all $\varphi \in C^\infty(M)$

$$P_g(\varphi u) = \varphi^{(N+4)/(N-4)} P_{\overline{g}}(u).$$

In particular,

$$P_g(\varphi) = \frac{N-4}{2} Q_{\overline{g}} \varphi^{(N+4)/(N-4)}.$$

Many interesting results on the Paneitz operator and related topics have been recently studied by Branson [5], Branson *et al.* [6], Chang and Yang [10], Gursky [18], Ben Ayed and El Mehdi [4], Chtioui and Rigane [11], Esposito and Robert [15], Sandeep [24] and many others. In particular, when $N \geq 5$, Djadli *et al.* [12] studied the coercivity of the Paneitz operator and the positivity of solutions. Moreover, Djadli *et al.* [13] and Hebey and Robert [19] studied the blow-up analysis of the Q curvature equation.

Let us now consider the question: *given a smooth function Q on $\mathbb{S}^N$ ($N \geq 5$), does there exist a metric g conformal to the standard metric g_0 such that $Q = Q_g$?*

If we assume a conformal transformation of the form $g = w^{4/(N-4)}g_0$, the answer to the above question is 'yes' if and only if we can solve for w in the equation

$$\begin{cases} P_{g_0} w = \dfrac{N-4}{2} Q(x) w^{(N+4)/(N-4)} & \text{in } \mathbb{S}^N, \\ w > 0 & \text{in } \mathbb{S}^N. \end{cases} \tag{1.1}$$

The problem of finding Q such that (1.1) possesses a solution can be seen as the generalization to the Paneitz operator of the so-called 'Nirenberg problem' Q; namely: which functions on $\mathbb{S}^N$ are the scalar curvature of a metric conformal to the standard one? The Nirenberg problem has been studied by several authors; we mention Ambrosetti *et al.* [2], Chang and Yang [10], Chang *et al.* [9] and Kazdan and Warner [20]. A detailed bibliography on the Nirenberg problem can be found in Ambrosetti and Malchiodi [3].

It can be checked that the Paneitz operator on $(\mathbb{S}^N, g_0)$ is given by

$$P_{g_0} w = \Delta^2_{\mathbb{S}^N} w - \frac{1}{2}(N^2 - 2N - 4)\Delta_{\mathbb{S}^N} w + \frac{(N-4)N(N^2-4)}{16} w. \tag{1.2}$$

Consider the inverse of the stereographic projection

$$\Pi : \mathbb{R}^N \to \mathbb{S}^N$$

given by

$$x \mapsto \left(\frac{2x}{1+|x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right).$$

The spherical metric g_0 is given in terms of the stereographic coordinate system as

$$g_0 = \frac{4\, dx^2}{(1+|x|^2)^2}.$$

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Hence, by a direct computation,

$$P_{g_0}\Phi(u) = \Big(\frac{1+|x|^2}{2}\Big)^{(N+4)/2}\Delta^2 u \quad \text{for all } u \in C^\infty(\mathbb{R}^N),$$

where

$$\Phi(u)(y) = u(\Pi(x))\Big(\frac{1+|\Pi(x)|^2}{2}\Big)^{(N-4)/2}, \quad y = \Pi(x).$$

Then (1.2) reduces to

$$\Delta^2 u = \tilde{Q}(x)u^{(N+4)/(N-4)} \quad \text{in } \mathbb{R}^4, \quad \text{where } \tilde{Q} = Q \circ \Pi. \tag{1.3}$$

Let us consider the problem (1.1) by taking Q to be a perturbation of a constant function. More precisely, we let $Q = (1+\varepsilon h)$, where h is a smooth function on $\mathbb{S}^N$ and $\varepsilon > 0$ is a small parameter. Using the stereographic projection from $\mathbb{S}^N$ to $\mathbb{R}^N$, we transform (1.3) (with f denoting the transformed function h) to the following problem:

$$\begin{cases} \Delta^2 u = (1+\varepsilon f(x))u^{(N+4)/(N-4)} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4}$$

But, in this paper, we consider the nonlinear perturbed problem

$$\begin{cases} \Delta^2 u = u^{(N+4)/(N-4)} + \varepsilon f(x)u^q & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.5}$$

with $f(\not\equiv 0) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, ε being a positive parameter and $1 < q \leq (N+4)/(N-4)$. Note that when $q = (N+4)/(N-4)$, then (1.5) reduces to (1.4). When $q = (N+4)/(N-4)$, it is enough to have $f \in L^\infty(\mathbb{R}^N)$.

Note that (1.5) is related to the entire space problem

$$\begin{cases} \Delta^2 U = U^{(N+4)/(N-4)} & \text{in } \mathbb{R}^N, \\ U \in \mathcal{D}^{2,2}(\mathbb{R}^N), \end{cases}$$

where $\mathcal{D}^{2,2}(\mathbb{R}^N) = \{u \in L^{2N/(N-4)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\Delta u|^2\,dx < +\infty\}$, and the solutions are given by Lin [21] as

$$U_{1,0}(x) = C_N\Big(\frac{1}{1+|x|^2}\Big)^{(N-4)/2},$$

$$U_{\lambda,\xi}(x) = \lambda^{-(N-4)/2}U_{1,0}\Big(\frac{x-\xi}{\lambda}\Big) \tag{1.6}$$

and

$$\langle (x-\xi), \nabla U_{\lambda,\xi}\rangle = -\Big(\lambda\frac{\partial U_{\lambda,\xi}}{\partial\lambda} + \frac{N-4}{2}U_{\lambda,\xi}\Big), \tag{1.7}$$

where $C_N = [N^2(N^2-4)(N-4)]^{(N-4)/8}$. Here

$$\|u\|^2_{\mathcal{D}^{2,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\Delta u|^2\,dx.$$

Note that when $1 < q < (N+4)/(N-4)$, we have interaction with the critical dimension as $U_{1,0}^{q+1}$ is integrable provided $q > 4/(N-4)$, that is, the cases $N = 5, 6, 7$ are the worst case scenario and that is the reason why we require $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Let us define a finite-dimensional functional $\mathcal{J}$, where

$$\mathcal{J}(\lambda, \xi) = \frac{1}{q+1} \int_{\mathbb{R}^N} f(x) U_{\lambda,\xi}{}^{q+1}(x)\, dx = \frac{\lambda^{N-\theta}}{q+1} \int_{\mathbb{R}^N} f(\xi + \lambda x) U_{1,0}^{q+1}(x)\, dx, \tag{1.8}$$

where $\theta = ((N-4)(q+1))/2$. Using the Hölder inequality in (1.8) and choosing $N/(N-4) < s < 2N/(N-4)$,

$$\begin{aligned} |\mathcal{J}(\lambda, \xi)| &\le C\Big(\int_{\mathbb{R}^N} |f(x)|^{s/(s-1)}\, dx \Big)^{(s-1)/s} \Big(\int_{\mathbb{R}^N} U_{\lambda,\xi}^{s}(x)\, dx \Big)^{(q+1)/s} \\ &\le c\lambda^{(N(q+1)/s)-\theta} \|f\|_{L^{(s-1)/s}} \|U_{1,0}\|_{L^s}^{q+1}. \end{aligned}$$

Hence,

$$|\mathcal{J}(\lambda, \xi)| \to 0 \quad \text{as } \lambda \to 0. \tag{1.9}$$

As a result, we can extend $\mathcal{J}(\lambda, \xi)$ on $\mathbb{R} \times \mathbb{R}^N$ in an odd way as

$$\tilde{\mathcal{J}}(\lambda, \xi) = -\mathcal{J}(-\lambda, \xi) \quad \text{for } \lambda < 0.$$

Without loss of generality, we consider $\tilde{\mathcal{J}}(\lambda, \xi) = \mathcal{J}(\lambda, \xi)$. Moreover, from (1.8) and the fact that $U_{1,0}$ is bounded,

$$\begin{aligned} \mathcal{J}(\lambda, \xi) &= \frac{\lambda^{N-\theta}}{q+1} \int_{\mathbb{R}^N} f(\xi + \lambda x) U_{1,0}^{q+1}(x) \\ &\le c\lambda^{N-\theta} \|f\|_{L^1}. \end{aligned}$$

Noting the fact that $N - \theta$ is negative, we conclude the fact that $\mathcal{J}(\lambda, \xi) \to 0$ as $|\lambda| \to \infty$. Furthermore, if $\lambda \to \lambda_\star > 0$ and $|\xi| \to \infty$, by the dominated convergence theorem,

$$\mathcal{J}(\lambda, \xi) = \frac{\lambda^{-\theta}}{q+1} \int_{\mathbb{R}^N} f(x) U^{q+1}\Big(\frac{x - \xi}{\lambda}\Big) \to 0.$$

Hence,

$$\lim_{|\lambda| + |\xi| \to \infty} \mathcal{J}(\lambda, \xi) = 0. \tag{1.10}$$

Hence, from (1.9) and (1.10), there exists (λ, ξ) with $\lambda > 0$ such that $\mathcal{J}$ has a critical point (a global maximum or a global minimum) at (λ, ξ). Let

$$J_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u|^2\, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\, dx - \frac{\varepsilon}{q+1} \int_{\mathbb{R}^N} f(x) |u|^{q+1}\, dx.$$

Hence, by Felli [16] as well as Lemma 2.2, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $J_\varepsilon \in C^2(\mathcal{D}^{2,2}(\mathbb{R}^N), \mathbb{R})$ admits a critical point $u_\varepsilon \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ near $\mathcal{M}$ and hence u_ε is a solution of (1.5), where $p + 1 = 2N/(N-4)$ and

$$\mathcal{M} = \{U_{\lambda,\xi} : (\lambda, \xi) \in \mathbb{R}^+ \times \mathbb{R}^N\}$$

is an $(N+1)$-dimensional manifold of solutions. Note that the existence of a solution is dependent on some sort of 'nondegeneracy' condition of the critical point of $\mathcal{J}$.

Let $K \subset \mathbb{R}^+ \times \mathbb{R}^N$ be a compact set and define

$$d(u, \mathcal{M}_K) = \inf_{(\lambda,\xi)\in K} \|u - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)}.$$

In this paper we discuss the existence, uniqueness and multiplicity of positive solutions of (1.5) under the assumption that $f \in L^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Now we state the following theorems motivated by [23].

Theorem 1.1. *Let (λ, ξ) be a nondegenerate critical point of $\mathcal{J}$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, (1.5) admits a positive solution u_ε. Moreover, $\|u_\varepsilon - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} = O(\varepsilon)$.*

Corollary 1.2. *Let u_ε be a sequence of solutions of (1.5) such that*

$$\|u_\varepsilon - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Then $\nabla\mathcal{J}(\lambda, \xi) = 0$.

Theorem 1.3 (Uniqueness). *Let (λ, ξ) be a nondegenerate critical point of $\mathcal{J}$. Furthermore, suppose $|\nabla f(x)| \le C$ and there exists two sequences of solutions $\{u_{\varepsilon,i}\}$ $(i = 1, 2)$ of (1.5) such that*

$$\|u_{\varepsilon,i} - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{1.11}$$

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $u_{\varepsilon,1} \equiv u_{\varepsilon,2}$.

Remark 1.4. Note that if $q = 1$ and $N > 8$, positive solutions of (1.5) are nonunique for ε sufficiently small. See Felli [16]. In fact, Esposito [14] proved existence of two positive solutions of the Paneitz operator on $\mathbb{S}^N$ (see (1.2))

$$Pu = \frac{N^2(N-4)(N^2-4)}{16}|u|^{8/(N-4)}u + (\varepsilon f + o(\varepsilon))|u|^{q-1}u$$

and $1 \le q \le (N+4)/(N-4)$ when f changes sign and $q \ge 4/(N-4)$ or $q < 4/(N-4)$ and $\int_{\mathbb{S}^N} f = 0$. Note that our uniqueness is different in this context.

Theorem 1.5 (Multiplicity). *Assume that there is a compact set $K \subset \mathbb{R}^+ \times \mathbb{R}^N$ with nonempty interior such that the critical points of $\mathcal{J}$ in K are finite and nondegenerate. Furthermore, suppose $|\nabla f(x)| \le C$. Then there exists $\rho_0 = \rho_0(K) > 0$ and $\varepsilon_0 = \varepsilon_0(\rho_0) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the number of solutions to the problem (1.5) with $d(u, \mathcal{M}_K) < \rho_0$ is the same as the number of nondegenerate critical points of $\mathcal{J}$.*

Corollary 1.6. *Furthermore, the conclusions of Theorems 1.1–1.5 hold for the equation*

$$(-\Delta)^m u = (1 + \varepsilon f(x))u^{(N+2m)/(N-2m)} \quad \text{in } \mathbb{R}^N$$

whenever $\|f\|_\infty + \|\nabla f\|_\infty \le C$, $N > 2m$ and $m \in \mathbb{N}$. The construction of positive solutions follows from Wei and Xu [25].

Remark 1.7. Note that the conclusions of Theorems 1.1–1.5 are not only applicable to the powers of Laplacians, but also applicable for the coercive Hardy equation $-\Delta u - (\mu/|x|^2)u = (1 + \varepsilon f(x))u^{(N+2)/(N-2)}$ with $N \geq 3$ and $\mu > 0$. Here proving the results becomes much easier as $Ker\{-\Delta - (\mu/|x|^2) - ((N+2)/(N-2))u^{4/(N-2)}\}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is one dimensional due to the scaling invariance of the operator.

2. Preliminaries

Lemma 2.1 (Nondegeneracy). *The kernel of the linearized operator*

$$\mathcal{L} = \Delta^2 - \frac{N+4}{N-4} U_{\lambda,\xi}{}^{4/(N-4)}$$

in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ *is* $N+1$ *dimensional and*

$$Ker(\mathcal{L}) = \left\{ \frac{\partial U_{\lambda,\xi}}{\partial \lambda}, \frac{\partial U_{\lambda,\xi}}{\partial \xi_1}, \frac{\partial U_{\lambda,\xi}}{\partial \xi_2}, \dots, \frac{\partial U_{\lambda,\xi}}{\partial \xi_N} \right\}.$$

Proof. This follows from Djadli *et al.* [13]. □

Let H be a Hilbert space and $J_\varepsilon(u) = J_0(u) - \varepsilon G(u)$ be a perturbed functional, where $J_0, G \in C^2(H, \mathbb{R})$. Moreover, assume that J_0 satisfies:

(f1) J_0 has a finite-dimensional manifold of critical points $\mathcal{M}$; let $c = J_0(z)$ for all $z \in \mathcal{M}$;

(f2) for all $z \in \mathcal{M}$, $J_0''(z)$ is a Fredholm operator of index zero;

(f3) for all $z \in \mathcal{M}$, $T_z\mathcal{M} = Ker\ J_0''(z)$. We denote $\mathcal{J} = G|_{\mathcal{M}}$.

Lemma 2.2. *Let* J_0 *satisfy (f1)–(f3) and suppose there exists* $z \in \mathcal{M}$ *which is a critical point of* $\mathcal{J}$ *such that one of the following conditions holds:*

(1) *z is nondegenerate;*

(2) *z is a global maximum or global minimum;*

(3) *z is isolated and the local degree of* $\nabla\mathcal{J}$ *at z is different from zero.*

Then there exists $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *the functional* J_ε *has a critical point* u_ε *such that* $u_\varepsilon \to z$ *as* $\varepsilon \to 0$.

Proof. The proof of this lemma follows from Ambrosetti and Badiale [1]. Also, see Ambrosetti *et al.* [2, page 122] and the book by Ambrosetti and Malchiodi [3]. Note that Lemma 2.2 is a very general theorem; it is not restricted to Laplacian operators only. Note that in Felli's proof [16], condition (2) of the lemma holds. □

Lemma 2.3 (Caristi and Mitidieri [7]). *Let* Ω *be an open subset of* $\mathbb{R}^N$ $(N \geq 5)$ *and* $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ *be a weak solution of*

$$\Delta^2 u = a(x)u \quad in\ \Omega,$$

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where $a \in L^{\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha > N/4$. Then, for any $0 < \beta < +\infty$, there exist $C > 0$ and $R > 0$ such that

$$\sup_{B(y,r)\cap\Omega} |u| \le C\Big[\frac{1}{r^N}\int_{B(y,2r)\cap\Omega} |u|^{\beta+1}\Big]^{1/(\beta+1)}$$

for any $y \in \mathbb{R}^N$ and $0 < r < R$.

Lemma 2.4. *Let u_ε be a sequence of solutions of (1.5) with $\|u_\varepsilon - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$ for some $(\lambda,\xi) \in \mathbb{R}^+ \times \mathbb{R}^N$. Then the asymptotic behavior for derivatives of u_ε at infinity is given by*

$$|\nabla^{(\beta)} u_\varepsilon(x)| = O(1)|x|^{4-N-|\beta|} \tag{2.1}$$

for $0 \le |\beta| \le 3$ whenever $|x| \gg 1$.

Proof. First note that if $u_\varepsilon \to U_{\lambda,\xi}$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} u_\varepsilon^{2N/(N-4)}(x)\,dx \to \int_{\mathbb{R}^N} U_{\lambda,\xi}^{2N/(N-4)}(x)\,dx$$

as $\varepsilon \to 0$. Moreover, as $f \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, by the Hölder inequality,

$$\Big|\int_{\mathbb{R}^N} f(x)u_\varepsilon^{q+1}(x)\,dx\Big| \le C,$$

$$\int_{\mathbb{R}^N} f(x)u_\varepsilon^{q+1}(x)\,dx \to \int_{\mathbb{R}^N} f(x)U_{\lambda,\xi}^{q+1}(x)\,dx.$$

Also, by elliptic regularity, $u_\varepsilon \to U_{\lambda,\xi}$ in $C^4_{\mathrm{loc}}(\mathbb{R}^N)$. Hence, u_ε is locally uniformly bounded. So, we need to study the decay of u_ε at infinity. Define the Kelvin transform of u_ε as

$$\hat{u}_\varepsilon(x) := |x|^{4-N} u_\varepsilon\Big(\frac{x}{|x|^2}\Big).$$

By the application of the Kelvin transform on (1.5),

$$\Delta^2 \hat{u}_\varepsilon = [\hat{u}_\varepsilon^{8/(N-4)} + \varepsilon \hat{f}(x)|x|^{-\tau}\hat{u}_\varepsilon^{q-1}]\hat{u}_\varepsilon \quad \text{in } \mathbb{R}^N\backslash\{0\},$$

where $\tau = N + 4 - q(N-4)$ and $\hat{f}(x) = f(x/|x|^2)$. Let $a_\varepsilon(x) = \hat{u}_\varepsilon^{8/(N-4)} + \varepsilon\hat{f}(x)|x|^{-\tau}\hat{u}_\varepsilon^{q-1}$. But $\hat{f}(x)|x|^{-\tau}$ is bounded near 0. Hence, by Lemma 2.3, there exist $R > 0$ and $C > 0$ independent of $\varepsilon > 0$ such that

$$\sup_{B_R(0)} |\hat{u}_\varepsilon(x)| \le C\Big[\frac{1}{R^N}\int_{B_{2R}} |\hat{u}_\varepsilon(z)|^{2N/(N-4)}\,dz\Big]^{(N-4)/2N} \le C.$$

This implies that, for $|x| \gg 1$,

$$u_\varepsilon(x) = O(|x|^{4-N}).$$

And, hence, by the Schauder estimates,

$$|\nabla^{(\beta)} u_\varepsilon| \le C|x|^{4-N-|\beta|}.$$

Note that in the above estimate $C > 0$ is independent of $\varepsilon > 0$. □

Lemma 2.5. *Let w_ε be a sequence of solutions of*

$$\begin{cases}\Delta^2 w = c_\varepsilon(x)w + \varepsilon f(x)d_\varepsilon(x)w & \text{in } \mathbb{R}^N\\ w \in \mathcal{D}^{2,2}(\mathbb{R}^N)\end{cases} \tag{2.2}$$

with $\|w_\varepsilon\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \le C$, where $u_{\varepsilon,i}$ $(i = 1, 2)$ are solutions of (1.5)

$$c_\varepsilon(x) = \int_0^1 [tu_{\varepsilon,1}(x) + (1-t)u_{\varepsilon,2}(x)]^{8/(N-4)}\,dt$$

and

$$d_\varepsilon(x) = \int_0^1 [tu_{\varepsilon,1}(x) + (1-t)u_{\varepsilon,2}(x)]^{q-1}\,dt.$$

Then, for $|x| \gg 1$, we have a uniform estimate

$$|\nabla^{(\beta)} w_\varepsilon(x)| = \mathcal{O}(1)|x|^{4-N-|\beta|} \tag{2.3}$$

for $0 \le |\beta| \le 3$.

Proof. By the standard regularity, w_ε is locally uniformly bounded. Let us consider the Kelvin transform of w_ε

$$\hat{w}_\varepsilon(x) := |x|^{4-N} w_\varepsilon\Big(\frac{x}{|x|^2}\Big),$$

$$\hat{u}_\varepsilon(x) = |x|^{4-N} u_\varepsilon\Big(\frac{x}{|x|^2}\Big), \quad \hat{w}_\varepsilon(x) = |x|^{4-N} w_\varepsilon\Big(\frac{x}{|x|^2}\Big), \quad x \in \mathbb{R}^N\backslash\{0\}.$$

Furthermore, define

$$\hat{c}_\varepsilon(x) = \int_0^1 [t\hat{u}_{\varepsilon,1} + (1-t)\hat{u}_{\varepsilon,2}]^{8/(N-4)}\,dt,$$

$$\hat{d}_n(x) = \int_0^1 [t\hat{u}_{\varepsilon,1} + (1-t)\hat{u}_{\varepsilon,2}]^{q-1}\,dt.$$

Then, by (2.2), $\hat{w}_\varepsilon$ satisfies

$$\Delta^2 \hat{w}_\varepsilon = \hat{c}_\varepsilon \hat{w}_\varepsilon + \varepsilon |x|^{-\tau} f\left(\frac{x}{|x|^2}\right)\hat{d}_\varepsilon \hat{w}_\varepsilon \quad \text{in } \mathbb{R}^N\backslash\{0\}. \tag{2.4}$$

So, we are going to study boundedness of (2.4) near a neighborhood of the origin. From Lemma 2.4, $\hat{c}_\varepsilon$, $|x|^{-\tau}\hat{d}_\varepsilon f(x/|x|^2)$ is uniformly bounded near the origin. Hence, by Lemma 2.3, there exist $C, R > 0$ such that

$$\sup_{B(y,R)\cap\Omega} |\hat{w}_\varepsilon| \le C\Big[\frac{1}{R^N}\int_{B(y,2R)\cap\Omega} |\hat{w}_\varepsilon(z)|^{2N/(N-4)}\,dz\Big]^{(N-4)/(2N)} \le C.$$

Hence, $\hat{w}_\varepsilon$ is uniformly bounded near the origin and hence $|w_\varepsilon(x)| \le C|x|^{4-N}$ when $|x| \gg 1$. The decay of higher derivatives follows from the standard elliptic estimates. □

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LEMMA 2.6 (Kazdan–Warner-type identities). *Let u_ε be a solution of (1.5) such that $\|u_\varepsilon - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$ for some $(\lambda, \xi) \in \mathbb{R}^+ \times \mathbb{R}^N$. Then, we have the following two types of Pohozaev identities:*

$$\int_{\mathbb{R}^N} f(x)u_\varepsilon^q \frac{\partial u_\varepsilon}{\partial x_i} = 0, \quad i = 1, 2 \tag{2.5}$$

and

$$\int_{\mathbb{R}^N} f(x)u_\varepsilon^q \Big[(x-\xi)\cdot\nabla u_\varepsilon + \Big(\frac{N-4}{2}\Big)u_\varepsilon\Big] = 0. \tag{2.6}$$

PROOF. In order to prove (2.5), we multiply (1.5) by $\partial u_\varepsilon(x)/\partial x_i$, $i = 1, 2, \ldots, N$, and integrate by parts on the ball $B_R(0)$ to get

$$\int_{B_R(0)} (u_\varepsilon^{(N+4)/(N-4)} + \varepsilon f(x)u_\varepsilon^q)\frac{\partial u_\varepsilon}{\partial x_i} = \int_{\partial B_R(0)} \frac{\partial \Delta u_\varepsilon}{\partial \nu}\frac{\partial u_\varepsilon}{\partial x_i} d\sigma - \int_{B_R(0)} \nabla \Delta u_\varepsilon \cdot \frac{\partial}{\partial x_i}(\nabla u_\varepsilon). \tag{2.7}$$

By (2.1), we obtain

$$\int_{\partial B_R(0)} \Big|\frac{\partial \Delta u_\varepsilon}{\partial \nu}\frac{\partial u_\varepsilon}{\partial x_i}\Big| d\sigma = O\Big(\frac{1}{R^{2(N-2)}}\Big) \quad \text{as } R \to \infty.$$

Again, by a suitable integration by parts and using (2.1) and Lemma 2.4, we get, as $R \to \infty$,

$$\int_{B_R(0)} \nabla\Delta u_\varepsilon \cdot \frac{\partial}{\partial x_i}(\nabla u_\varepsilon) = \int_{\partial B_R(0)} \Big(\Delta u_\varepsilon \frac{\partial}{\partial \nu}\Big(\frac{\partial u_\varepsilon}{\partial x_i}\Big) - \frac{1}{2R}x_i|\Delta u_\varepsilon|^2\Big) d\sigma = O\Big(\frac{1}{R^{2(N-2)}}\Big).$$

Hence, from the last two relations,

$$\lim_{R\to\infty} \{\text{Right-hand side of (2.7)}\} = 0. \tag{2.8}$$

We note that, again integrating by parts,

$$\int_{B_R(0)} (u_\varepsilon^{(N+4)/(N-4)} + \varepsilon f(x)u_\varepsilon^q)\frac{\partial u_\varepsilon}{\partial x_i} = \frac{1}{R}\int_{\partial B_R(0)} x_i u_\varepsilon^{2N/(N-4)} d\sigma + \varepsilon \int_{B_R(0)} f(x)u_\varepsilon^q \frac{\partial u_\varepsilon}{\partial x_i}.$$

Using (2.1) and letting $R \to \infty$ in the above equation,

$$\lim_{R\to\infty} \int_{B_R(0)} (u_\varepsilon^{(N+4)/(N-4)} + \varepsilon f(x)u_\varepsilon^q)\frac{\partial u_\varepsilon}{\partial x_i} = \varepsilon \int_{\mathbb{R}^N} f(x)u_\varepsilon^q \frac{\partial u_\varepsilon}{\partial x_i}. \tag{2.9}$$

Therefore, we obtain, using (2.9) and (2.8),

$$\varepsilon \int_{\mathbb{R}^N} f(x)u_\varepsilon^q \frac{\partial u_\varepsilon}{\partial x_i} = \lim_{R\to\infty} \{\text{Left-hand side of (2.7)}\} = 0,$$

which proves (2.5).

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For (2.6), we multiply (1.5) by $(x-\xi)\cdot\nabla u_\varepsilon + ((N-4)/2)u_\varepsilon$ on either side and integrate on the ball $B_R(y)$ as before to obtain

$$\begin{aligned}\int_{B_R(y)} &(u_\varepsilon^{(N+4)/(N-4)} + \varepsilon f(x)u_\varepsilon^q)\Big((x-\xi)\cdot\nabla u_\varepsilon + \Big(\frac{N-4}{2}\Big)u_\varepsilon\Big)\\ &= \int_{B_R(y)} \Delta^2 u_\varepsilon\Big((x-\xi)\cdot\nabla u_\varepsilon + \Big(\frac{N-4}{2}\Big)u_\varepsilon\Big).\end{aligned} \tag{2.10}$$

Integrating by parts,

$$\begin{aligned}\text{Left-hand side of (2.10)} = R\int_{\partial B_R(y)} &u_\varepsilon^{(N+4)/(N-4)}\,d\sigma\\ &+\varepsilon\int_{B_R(y)} f(x)u_\varepsilon^q\Big((x-\xi)\cdot\nabla u_\varepsilon + \Big(\frac{N-4}{2}\Big)u_\varepsilon\Big).\end{aligned}$$

Again integrating by parts suitably,

$$\begin{aligned}\text{Right-hand side of (2.10)} = \int_{\partial B_R(y)} \Big(&|x-\xi|\Big[\frac{1}{2}|\Delta u_\varepsilon|^2 + \frac{\partial u_\varepsilon}{\partial r}\frac{\partial}{\partial r}(\Delta u_\varepsilon)\Big]\\ &- \Delta u_\varepsilon \frac{\partial}{\partial r}\Big(r\frac{\partial u_\varepsilon}{\partial r}\Big)\Big)\,d\sigma.\end{aligned}$$

Using the decay estimate (2.1),

$$\lim_{R\to\infty}\{\text{Left-hand side of (2.10)}\} = \varepsilon\int_{\mathbb{R}^N} f(x){u_\varepsilon}^q\Big((x-\xi)\cdot\nabla u_\varepsilon + \Big(\frac{N-4}{2}\Big)u_\varepsilon\Big)$$

and

$$\lim_{R\to\infty}\{\text{Right-hand side of (2.10)}\} = 0.$$

Hence, (2.6) follows. □

Remark 2.7. Note that when $q = (N+4)/(N-4)$ one can derive the Kazdan and Warner [20] kind of identities using the concept of an integral equation in $\mathcal{D}^{2,2}(\mathbb{R}^N)$;

$$u_\varepsilon(x) = \int_{\mathbb{R}^N} (1+\varepsilon f(y))F(x,y)u_\varepsilon^{(N+4)/(N-4)}(y)\,dy, \tag{2.11}$$

where $F(x,y) = 1/(4-N)\sigma_N|x-y|^{N-4}$ is the fundamental solution of Δ^2 and σ_N is the area of the unit sphere in $\mathbb{R}^N$. The main idea is the fact that

$$\Delta^2 u = f \quad \text{in } \mathbb{R}^N$$

can be written as $u = u_1 + u_2$, where $u_i \in \mathcal{D}^{2,2}(\mathbb{R}^N)$; $i = 1, 2$, $u_1(x) = \int_{\mathbb{R}^N} F(x,y)g(y)\,dy$ and $\Delta^2 u_2 = 0$. But this implies $u_2 = 0$. As a result, we end up getting (2.11).

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Proof of Corollary 1.2. By the Schauder estimates, $u_\varepsilon \to U_{\lambda,\xi}$ in $C^4_{\mathrm{loc}}(\mathbb{R}^N)$, and by Lemma 2.6 and the dominated convergence theorem we can pass to the limit in (2.5) and (2.6). Using (1.7),

$$\int_{\mathbb{R}^3} f(x)U^q_{\lambda,\xi}\frac{\partial U_{\lambda,\xi}}{\partial x_i} = 0, \quad i = 1, 2, \ldots, N \tag{2.12}$$

and

$$\int_{\mathbb{R}^3} f(x)U^q_{\lambda,\xi}\frac{\partial U_{\lambda,\xi}}{\partial \lambda} = 0. \tag{2.13}$$

Hence, we obtain $\nabla\mathcal{J}(\lambda,\xi) = 0$. □

Lemma 2.8. *If (λ_0, ξ_0) is a critical point of $\mathcal{J}$, then*

$$\begin{aligned}\lambda_0\frac{\partial^2\mathcal{J}}{\partial\lambda^2}(\lambda_0,\xi_0) = -\theta &\int_{\mathbb{R}^N} f(z)U^q_{\lambda_0,\xi_0}(z)\frac{\partial U_{\lambda_0,\xi_0}}{\partial\lambda}(z)\,dz\\ &- N\int_{\mathbb{R}^N} f(z)U^q_{\lambda_0,\xi_0}(z)\Big\langle z-\xi_0, \nabla\frac{\partial U_{\lambda_0,\xi_0}}{\partial\lambda}(z)\Big\rangle\,dz\\ &- Nq\int_{\mathbb{R}^N} f(z)U^{q-1}_{\lambda_0,\xi_0}(z)\langle z-\xi_0, \nabla U_{\lambda_0,\xi_0}\rangle\frac{\partial U_{\lambda_0,\xi_0}}{\partial\lambda}(z)\,dz.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{\partial^2\mathcal{J}}{\partial\lambda\partial\xi_i}(\lambda_0,\xi_0) = -&\int_{\mathbb{R}^N} f(z)U^q_{\lambda_0,\xi_0}(z)\frac{\partial}{\partial z_i}\Big(\frac{\partial U_{\lambda_0,\xi_0}}{\partial\lambda}(z)\Big)\,dz\\ &- q\int_{\mathbb{R}^N} f(z)U^{q-1}_{\lambda_0,\xi_0}(z)\frac{\partial U_{\lambda_0,\xi_0}}{\partial\lambda}(z)\frac{\partial U_{\lambda_0,\xi_0}}{\partial z_i}(z)\,dz.\end{aligned}$$

Moreover, for $1 \le i, j \le N$,

$$\begin{aligned}\frac{\partial^2\mathcal{J}}{\partial\xi_i\partial\xi_j}(\lambda_0,\xi_0) = -&\int_{\mathbb{R}^N} f(z)U^q_{\lambda_0,\xi_0}(z)\frac{\partial}{\partial z_i}\Big(\frac{\partial U_{\lambda_0,\xi_0}}{\partial z_j}(z)\Big)\,dz\\ &- q\int_{\mathbb{R}^N} f(z)U^{q-1}_{\lambda_0,\xi_0}(z)\frac{\partial U_{\lambda_0,\xi_0}}{\partial z_j}(z)\frac{\partial U_{\lambda_0,\xi_0}}{\partial z_i}(z)\,dz,\end{aligned}$$

where $z = \xi + \lambda x$.

Proof. As $U_{\lambda,\xi}$ satisfies (1.6) and (1.7),

$$\begin{aligned}\frac{\partial\mathcal{J}}{\partial\lambda}(\lambda,\xi) &= \frac{\lambda^{N-\theta}}{q+1}\int_{\mathbb{R}^N}\langle x, \nabla f(\lambda x+\xi)\rangle U^{q+1}_{1,0}(x)\,dx\\ &\quad + \frac{N-\theta}{q+1}\lambda^{N-\theta-1}\int_{\mathbb{R}^N} f(\lambda x+\xi)U^{q+1}_{1,0}(x)\,dx,\\ \frac{\partial\mathcal{J}}{\partial\xi_i}(\lambda,\xi) &= \frac{\lambda^{N-\theta}}{(q+1)\lambda}\int_{\mathbb{R}^N}\frac{\partial f(\lambda x+\xi)}{\partial x_i}U^{q+1}_{1,0}(x)\,dx.\end{aligned}$$

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Also, note that $\theta = (N-4)(q+1)/2$. Integrating by parts,

$$
\begin{aligned}
\lambda \frac{\partial \mathcal{J}}{\partial \lambda}(\lambda, \xi) = &- \frac{N}{q+1}\lambda^{N-\theta} \int_{\mathbb{R}^N} f(\lambda x + \xi) U_{1,0}^{q+1}(x)\, dx \\
&- N\lambda^{N-\theta} \int_{\mathbb{R}^N} f(\lambda x + \xi) U_{1,0}^{q} \langle x, \nabla U_{1,0}(x)\rangle\, dx \\
&+ \frac{N-\theta}{q+1}\lambda^{N-\theta} \int_{\mathbb{R}^N} f(\lambda x + \xi) U_{1,0}^{q+1}(x)\, dx \\
= &- \frac{\theta}{q+1}\lambda^{N-\theta} \int_{\mathbb{R}^N} f(\lambda x + \xi) U_{1,0}^{q+1}(x)\, dx \\
&- N\lambda^{N-\theta} \int_{\mathbb{R}^N} f(\lambda x + \xi) U_{1,0}^{q} \langle x, \nabla U_{1,0}(x)\rangle\, dx
\end{aligned}
$$

and

$$
\frac{\partial \mathcal{J}}{\partial \xi_i}(\lambda, \xi) = -\lambda^{N-\theta-1} \int_{\mathbb{R}^N} f(\lambda x + \xi) U_{1,0}^{q}(x) \frac{\partial U_{1,0}}{\partial x_i}\, dx.
$$

Since (λ_0, ξ_0) is a critical point of $\mathcal{J}$, we must have $(\partial \mathcal{J}/\partial\lambda)(\lambda_0, \xi_0) = 0$ and $(\partial \mathcal{J}/\partial \xi_i)(\lambda_0, \xi_0) = 0$. Hence, letting $z = \xi + \lambda x$,

$$
\begin{aligned}
\lambda_0 \frac{\partial^2 \mathcal{J}}{\partial \lambda^2}(\lambda_0, \xi_0) = &-\theta \int_{\mathbb{R}^N} f(z) U_{\lambda_0,\xi_0}^{q}(z) \frac{\partial U_{\lambda_0,\xi_0}}{\partial \lambda}(z)\, dz \\
&- N \int_{\mathbb{R}^N} f(z) U_{\lambda_0,\xi_0}^{q}(z) \Big\langle z - \xi_0, \nabla \frac{\partial U_{\lambda_0,\xi_0}}{\partial \lambda}(z) \Big\rangle\, dz \\
&- Nq \int_{\mathbb{R}^N} f(z) U_{\lambda_0,\xi_0}^{q-1}(z) \langle z - \xi_0, \nabla U_{\lambda_0,\xi_0}\rangle \frac{\partial U_{\lambda_0,\xi_0}}{\partial \lambda}(z)\, dz.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\frac{\partial^2 \mathcal{J}}{\partial \lambda \partial \xi_i}(\lambda_0, \xi_0) = &- \int_{\mathbb{R}^N} f(z) U_{\lambda_0,\xi_0}^{q}(z) \frac{\partial}{\partial z_i}\Big(\frac{\partial U_{\lambda_0,\xi_0}}{\partial \lambda}(z)\Big)\, dz \\
&- q \int_{\mathbb{R}^N} f(z) U_{\lambda_0,\xi_0}^{q-1}(z) \frac{\partial U_{\lambda_0,\xi_0}}{\partial \lambda}(z) \frac{\partial U_{\lambda_0,\xi_0}}{\partial z_i}(z)\, dz.
\end{aligned}
$$

Moreover, for $1 \le i, j \le N$,

$$
\begin{aligned}
\frac{\partial^2 \mathcal{J}}{\partial \xi_i \partial \xi_j}(\lambda_0, \xi_0) = &- \int_{\mathbb{R}^N} f(z) U_{\lambda_0,\xi_0}^{q}(z) \frac{\partial}{\partial z_i}\Big(\frac{\partial U_{\lambda_0,\xi_0}}{\partial z_j}(z)\Big)\, dz \\
&- q \int_{\mathbb{R}^N} f(z) U_{\lambda_0,\xi_0}^{q-1}(z) \frac{\partial U_{\lambda_0,\xi_0}}{\partial z_j}(z) \frac{\partial U_{\lambda_0,\xi_0}}{\partial z_i}(z)\, dz.
\end{aligned}
$$

□

3. Proof of the main theorems

PROOF OF THEOREM 1.1. Let (λ, ξ) be a nondegenerate critical point of $\mathcal{J}$. Then $\nabla \mathcal{J}(\lambda, \xi) = 0$ and $det(\nabla^2 \mathcal{J}(\lambda, \xi)) \neq 0$. Hence, $\nabla^2\, \mathcal{J}(\lambda, \xi)$ is an invertible matrix of

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order $N+1$. Our aim is to obtain a solution of (1.5) which is of the form $u_\varepsilon = U_{\lambda,\xi} + \phi_\varepsilon$. Note that

$$J_\varepsilon(u) = J_0(u) - \frac{\varepsilon}{q+1}\int_{\mathbb{R}^N} f(x)|u|^{q+1}\,dx,$$

where

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx$$

and $Ker(\mathcal{L})$ is $N+1$-dimensional; see Lemma 2.1. Moreover, it is easy to check that J_0 satisfies (f1)–(f3). Hence, by Lemma 2.2, (1) holds and we obtain a solution of (1.5) for sufficiently small $\varepsilon > 0$.

Proof of Theorem 1.3. If possible, let there exist a sequence $\varepsilon_n \to 0$ and two distinct functions $u_{1,\varepsilon_n} \equiv u_{1,n}$, $u_{2,\varepsilon_n} \equiv u_{2,n}$ which solve (1.5) with $\varepsilon = \varepsilon_n$ and $\|u_{i,n} - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$ for $i = 1, 2$. Set $\tilde{w}_n = u_{1,n} - u_{2,n}$. Then $\|\tilde{w}_n\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Hence, by Lemma 2.4, $\|\tilde{w}_n\|_{L^\infty(\mathbb{R}^N)} \le C$.

Define $w_n = \tilde{w}_n/\|\tilde{w}_n\|_{L^\infty(\mathbb{R}^N)}$. Then there exists $x_n \in \mathbb{R}^N$ such that $|w_n(x_n)| \ge \frac{1}{2}$. Then w_n satisfies

$$\Delta^2 w_n = c_n(x)w_n + \varepsilon f(x)d_n(x)w_n \quad \text{with } c_n(x) = \int_0^1 [tu_{1,n} + (1-t)u_{2,n}]^{8/(N-4)}\,dt$$

and

$$d_n(x) = \int_0^1 [tu_{1,n} + (1-t)u_{2,n}]^{q-1}\,dt.$$

Using Schauder estimates, we obtain $w_n \to w$ in $C^4_{\mathrm{loc}}(\mathbb{R}^N)$, where w satisfies the entire problem

$$\Delta^2 w = \frac{N+4}{N-4}U_{\lambda,\xi}^{8/(N-4)} w \quad \text{in } \mathbb{R}^N.$$

By the nondegeneracy result in Lemma 2.1,

$$w = c_0\frac{\partial U_{\lambda,\xi}}{\partial\lambda} + \sum_{j=1}^N c_j\frac{\partial U_{\lambda,\xi}}{\partial x_j}$$

for some $c_j \in \mathbb{R}$, $j = 1, \ldots, N$. We claim that $c_j = 0$ for all $j = 0, 1, \ldots, N$. By the identity (2.5),

$$\int_{\mathbb{R}^N} f(x){u_{i,n}}^q\frac{\partial u_{i,n}}{\partial x_j} = 0, \quad j = 1, 2, \ldots, N. \tag{3.1}$$

We derive from (3.1) and (2.1)

$$\int_{\mathbb{R}^N}\frac{\partial f}{\partial x_j}u_{\varepsilon,i}^{q+1} = 0, \quad i = 1, 2 \text{ and } j = 1, 2, \ldots, N.$$

Therefore,

$$\int_{\mathbb{R}^N}\left(\frac{\partial f}{\partial x_j}{u_{1,n}}^{q+1} - \frac{\partial f}{\partial x_j}{u_{2,n}}^{q+1}\right) = 0 \quad \text{for } j = 1, 2, \ldots, N$$

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and, using the fundamental theorem of integral calculus,

$$\int_{\mathbb{R}^N} \frac{\partial f}{\partial x_j}\Big(\int_0^1 [tu_{1,n} + (1-t)u_{2,n}]^q\,dt\Big)\tilde{w}_n\,dx = 0 \quad \text{for } j = 1, 2, \ldots, N. \tag{3.2}$$

Letting $\varepsilon \to 0$ in (3.2),

$$\int_{\mathbb{R}^N} \frac{\partial f}{\partial x_j} U_{\lambda,\xi}{}^q \left(c_0 \frac{\partial U_{\lambda,\xi}}{\partial \lambda} + \sum_{i=1}^N c_i \frac{\partial U_{\lambda,\xi}}{\partial x_i}\right) = 0, \quad j = 1, 2, \ldots, N.$$

That is, integrating by parts again,

$$\int_{\mathbb{R}^N} f \frac{\partial}{\partial x_j}(U_{\lambda,\xi}{}^q w) = 0, \quad j = 1, 2, \ldots, N.$$

This implies that

$$q \int_{\mathbb{R}^N} f(x) U_{\lambda,\xi}^{q-1} \frac{\partial U_{\lambda,\xi}}{\partial x_j} w + \int_{\mathbb{R}^N} f(x) U_{\lambda,\xi}^q \frac{\partial w}{\partial x_j} = 0. \tag{3.3}$$

Furthermore, we obtain by integrating on $B_R(y)$

$$\int_{B_R(y)} (x-\xi)\cdot\nabla(f u_{i,n}^{q+1}) = R\int_{\partial B_R(y)} f(x)u_{i,n}^{q+1} - N\int_{B_R(y)} f(x)u_{i,n}^{q+1} \quad \text{for } i = 1, 2.$$

This implies that as $R \to +\infty$

$$\int_{\mathbb{R}^N} (x-\xi)\cdot\nabla(f u_{i,n}^{q+1}) = -N\int_{\mathbb{R}^N} f(x)u_{i,n}^{q+1} \quad \text{for } i = 1, 2.$$

And, as a result,

$$\int_{\mathbb{R}^N} \langle (x-\xi), \nabla f(x)\rangle u_{i,n}^{q+1} + (q+1)\int_{\mathbb{R}^N} f(x)\langle (x-\xi), \nabla u_{i,n}\rangle u_{i,n}^q = -N\int_{\mathbb{R}^N} f(x)u_{i,n}^{q+1}.$$

Hence, by the Pohozaev identity (2.6), we have for $i = 1, 2$

$$\begin{aligned}\int_{\mathbb{R}^N} \langle (x-\xi), \nabla f(x)\rangle u_{i,n}^{q+1} &= \left[\frac{(N-4)(q+1)-2N}{2}\right]\int_{\mathbb{R}^N} f(x)u_{i,n}^{q+1}\\ &= \gamma\int_{\mathbb{R}^N} f(x)u_{i,n}^{q+1},\end{aligned}$$

where $\gamma = (N-4)(q+1) - 2N/2$. This implies that

$$\int_{\mathbb{R}^N} \langle (x-\xi), \nabla f(x)\rangle u_{1,n}^{q+1} - \int_{\mathbb{R}^N} \langle (x-\xi)\cdot\nabla f(x)\rangle u_{2,n}^{q+1} = \gamma\int_{\mathbb{R}^N} f(x)[u_{1,n}^{q+1} - u_{2,n}^{q+1}]$$

and, by the application of the mean value theorem,

$$\begin{aligned}&\int_{\mathbb{R}^N} \langle (x-\xi), \nabla f(x)\rangle\Big(\int_0^1 (tu_{1,n} + (1-t)u_{1,n})^q\,dt\Big)w_n\\ &\quad= \gamma\int_{\mathbb{R}^N} f(x)\Big(\int_0^1 (tu_{1,n} + (1-t)u_{1,n})^q\,dt\Big)w_n.\end{aligned}$$

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And, letting $n \to \infty$,

$$\int_{\mathbb{R}^N} \langle (x-\xi), \nabla f(x) \rangle U_{\lambda,\xi}^q w = \gamma \int_{\mathbb{R}^N} f(x) U_{\lambda,\xi}^q w = 0 \tag{3.4}$$

because of (2.5) and (2.6) and passing to the limit as $\varepsilon \to 0$. Again, integrating by parts (3.4),

$$\int_{\mathbb{R}^N} f(x) U_{\lambda,\xi}^q [Nw + \langle (x-\xi), \nabla w \rangle] + q \int_{\mathbb{R}^N} f(x) U_{\lambda,\xi}^{q-1} w \langle (x-\xi), \nabla U_{\lambda,\xi} \rangle = 0. \tag{3.5}$$

From (3.3), (3.5), Corollary 1.2 and Lemma 2.8, $\nabla \mathcal{J}(\lambda, \xi) = 0$ and

$$\nabla^2 \mathcal{J}(\lambda, \xi)(c_0, c_1, \dots, c_N)^T = 0$$

with $\nabla^2 \mathcal{J}(\lambda, \xi)$ an invertible matrix, which implies $c_0 = c_1 = c_2 \cdots = c_N = 0$. Also, note that there will be some cancelation in Lemma 2.8 due to (2.12) and (2.13). This proves that $w \equiv 0$ in $\mathbb{R}^N$ and hence $w_n \to 0$ in $C^4_{\mathrm{loc}}(\mathbb{R}^N)$. Hence, we must have $|x_n| \to \infty$. As usual, we define the Kelvin transform of the functions $u_{i,n}(x)$ and $w_n(x)$ as

$$\hat{u}_{i,n}(x) = |x|^{4-N} u_{i,n}\Big(\frac{x}{|x|^2}\Big), \quad i = 1, 2, \quad \hat{w}_n(x) = |x|^{4-N} w_n\Big(\frac{x}{|x|^2}\Big), \quad x \in \mathbb{R}^N \backslash \{0\}.$$

Furthermore, define

$$\hat{c}_n(x) = \int_0^1 [t\hat{u}_{1,n} + (1-t)\hat{u}_{2,n}]^{8/(N-4)}\, dt,$$
$$\hat{d}_n(x) = \int_0^1 [t\hat{u}_{1,n} + (1-t)\hat{u}_{2,n}]^{q-1}\, dt.$$

Clearly, we have $|\hat{w}_n(x_n/|x_n|^2)| \geq \frac{1}{2}$ for all large n. It is easily seen that $\hat{w}_n$ satisfies the following equation:

$$\Delta^2 \hat{w}_n = \hat{c}_n \hat{w}_n + \varepsilon f\left(\frac{x}{|x|^2}\right) |x|^{-(N+4)+q(N-4)} \hat{d}_n \hat{w}_n.$$

By the decay estimate, we obtain $|\hat{w}_n(x)| \leq 1$ for all n and all $x \in B_1(0) \backslash \{0\}$. Since $\hat{w}_n \to 0$ in $C^4_{\mathrm{loc}}(\mathbb{R}^N \backslash \{0\})$, by the dominated convergence theorem, we obtain $\hat{w}_n \to 0$ in $L^p(B_1(0))$ for all $p \geq 1$. Using the assumption $f \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and the estimate (2.3),

$$\hat{c}_n(x), f\Big(\frac{x}{|x|^2}\Big)|x|^{-\tau} \hat{d}_n(x)$$

are bounded sequences in $L^2(B_1(0))$. Using L^p theory on $\hat{w}_n$ [17, Corollary 2.23, page 45],

$$\|\hat{w}_n\|_{L^\infty(B_{\frac{1}{2}}(0))} \leq C \|\hat{w}_n\|_{L^p(B_1(0))} \to 0.$$

This gives a contradiction, since

$$\|\hat{w}_n\|_{L^\infty(B_{\frac{1}{2}}(0))} \geq \left| \hat{w}_n\Big(\frac{x_n}{|x_n|^2}\Big) \right| \geq \frac{1}{2}$$

for all large n. This proves the theorem.

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Proof of Theorem 1.5. By the assumptions, the nondegenerate critical points of $\mathcal{J}$ are contained in the interior of a ball $K = \overline{B}_R(0) \subset \mathbb{R}^+ \times \mathbb{R}^N$ for some $R > 0$. Let (λ_i, ξ_i) be the nondegenerate critical points of $\mathcal{J}$ $(i = 1, 2, \ldots, s)$ contained in K. Then, by Theorem 1.1 and Corollary 1.2, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the problem (1.5) has at least s solutions $u_{\varepsilon,i}$ and s points (λ_i, ξ_i) such that $u_{\varepsilon,i} - U_{\lambda_i,\xi_i} \to 0$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$. For any $\mu > 0$, define

$$\mathcal{S}_\mu = \{u \text{ solves (1.5) for } \varepsilon \in (0, \mu)\} \backslash \{u_{\varepsilon,i}\}_{0<\varepsilon<\mu, 1\le i\le s}.$$

Let

$$\theta_\mu = \inf_{u\in\mathcal{S}_\mu} d(u, \mathcal{M}_K).$$

We now claim that

$$\theta_0 = \liminf_{\mu\to 0} \theta_\mu > 0.$$

If possible, let $\theta_0 = 0$; then there exist sequences $\{u_n\} \subset \mathcal{S}_\mu$ and $\{(\lambda_n, \xi_n)\} \subset K$ such that $\|u_n - U_{\lambda_n,\xi_n}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Let $(\lambda_n, \xi_n) \to (\lambda, \xi)$. Then $(\lambda, \xi) \in K$ and $\nabla\mathcal{J}(\lambda, \xi) = 0$ and hence $\{u_n\}$ is a sequence of solutions bifurcating from (λ, ξ). But, by the uniqueness theorem (Theorem 1.3) and $\{u_n\} \subset \mathcal{S}_\mu$, we obtain a contradiction. This proves the claim.

As a result, we can choose $\mu_0 > 0$ small such that $\theta_\mu \ge \theta_0/2$ for all $\mu < \mu_0$. By Theorem 1.1, there exist some $C > 0$ and $\varepsilon' > 0$ such that

$$d(u_{\varepsilon,i}, \mathcal{M}_K) \le C\varepsilon, \quad i = 1, \ldots, s, \quad \varepsilon \in (0, \varepsilon').$$

Choosing $\rho_0 = \theta_0/2$ and $\varepsilon_1 = \min\{\theta_0/2C, \mu_0, \varepsilon'\}$, we obtain the required result.

References

[1] A. Ambrosetti and M. Badiale, 'Variational perturbative methods and bifurcation of bound states from the essential spectrum', *Proc. Roy. Soc. Edinburgh Sect. A* **128**(6) (1998), 1131–1161.

[2] A. Ambrosetti, A. Garcia and I. Peral, 'Perturbation of $\Delta u + u^{N+2N-2} = 0$, the scalar curvature problem in $\mathbb{R}^N$ and related topics', *J. Funct. Anal.* **165** (1999), 117–149.

[3] A. Ambrosetti and A. Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on* $\mathbb{R}^N$, Progress in Mathematics, 240 (Birkhäuser, Basel, 2006).

[4] M. Ben Ayed and K. El Mehdi, 'The Paneitz curvature problem on lower-dimensional spheres', *Ann. Global Anal. Geom.* **31**(1) (2007), 1–36.

[5] T. Branson, 'Differential operators canonically associated to a conformal structure', *Math. Scand.* **57** (1985), 293–345.

[6] T. Branson, A. Chang and P. Yang, 'Estimates and extremals for zeta function determinants on four-manifolds', *Comm. Math. Phys.* **149**(2) (1992), 241–262.

[7] G. Caristi and E. Mitidieri, 'Harnack inequality and applications to solutions of biharmonic equations', in: *Partial Differential Equations and Functional Analysis*, Operator Theory Advances and Applications, 168 (Birkhäuser, Basel, 2006), 1–26.

[8] A. Chang, 'On Paneitz operator—a fourth-order partial differential equation in conformal geometry', in: *Harmonic Analysis and Partial Differential Equations; Essays in honor of Alberto P. Calderon*, Chicago Lectures in Mathematics, 1999 (eds. M. Christ, C. Kenig and C. Sadorsky) (University of Chicago Press, 1996), Ch. 8, 127–150.

https://doi.org/10.1017/S1446788714000196 Published online by Cambridge University Press

[9] A. Chang, M. Gursky and P. Yang, 'The scalar curvature equation on 2- and 3-spheres', *Calc. Var. Partial Differential Equations* **1** (1993), 205–229.

[10] A. Chang and P. Yang, 'On a fourth order curvature invariant', in: *Spectral Problems in Geometry and Arithmetic*, Contemporary Mathematics, 237 (American Mathematical Society, Providence, RI), 9–28.

[11] H. Chtioui and A. Rigane, 'On the prescribed Q-curvature problem on $\mathbb{S}^N$', *J. Funct. Anal.* **261**(10) (2011), 2999–3043.

[12] Z. Djadli, E. Hebey and M. Ledoux, 'Paneitz-type operators and applications', *Duke Math. J.* **104**(1) (2000), 129–169.

[13] Z. Djadli, A. Malchiodi and M. O. Ahmedou, 'Prescribing a fourth order conformal invariant on the standard sphere, Part I: a perturbation result', *Commun. Contemp. Math.* **4** (2002), 1–34.

[14] P. Esposito, 'Perturbations of Paneitz–Branson operators on $\mathbb{S}^N$', *Rend. Semin. Mat. Univ. Padova* **107** (2002), 165–184.

[15] P. Esposito and F. Robert, 'Mountain pass critical points for Paneitz–Branson operators', *Calc. Var. Partial Differential Equations* **15**(4) (2002), 493–517.

[16] V. Felli, 'Existence of conformal metrics on $\mathbb{S}^N$ with prescribed fourth-order invariant', *Adv. Differential Equations* **7**(1) (2002), 47–76.

[17] F. Gazzola, H. C. Grunau and G. Sweers, 'Polyharmonic boundary value problems', in: *Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains*, Lecture Notes in Mathematics, 1991 (Springer, Berlin, 2010).

[18] M. Gursky, 'The Weyl functional, de Rham cohomology, and Kahler–Einstein metrics', *Ann. of Math.* (2) **148** (1998), 315–337.

[19] E. Hebey and F. Robert, 'Asymptotic analysis for fourth order Paneitz equations with critical growth', *Adv. Calc. Var.* **4**(3) (2011), 229–275.

[20] J. L. Kazdan and F. W. Warner, 'Curvature functions for compact 2-manifolds', *Ann. of Math.* (2) **99**(1) (1974), 14–47.

[21] C. S. Lin, 'A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^N$', *Comment. Math. Helv.* **73**(2) (1998), 206–231.

[22] S. Paneitz, 'A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds', *SIGMA Symmetry Integrability Geom. Methods Appl.* **4** (2008), 1–3.

[23] S. Prashanth, S. Santra and A. Sarkar, 'On the perturbed Q-curvature problem on $\mathbb{S}^4$', *J. Differential Equations* **255**(8) (2013), 2363–2391.

[24] K. Sandeep, 'A compactness type result for Paneitz–Branson operators with critical nonlinearity', *Differential Integral Equations* **18**(5) (2005), 495–508.

[25] J. Wei and X. Xu, 'Classification of solutions of higher order conformally invariant equations', *Math. Ann.* **313**(2) (1999), 207–228.

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