

A CRITERION FOR THE EXISTENCE OF BIHARMONIC GREEN'S FUNCTIONS

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Abstract

The biharmonic Green's function of a simply supported plate is generalized to Riemannian manifolds and shown to exist if and only if the harmonic measure of the ideal boundary is square integrable.

The harmonic Green's function was originally introduced as the electrostatic potential of a point charge in a grounded system. Its characterization by the fundamental singularity and vanishing boundary values permitted its generalization to regular subregions Ω of an abstract Riemann surface or Riemannian manifold R . The Green's function g on R was then defined as the directed limit, if it exists, of the Green's function g_n on Ω as $\{\Omega\}$ exhausts R . The distinction of Riemann surfaces and Riemannian manifolds into hyperbolic and parabolic types according as g does or does not exist is still a cornerstone of the harmonic classification theory.

The biharmonic Green's function γ also has an important physical meaning: it is the deflection of a thin elastic plate under a point load. However, in sharp contrast with the harmonic case, nothing seems to be known about its existence on noncompact spaces. The purpose of the present paper is to initiate research on this fundamental problem of biharmonic classification theory.

Biharmonicity being not meaningful on abstract Riemann surfaces, our aim is to generalize the definition of the biharmonic Green's function to Riemannian manifolds R and to explore its existence on them. On a regular subregion Ω of R , there exist two biharmonic Green's functions, to be denoted by β and γ , with a biharmonic fundamental singularity, and with boundary data $\beta = \partial\beta/\partial n = 0$ and $\gamma = \Delta\gamma = 0$. For dimension 2, both functions give the deflection under a point load of a thin plate which is clamped or simply supported at the edges,

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respectively. Our present investigation deals exclusively with γ . The corresponding function γ_Ω on Ω increases with Ω , and we set $\gamma = \lim_{\Omega \rightarrow R} \gamma_\Omega$ on R .

We first study the existence of γ on the Euclidean N -space R^N . The result is fascinatingly simple: γ exists if and only if $N > 4$. By way of preparation we recall the peculiar behavior of the biharmonic fundamental singularity at the origin: $r^2 \log r$ for $N = 2$, r for $N = 3$, $\log r$ for $N = 4$, and r^{4-N} for $N > 4$.

No parabolic Riemannian manifold carries γ . For a hyperbolic Riemannian manifold R we deduce a natural criterion: If $\bar{R}_0 \subset \Omega \subset R$, and ω_Ω is harmonic on $\Omega - \bar{R}_0$ with boundary values 1 on ∂R_0 , 0 on $\partial \Omega$, and we denote by $\omega = \lim_{\Omega \rightarrow R} \omega_\Omega$ the harmonic measure of ∂R_0 on $R - R_0$, then γ exists on R if and only if $\omega \in L^2(R - R_0)$. An essential step of the reasoning is the proof of the independence of the existence of γ on the choice of the fundamental singularity. This property allows us to introduce the class O_R of Riemannian manifolds which do not carry γ , in analogy with the class O_G of parabolic manifolds.

As a simple illustration of our criterion we generalize the above result that $R^N \in O_R$ if and only if $N \leq 4$. We ask whether one could induce γ to exist even for these low dimensions by replacing the Euclidean metric $ds = |dx|$ by $ds = (1 + r^2)^\alpha |dx|$, with the constant α sufficiently large. The answer is intriguing: the resulting space is in O_R for $N \leq 4$ regardless of what α is chosen. For $N > 4$, γ continues to exist if and only if $\alpha > -\frac{1}{2}$.

The usefulness of the criterion $\omega \in L^2$ for $R \notin O_R$ lies in the fact that it also applies if there is no way of obtaining an expression for the approximating Green's functions γ_Ω , and even if nothing is known about the metric of an arbitrarily small neighborhood of the ideal boundary of the Riemannian manifold.

Applications of our criterion to relations between O_R and other biharmonic null classes will be discussed in later studies [see Ralston and Sario (to appear), Sario (to appear), Wang (1974), Wang (to appear), in the attached Bibliography of recent work in the field].

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Throughout this paper, Δ will stand for the Laplace-Beltrami operator $d\delta + \delta d$, biharmonicity being defined by $\Delta^2 \cdot = 0$. Let Ω be a regular subregion of a Riemannian manifold R , carrying the biharmonic Green's function $\gamma_\Omega(x, y)$ on $\bar{\Omega}$, with the biharmonic fundamental singularity at $y \in \Omega$ and with the boundary data

$$(1) \quad \gamma_\Omega|_{\partial \Omega} = 0, \quad \Delta \gamma_\Omega|_{\partial \Omega} = 0.$$

In terms of the harmonic Green's function $g_\Omega(x, y)$ on Ω with singularity y , the function $\gamma_\Omega(x, y)$ has the well known integral representation

$$(2) \quad \gamma_\Omega(x, y) = \int_\Omega g_\Omega(x, z)g_\Omega(z, y)dz,$$

with dz the Riemannian volume element at z . To see this, it suffices to verify

$$(3) \quad \Delta \int_{\Omega} g_{\Omega}(x, z)g_{\Omega}(z, y)dz = g_{\Omega}(x, y),$$

where Δ is taken with respect to x . For every $\phi \in C_0^{\infty}$,

$$\begin{aligned} \int_{\Omega} \Delta \left(\int_{\Omega} g_{\Omega}(x, z)g_{\Omega}(z, y)dz \right) \cdot \phi(x)dx &= \int_{\Omega} \left(\int_{\Omega} g_{\Omega}(x, z)g_{\Omega}(z, y)dz \right) \Delta \phi(x)dx \\ &= \int_{\Omega} \left(\int_{\Omega} g_{\Omega}(x, z)\Delta \phi(x)dx \right) g_{\Omega}(z, y)dz = \int_{\Omega} g_{\Omega}(z, y)\phi(z)dz. \end{aligned}$$

Therefore (3) is true in the sense of distributions and a fortiori in the conventional sense.

We introduce the biharmonic Green's function $\gamma(x, y)$ on a Riemannian manifold R by setting

$$(4) \quad \gamma(x, y) = \lim_{\Omega \rightarrow R} \gamma_{\Omega}(x, y)$$

provided the limit exists for some exhaustion $\{\Omega\}$. We shall show in No. 4 that the existence is independent of the exhaustion $\{\Omega\}$ and the choice of the singularity y .

2

We start by examining the existence of $\gamma(x, y)$ in an illuminating special case, the Euclidean N -space R^N . Here the computation is elementary. A function $h(r)$ is harmonic if $\Delta h(r) = 0$, i.e., $-r^{-N+1}(r^{N-1}h'(r))' = 0$. We obtain

$$(5) \quad h(r) = \begin{cases} a \log r + b, & N = 2, \\ ar^{-N+2} + b, & N > 2, \end{cases}$$

where a, b are arbitrary constants. If $u(r)$ belongs to the class H^2 of nonharmonic biharmonic functions on R^N , then $\Delta u(r) = h(r)$. A straightforward integration yields the biharmonic, quasiharmonic, harmonic, and constant components of $u(r)$:

$$(6) \quad u(r) = \begin{cases} ar^2 \log r + br^2 + c \log r + d, & N = 2, \\ ar + br^2 + cr^{-1} + d, & N = 3, \\ a \log r + br^2 + cr^{-2} + d, & N = 4, \\ ar^{-N+4} + br^2 + cr^{-N+2} + d, & N > 4, \end{cases}$$

with a, b, c, d arbitrary constants.

Let B_ρ be a ball $\{r < \rho\}$ and take $x \in B_\rho, |x| = r$. In view of (6) and $\Delta r^2 = -2N$, the biharmonic Green's function on B_ρ with singularity 0 is

$$(7) \quad \gamma_\rho(x, 0) = \begin{cases} r^2 \log \frac{r}{\rho} - (r^2 - \rho^2), & N = 2, \\ -r + \rho + \frac{1}{3}(r^2 - \rho^2)\rho^{-1}, & N = 3, \\ -\log \frac{r}{\rho} + \frac{1}{4}(r^2 - \rho^2)\rho^{-2}, & N = 4, \\ r^{-N+4} - \rho^{-N+4} + \frac{N-4}{N}(r^2 - \rho^2)\rho^{-N+2}, & N > 4, \end{cases}$$

where the signs have been so chosen that $\gamma_\rho > 0$. As $\rho \rightarrow \infty, \{B_\rho\}$ exhausts R^N , and we obtain

$$(8) \quad \gamma(x, 0) = r^{-N+4}, \quad N > 4,$$

whereas $\lim_{\rho \rightarrow \infty} \gamma_\rho = \infty$ for $N = 2, 3, 4$.

We have proved:

THEOREM 1. *The biharmonic Green's function exists on the Euclidean N -space if and only if $N > 4$.*

3

For further illustration we recall the simple computation showing the significance of the biharmonic Green's function in expressing values of a biharmonic function u in terms of the boundary values of u and Δu . Given a regular subregion Ω of a Riemannian manifold, let $y \in \Omega, u \in H^2(\bar{\Omega})$, and $\gamma \in H^2(\Omega - \{y\}) \cap C^3(\bar{\Omega} - \{y\})$. Take a compact hypersurface α enclosing y and suppose that as α shrinks to y ,

$$(9) \quad \int_\alpha \Delta \gamma * du \rightarrow 0, \int_\alpha \Delta u * d\gamma \rightarrow 0, \int_\alpha \gamma * d\Delta u \rightarrow 0.$$

Then

$$(10) \quad u(y) = \frac{1}{F(\Delta \gamma)} \int_{\partial \Omega} u * d\Delta \gamma - \Delta \gamma * du + \Delta u * d\gamma - \gamma * d\Delta u,$$

with $F(\Delta \gamma) = \int_\alpha * d\Delta \gamma$ the flux of $\Delta \gamma$. In fact, by Stokes' formula the integral on the right taken along $\partial \Omega - \alpha$ is

$$\begin{aligned} & (du, d\Delta \gamma) - (u, \Delta^2 \gamma) - (d\Delta \gamma, du) + (\Delta \gamma, \Delta u) \\ & + (d\Delta u, d\gamma) - (\Delta u, \Delta \gamma) - (d\gamma, d\Delta u) + (\gamma, \Delta^2 u) = 0, \end{aligned}$$

where the inner products are taken over the region bounded by $\partial\Omega \cup \alpha$. If $\langle \varepsilon \rangle$ stands for a quantity which $\rightarrow 0$ as $\alpha \rightarrow y$, then

$$\int_{\alpha} u * d\Delta\gamma = \int_{\alpha} (u(y) + \langle \varepsilon \rangle) * d\Delta\gamma \rightarrow u(y)F(\Delta\gamma),$$

and (10) follows.

In the Euclidean case of No. 2, Ω is the ball $\{r < \rho\}$, y the origin, and $\gamma = \gamma_{\rho}$. Accordingly,

$$(11) \quad u(0) = \frac{1}{F(\Delta\gamma)} \int_{r=\rho} u * d\Delta\gamma_{\rho} + \Delta u * d\gamma_{\rho},$$

provided (9) holds.

Denote by dS the surface element on the sphere $S_{\rho} = \{r = \rho\}$ and let α be the sphere $\{r = \delta < \rho\}$. For $N = 2$, (7) gives

$$\begin{aligned} *d\gamma_{\rho} &= \left(2r \log \frac{r}{\rho} - r\right) dS, \quad \Delta\gamma_{\rho} = -4 \log \frac{r}{\rho}, \\ *d\Delta\gamma_{\rho} &= -4r^{-1}dS, \quad F(\Delta\gamma_{\rho}) = -4 \int_{\alpha} r^{-1}dS = -\int_0^{2\pi} 4d\phi = -8\pi. \end{aligned}$$

On the other hand, $|u|$, $|\partial u / \partial n|$, $|\Delta u|$ are bounded on B_{ρ} , and as $\delta \rightarrow 0$,

$$\begin{aligned} \left| \int_{\alpha} \Delta\gamma_{\rho} * du \right| &< M \left| \delta \log \frac{\delta}{\rho} \right| \rightarrow 0, \\ \left| \int_{\alpha} \Delta u * d\gamma_{\rho} \right| &< M \left| 2\delta \log \frac{\delta}{\rho} - \delta \right| \delta \rightarrow 0, \\ \left| \int_{\alpha} \gamma_{\rho} * d\Delta u \right| &= c(\delta)F(\Delta u) = 0. \end{aligned}$$

Equation (11) follows, with $F(\Delta\gamma_{\rho}) = -8\pi$.

For $N = 3$, (7) gives

$$\begin{aligned} *d\gamma_{\rho} &= (-1 + \frac{2}{3}r\rho^{-1})dS, \quad \Delta\gamma_{\rho} = 2r^{-1} - 2\rho^{-1}, \\ *d\Delta\gamma_{\rho} &= -2r^{-2}dS, \quad F(\Delta\gamma_{\rho}) = -2 \int r^{-2}r^2d\omega = -8\pi, \end{aligned}$$

where $d\omega$ is the area element of the unit sphere. As $\delta \rightarrow 0$,

$$\begin{aligned} \left| \int_{\alpha} \Delta\gamma_{\rho} * du \right| &< M(2\delta^{-1} - 2\rho^{-1}) \cdot 4\pi\delta^2 \rightarrow 0, \\ \left| \int_{\alpha} \Delta u * d\gamma_{\rho} \right| &< M \left| -1 + \frac{2}{3}\delta\rho^{-1} \right| \cdot 4\pi\delta^2 \rightarrow 0, \\ \left| \int_{\alpha} \gamma_{\rho} * d\Delta u \right| &= c(\delta)F(\Delta u) = 0, \end{aligned}$$

and (11) follows, with $F(\Delta\gamma_{\rho}) = -8\pi$.

For $N = 4$, (7) gives

$$\begin{aligned} *d\gamma_\rho &= (-r^{-1} + \frac{1}{2}r\rho^{-2})dS, \quad \Delta\gamma_\rho = 2r^{-2} - 2\rho^{-2}, \\ *d\Delta\gamma_\rho &= -4r^{-3}dS, \quad F(\Delta\gamma_\rho) = -4 \int r^{-3}r^3d\omega = -4\omega_3, \end{aligned}$$

where ω_3 is the area of the unit 3-sphere. As $\delta \rightarrow 0$,

$$\begin{aligned} \left| \int_\alpha \Delta\gamma_\rho * du \right| &< M(2\delta^{-2} - 2\rho^{-2})\omega_3\delta^3 \rightarrow 0, \\ \left| \int_\alpha \Delta u * d\gamma_\rho \right| &< M | -\delta^{-1} + \frac{1}{2}\delta\rho^{-2} | \omega_3\delta^3 \rightarrow 0, \\ \left| \int_\alpha \gamma_\rho * d\Delta u \right| &= c(\delta)F(\Delta u) = 0, \end{aligned}$$

and (11) follows, with $F(\Delta\gamma_\rho) = -4\omega_3$.

Finally, for $N > 4$, (7) gives

$$\begin{aligned} *d\gamma_\rho &= (N-4)r^{-N+3} \left(-1 + \frac{2}{N} \left(\frac{r}{\rho} \right)^{N-2} \right) dS, \\ \Delta\gamma_\rho &= (N-4) \left[-(N-3) \left(r^{-N+2} + \frac{2}{N(N-3)} \rho^{-N+2} \right) \right. \\ &\quad \left. - (N-1) \left(-r^{-N+2} + \frac{2}{N} \rho^{-N+2} \right) \right], \\ *d\Delta\gamma_\rho &= -2(N-2)(N-4)r^{-N+1}dS, \\ F(\Delta\gamma_\rho) &= -2(N-2)(N-4)\omega_{N-1}, \end{aligned}$$

where ω_{N-1} is the area of the unit $(N-1)$ -sphere. As $\delta \rightarrow 0$,

$$\begin{aligned} \left| \int_\alpha \Delta\gamma_\rho * du \right| &< M | c_1\delta^{-N+2} + c_2 | c_3\delta^{N-1} \rightarrow 0, \\ \left| \int_\alpha \Delta u * d\gamma_\rho \right| &< M | c_1\delta^{-N+3} + c_2\delta | c_3\delta^{N-1} \rightarrow 0, \\ \left| \int_\alpha \gamma_\rho * d\Delta u \right| &= c(\delta)F(\Delta u) = 0, \end{aligned}$$

and (11) follows, with $F(\Delta\gamma_\rho) = -2(N-2)(N-4)\omega_{N-1}$.

In summary, (11) is true for all N . For our present purpose of illustrating the use of γ this will suffice. Equation (11) generalizes, however, in various directions. First, since the fundamental singularity is locally defined, the above reasoning continues to hold when $u(0)$ is replaced by $u(y)$ with $y \in B_\rho$, and $\gamma_\rho(x, 0)$ by $\gamma_\rho(x, y)$. On a Riemannian manifold R , the geodesic distance d is, in a

sufficiently small neighborhood of y , a constant multiple of the Euclidean r in a parametric ball, added by a function which together with its derivatives is bounded. Therefore, the above elementary reasoning, *mutatis mutandis*, remains valid and (11) holds for $u(y)$ when S_p is replaced by the boundary $\partial\Omega$ of a regular subregion of R , and γ_p by the biharmonic Green's function $\gamma_\Omega(x, y)$ on Ω .

4

We return to our existence problem. The proof of the following intermediate result is essential.

PROPOSITION. *The existence of a biharmonic Green's function γ on a Riemannian manifold is independent of the exhaustion and the choice of the singularity.*

PROOF. Let $\{\Omega\}$ be some exhaustion of R as in No. 1. By (2) and Lebesgue's Dominated Convergence Theorem, if a biharmonic Green's function γ on R with singularity y exists, it can be written as

$$(12) \quad \gamma(x, y) = \int_R g(x, z)g(z, y)dz,$$

where g is the harmonic Green's function on R , and the integral is the directed limit of $\int_\Omega g_\Omega(x, z)g_\Omega(z, y)dz$ as $\Omega \rightarrow R$. Thus we have immediately the independence of the exhaustion. It remains to show that if γ exists for some (x_1, y_1) , then it exists for any (x_2, y_2) .

From (12), we see that γ can exist only on hyperbolic manifolds, and

$$(13) \quad \int_R g(x_1, z)g(z, y_1)dz < \infty.$$

Choose a regular subregion R_0 of R containing x_1, x_2, y_1 , and y_2 , with boundary α . Clearly (13) is equivalent to

$$\int_{R-R_0} g(x_1, z)g(z, y_1)dz < \infty,$$

which in turn is equivalent to

$$\int_{R-R_0} g(x_1, z)g(y_1, z)dz < \infty,$$

by virtue of the symmetry of the harmonic Green's function. Since $g(x, z), g(y, z) > 0$ on R_0 for each $z \in R - \bar{R}_0$, Harnack's inequality gives

$$g(y, z) < K g(x, z)$$

for all x, y in R_0 , with the constant K depending only on R and R_0 . Therefore,

$$K^{-1} \int_{R-R_0} g(x, z)^2 dz < \int_{R-R_0} g(x, z)g(y, z) dz < K \int_{R-R_0} g(x, z)^2 dz,$$

and we conclude that $\gamma(x_1, y_1)$ exists if and only if $\int_{R-R_0} g(x_1, z)^2 dz < \infty$, that is,

$$g(\cdot, x_1) \in L^2(R - R_0).$$

Next we shall show that $g(\cdot, x_1) \in L^2(R - R_0)$ if and only if the harmonic measure ω of α on $R - R_0$ belongs to $L^2(R - R_0)$. Recall that ω is the limit of the harmonic function ω_α on $\Omega - \bar{R}_0$ with boundary values 1 on $\alpha, 0$ on $\partial\Omega$, for some and hence every exhaustion $\{\Omega\}$ of R with $\bar{R}_0 \subset \Omega$. The function ω always exists, and $\omega \equiv 1$ if and only if R is parabolic. Set

$$m_\alpha = \min_\alpha g_\alpha(\cdot, x_1), \quad M_\alpha = \max_\alpha g_\alpha(\cdot, x_1).$$

By the maximum principle for harmonic functions,

$$m_\alpha \omega_\alpha \leq g_\alpha(\cdot, x_1) \leq M_\alpha \omega_\alpha$$

on $\bar{\Omega} - R_0$. On letting $\Omega \rightarrow R$, we obtain

$$(14) \quad m\omega \leq g(\cdot, x_1) \leq M\omega$$

on $R - R_0$, with

$$m = \min_\alpha g(\cdot, x_1), \quad M = \max_\alpha g(\cdot, x_1).$$

Thus $g(\cdot, x_1) \in L^2(R - R_0)$ if and only if $\omega \in L^2(R - R_0)$. Since $g(\cdot, x_2) \in L^2(R - R_0)$ is characterized by the same condition, we have proved the existence of $\gamma(x_2, y_2)$ as a consequence of that of $\gamma(x_1, y_1)$.

We also conclude from the above reasoning that $\lim_{\Omega \rightarrow R} \gamma_\Omega$ is either identically infinite or a biharmonic function γ on $R - \{y\}$ with the fundamental singularity y .

5

In view of the Proposition, we may introduce the class of Riemannian manifolds

$$(15) \quad O_\Gamma = \{R \mid \gamma \text{ does not exist}\}.$$

Let \tilde{O}_Γ be the class of manifolds which do carry γ . From the proof of the Proposition, we have the following criterion:

THEOREM 2. *Every parabolic Riemannian manifold R belongs to O_Γ . A hyperbolic R belongs to \tilde{O}_Γ if and only if $\omega \in L^2(R - R_0)$ for some and hence every R_0 .*

6

Using Theorem 2, we can extend Theorem 1. We ask whether γ exists even for the low dimensions if R^N is generalized to the Riemannian manifold R_α^N with the same base manifold $\{r < \infty\}$ but with the metric

$$(16) \quad ds = (1 + r^2)^\alpha |dx|,$$

α a constant. For a harmonic $h(r)$ we have by direct computation

$$h(r) = a \int_1^r r^{-N+1} (1 + r^2)^{-(N-2)\alpha} dr + b$$

as $r \rightarrow \infty$. If $\alpha \leq -\frac{1}{2}$, this is unbounded, hence $R_\alpha^N \in O_G \subset O_r$. For $\alpha > -\frac{1}{2}$, the harmonic measure ω satisfies

$$\omega(r) \sim ar^{-(N-2)(2\alpha+1)},$$

and the L^2 -norm of $\omega(r)$ over the annulus $(1, r)$ is

$$\|\omega\|_2^2 \sim c \int_1^r r^{-2(N-2)(2\alpha+1)+N-1+2N\alpha} dr = c \int_1^r r^{-2(N-4)\alpha-(N-3)} dr.$$

This is bounded if and only if $N > 4$ and $\alpha > -\frac{1}{2}$. We have proved that R_α^N fails to carry γ for $N \leq 4$ regardless of how rapidly or slowly the metric $(1 + r^2)^\alpha |dx|$ grows:

THEOREM 3. $R_\alpha^N \in \tilde{O}_r$ if and only if $N > 4$ and $\alpha > -\frac{1}{2}$.

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