FINITE GROUPS GENERATED BY SUBNORMAL T-SUBGROUPS

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1. Introduction. Our aim in this paper is to investigate the restrictions placed on the structure of a finite group if it can be generated by subnormal T-subgroups (a T-group is a group in which every subnormal subgroup is normal). For notational convenience we denote by $\mathcal{K}$ the class of finite groups that can be generated by subnormal T-subgroups and by $\mathcal{K}^*$ the subclass of $\mathcal{K}$ of those finite groups generated by normal T-subgroups; and for the remainder of this paper we will only consider finite groups.

T-groups may be regarded as a generalisation of abelian groups. We know that a group generated by subnormal abelian subgroups is nilpotent; and a group generated by normal abelian subgroups has class bounded by the number of abelian normal subgroups needed to generate it. We will be seeking results analogous to these for groups in $\mathcal{K}$ and $\mathcal{K}^*$.

We begin by considering soluble $\mathcal{K}$-groups. One of the basic properties of T-groups is that they are supersoluble and we will show that this property nearly carries over to $\mathcal{K}$-groups; indeed $\mathcal{K}$-groups of odd order are supersoluble, as are soluble $\mathcal{K}^*$-groups.

THEOREM 2. *Suppose that G is a soluble $\mathcal{K}$-group. Then G is metanilpotent, the odd order Sylow subgroups of $G/F(G)$ are abelian, and $O^2(G)$ is a supersoluble group of odd order. Moreover if G is a $\mathcal{K}^*$-group, G is supersoluble.*

We show that the subnormal structure of a soluble group in $\mathcal{K}^*$ is controlled by the number of normal T-subgroups needed to generate it. We use the Wielandt length of a group as a measure of the complexity of its subnormal subgroup structure. Here the Wielandt subgroup $\omega(G)$ of a group G is the intersection of the normalisers of all subnormal subgroups and the Wielandt series is defined by $\omega_1(G) = \omega(G)$ and $\omega_{i+1}(G)/\omega_i(G) = \omega(G/\omega_i(G))$; the Wielandt length of G is then the smallest n such that $\omega_n(G) = G$. As an easy consequence of Theorem 2 and Theorem 1, which gives an estimate for the Wielandt length of a supersoluble group in terms of the classes of its Sylow subgroups, we get the following theorem.

THEOREM 4. *Let G be a soluble $\mathcal{K}^*$-group generated by normal T-subgroups $N_i, i = 1, \dots, n$. Then G has Wielandt length at most $n+1$ and this bound is best possible.*

If G is nilpotent we can improve this bound a little. For odd primes p, T-groups of p-power order are abelian and so we are considering the product of abelian normal subgroups. It is easy to produce examples where the class and the Wielandt length are both just the number of factors and to show that the class and Wielandt length must be at most the number of factors. Since T-groups of 2-power order may be nonabelian, the 2-group case is more complicated and is given by the next theorem.

THEOREM 3. *Suppose that G is a 2-group such that $G = N_1 \dots N_n$, with each N_i a normal T-subgroup of G.*

(i) *If $n=2$ then G has class and Wielandt length at most* 3 *and this bound is best possible.*

(ii) *If $n>2$ then G has class and Wielandt length at most n and this bound is best possible.*

We then turn to insoluble groups in $\mathscr{X}$. The layer of a group G is the subgroup generated by the subnormal single headed perfect T-subgroups of G and is denoted by $E(G)$. If G can be generated by subnormal T-subgroups we show that $G/E(G)$ can be generated by subnormal soluble T-subgroups. Since $E(G)$ is contained in the Wielandt subgroup, we can now use the results for soluble groups to give analogues of the theorems above. In particular if G can be generated by n normal T-subgroups, G will have Wielandt length at most $n+2$. It seems to be difficult to determine if this bound is best possible.

Finally we consider possible generalisations of these results. An example due to Casolo shows that the natural generalisations do not hold.

2. Preliminaries. An account of the basic structure theorems and properties for T-groups and the Wielandt subgroup can be found in Robinson [**12**], Lennox and Stonehewer [**9**] or Bryce and Cossey [**2**]; any unreferenced properties of T-groups or the Wielandt subgroup may be found in any of these.

Recall that a Dedekind group is one in which every subgroup is normal and a Hamiltonian group is a nonabelian Dedekind group. A Hamiltonian group is the direct product of a quaternion group of order 8 and an abelian torsion group with no elements of order 4 (see [**7**, Satz 3.7.12]).

We will need the following lemma about irreducible modules for p-groups.

Lemma 1. *Let p,q be distinct primes, F an algebraically closed field of characteristic q, P a p-group and U a faithful irreducible $GF(q)P$-module. Then if X is a noncentral cyclic subgroup of P, U_X contains at least p nonisomorphic composition factors.*

Proof. We proceed by induction on $|P|$. There is nothing to prove if P is abelian. Hence suppose the result is true for p-groups of order less than $|P|$. If possible choose M to be a maximal subgroup of P containing X with X noncentral in M and consider U_M. If U_M is irreducible the result follows by induction, since M is then nonabelian and faithfully and irreducibly represented on U. Hence we suppose U_M is reducible, say $U_M=U_1+\ldots+U_k$. Since M is nonabelian the dimensions of the U_i are greater than 1 and moreover for some J we have $XC_M(U_j)/C_M(U_j)$ not central in $M/C_M(U_j)$. It then follows that U_{jX} has at least p nonisomorphic composition factors. Hence we may suppose that X is central in every maximal subgroup of P containing it. Thus if $y\notin C_P(X)$, $P=\langle X,y\rangle$ and $C=C_P(X)=\langle X,y^p\rangle$ is an abelian maximal subgroup of P. We then have $U_C=U_1+\ldots+U_p$, with the U_i all nonisomorphic and one dimensional. Since y^p is central in P the $U_{i\langle y^p\rangle}$ are all isomorphic and so, if U_{iX} were isomorphic to U_{jX}, we would have U_i isomorphic to U_j. Thus we have that U_X has p distinct composition factors. This completes the proof. □

Lemma 2. *Let G be supersoluble and suppose that a Sylow p-subgroup of G is nonabelian, of class n say. Then $\gamma_n(P)$ is normal in G and contained in $\omega(G)$.*

Proof. Put $N = \gamma_n(P)$. Since N clearly centralises every chief factor of G we have $N \le F(G)$. Then if Q is the Hall p'-subgroup of $F(G)$, QN/Q is characteristic in the normal Sylow p-subgroup of G/Q and so QN is normal in G. But then N is the Sylow p-subgroup of QN and so is normal in G.

If S is a subnormal subgroup of G, then $S/C_S(N)$ is a p'-group (since P centralises N) and so, by a lemma of Higman [**6**], $N = U \times V$ where $[U, S] = U$ and $[V, S] = 1$. It follows that $U \le S$ and hence N normalises S, proving $N \le \omega(G)$. □

We have as an easy consequence of this lemma, that in a supersoluble group the Wielandt length is bounded by the classes of the Sylow subgroups.

THEOREM 1. *Let G be a supersoluble group, let π be the set of primes dividing $|G^{\mathcal{N}}|$ (where $G^{\mathcal{N}}$ is the nilpotent residual of G) and σ the set of primes dividing $|G|$ but not $|G^{\mathcal{N}}|$. Suppose all the Sylow p-subgroups with $p \in \pi$ have class at most n and that the Sylow p-subgroups with $p \in \sigma$ have class at most m. Then G has Wielandt length at most* $\max(m, n+1)$.

Proof. Put $t = \max(m, n+1)$. Suppose that the result is not true and t is chosen minimal such that there is a group G of Wielandt length greater than t. If $t > 2$, let Z be the subgroup of G generated by $\gamma_m(P_p)$, where P_p is a Sylow p-subgroup of $G, p \in \sigma$, and $\gamma_n(P_q)$, where P_q is a Sylow q-subgroup of G, $q \in \pi$. Then we have $Z \le \omega(G)$ by Lemma 2 and by induction G/Z has Wielandt length at most $t-1$, giving G of Wielandt length at most t, a contradiction. Thus we must have $t \le 2$. If $t = 1$ then G is an abelian σ-group and so has Wielandt length 1, again a contradiction. Thus we must have $t = 2$.

It follows that the Sylow p-subgroups of G are abelian, $p \in \pi$, while the Sylow p-subgroups, $p \in \sigma$, are of class at most 2. Consider G'. A Sylow p-subgroup A of G' is then central in any Sylow p-subgroup of G containing it for $p \in \sigma$; the supersolubility of G then gives A central in G and so $A \le \omega(G)$. For a prime $p \in \pi$, a Sylow p-subgroup B of G' has centraliser of p' index and by an argument similar to that in Lemma 2 we have $b \le \omega(G)$. It follows that $G' \leqq \omega(G)$ and hence G has Wielandt length at most 2, a contradiction. The result now follows. □

3. Soluble $\mathcal{X}$-groups. We will show in this section that a soluble group G in $\mathcal{X}$ is close to being supersoluble. A corollary of our theorem is that a soluble group G in $\mathcal{X}$ of odd order will be supersoluble and so the Sylow subgroups of $G/F(G)$ will be abelian. The Sylow 2 subgroup of $G/F(G)$ for a soluble group G in $\mathcal{X}$ however is less restricted. The following examples show that neither the class nor the exponent need be bounded. Let S_0 be a cyclic group of order 2, and if S_i has been defined set $S_{i+1} = S_0 wr S_i$. Then S_i can be generated by $i+1$ subgroups of order 2, $X_1, \ldots, X_{i+1}$ say. It follows immediately from [**7**, Satz 1.15.9] that S_i contains a cyclic subgroup of order 2^{i+1} and so clearly S_i has exponent 2^{i+1}. Moreover S_i contains a subgroup isomorphic to the wreath product of S_0 and a cyclic group of order 2^i, which has class 2^i ([**10**, Lemma 2.1]) and hence the class of S_i is at least 2^i. Let p be an odd prime and let U be a faithful irreducible $GF(p)S_i$-module. Since X_j has order 2, U as an X_j-module can be written as the direct sum of a trivial module and a module V_j on which X_j acts nontrivially and homogeneously. We then have that V_jX_j is a normal subgroup of UX_j which is subnormal in US_i and so V_jX_j is subnormal in S_i. These 2-groups are generated by elements of order two and it seems likely that elements of order two play an important role. However, given an integer n, we can

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construct soluble groups in $\mathcal{K}$ for which the Sylow 2-subgroup cannot be generated by elements of 2^n. To see this let p be a prime such that $p-1$ is divisible by 2^{n+1}. If A is a cyclic group of order p and B is a cyclic subgroup of $\mathrm{Aut}(A)$ of order 2^{n+1} then $H = ABwrS_0$ has AB as a subnormal T-subgroup. Further if P is a Sylow p-subgroup of H then $P = P_0 \times P_1$ where P_1 is centralised by S_0 and so P_0S_0 is also a subnormal T-subgroup. It is easy to see that H is generated by AB and P_0S_0 and that a Sylow 2-subgroup of H cannot be generated by elements of order at most 2^n.

THEOREM 2. *Suppose that G is a soluble $\mathcal{K}$-group. Then G is metanilpotent, the odd order Sylow subgroups of $G/F(G)$ are abelian, and $O^2(G)$ is a supersoluble group of odd order. Moreover if G can be generated by normal T-subgroups, G is supersoluble.*

Proof. Let N_i, $i = 1, \ldots, n$, be subnormal T-subgroups generating G and note that it will not affect the hypothesis if we assume that the set we take includes all the conjugates in G of any of its members. Thus we suppose throughout this proof that the set of N_j is a union of G-conjugacy classes. That G is metanilpotent follows immediately from the fact that T-groups are metanilpotent and metanilpotent groups form a Fitting class.

Suppose now that p is an odd prime dividing $|G/F(G)|$ and that the Sylow p-subgroup of $G/F(G)$ is nonabelian. We suppose also that G has been chosen minimal with this property. We must then have for some i, j with $i \neq j$ that N_i contains an element x of p-power order and N_j contains an element y of p-power order with $[x, y]$ not in $F(G)$. It then follows that for some chief factor U/V of G with U contained in $F(G)$ that $[x, y]$ is not in the centraliser of U/V. By the minimality of G we must have $V = 1$. Since $[x, y]$ does not centralise U, neither does x. We then have $1 \neq [U, x] = [U, x, x] \leq N_i$ and so x acts as a (nontrivial) power automorphism on $[U, x]$. It follows that $U_{\langle x \rangle}$ is a direct sum of one dimensional submodules. Let N be the subgroup generated by $F(N_i)$ and x and let M be the subgroup generated by $F(N_j)$ and y. Then M and N are subnormal T-subgroups of G. Let H be the subgroup by M and N. Since $[x, y]$ is in H so is $[U, [x, y]]$ and then $[U, [x, y], [x, y]] \neq 1$. Again by the minimality of G, we must have $H = G$. Now, regarding U as a faithful irreducible $GF(q)(G/C_G(U))$-module, Lemma 1 tells us that $U^F_{\langle x \rangle}$ contains at least p nonisomorphic composition factors. But we have shown that $U_{\langle x \rangle}$ has all its composition factors of dimension 1 and so $U_{\langle x \rangle}$ has all its composition factors absolutely irreducible. Since p is an odd prime and $[U, x]$ contains all the nontrivial composition factors, $[U, x]$ has at least two nonisomorphic composition factors. This contradiction shows that $[x, y] \in F(G)$ and so the Sylow p-subgroup of $G/F(G)$ is abelian.

Note that the proof above also show that if U/V is a chief factor of G on which G acts as a $2'$-group, then U/V is cyclic. Now set $M = O^2(G)$. We have $O^2(N_i) \leq M$ and hence if H is the subgroup generated by the $O^2(N_i)$, $H \leq M$. Since our generating subnormal T-subgroups contain all the conjugates of the N_i we have H is normal in G. But then G/H is generated by subnormal 2-subgroups N_iH/H and so is a 2-group. It follows that $H = M$; that is $O^2(G)$ can be generated by subnormal T-subgroups which are $2'$-groups and hence is a $2'$-group. It then follows immediately that $O^2(G)$ is a supersoluble group of odd order.

Finally suppose that G is generated by normal T-subgroups N_i, $1 = 1, \ldots, n$, and that G is not supersoluble. Suppose that G has been chosen minimal with this property. We have $F(G)/\Phi(F(G))$ must contain a noncyclic chief factor and so $\Phi(F(G)) = 1$. Since $F(G)$ is then a direct product of minimal normal subgroups of G we must have by the minimality of G that $F(G)$ is the unique minimal normal subgroup of G. Since N_i is normal in G we have $F(G) \leq N_i$, $i = 1, \ldots, n$. This means that N_i must act on $F(G)$ as a subgroup of the group of power automorphisms of $F(G)$ and hence G itself must act on $F(G)$ as a group of power automorphisms. It now follows easily from [**9**, Proposition 6.4.8] that $F(G)$ must be cyclic. This contradiction completes the proof. □

4. Soluble $\mathscr{K}^*$-groups. We begin by considering nilpotent $\mathscr{K}^*$-groups; clearly it will be enough to consider p-groups. If G is a p-group which is the product of n normal T-subgroups, then if p is an odd prime G is the product of n abelian normal subgroups and so by [**5**, Theorem 10.3.2] G has class at most n. We show below that the Sylow p-subgroups of $GL(n+1, p)$ give examples for which the class and Wielandt length are precisely n. If $p = 2$, some of the factors may be Hamiltonian groups and so we have ([**5**, Theorem 10.3.2]) G has class at most $2n$; it is easy to see that this can be reduced to $n + 1$. We show that in most cases the bound can be reduced to n.

THEOREM 3. *Suppose that G is a 2-group such that $G = N_1 \ldots N_n$, with each N_i a normal T-subgroup of G.*

(i) *If $n = 2$ then G has class and Wielandt length at most* 3 *and this bound is best possible.*
(ii) *If $n > 2$ then G has class and Wielandt length at most n and this bound is best possible.*

Proof. Since N_i is a 2-group in which every subgroup is normal it is either abelian or the direct product of a quaternion group with an elementary abelian group ([**5**, Theorem 12.5.4]). It follows that N_n' is a normal subgroup of G of order at most 2 and hence is a central subgroup of G. It then follows that $N_1' \ldots N_n'$ is central in G and so by [**5**, Theorem 10.3.2] $G/(N_1' \ldots N_n')$ has class at most n, giving G of class at most $n+1$.

If $n = 2$, the generalised quaternion group H of order 16 can be written as the product of a normal quaternion subgroup and a cyclic normal subgroup and the dihedral group D of order 8 can be written as the product of two elementary abelian normal subgroups of order 4. Thus $G = H \times D$ can be written as the normal product of two normal T-subgroups. Since G has class 3 and Wielandt length 3, the bound in (i) is best possible.

For $n > 2$, to show that G has class n it will be enough to show that every commutator of weight $n+1$ with each entry from some N_i is trivial. Observe that $N_i/\zeta(N_i)$ has order at most 4 and hence any commutator of weight at least 3 with an entry from N_i will be contained in $\zeta(N_i)$. Suppose $c = [a_1, \ldots, a_{n+1}]$ is a commutator with each entry from some N_i; then at least 2 entries of c come from the same N_i, a_k and a_ℓ say, with $k < \ell$. If $\ell < n+1$, then $[a_1, \ldots, a_{\ell-1}] \in N_i$ and so $[a_1, \ldots, a_\ell] \in N_i'$ and is therefore central in G. But then $[a_1, \ldots, a_{\ell+1}] = 1$ and $c = 1$. Hence suppose $\ell = n+1$. Since $n + 1 \geq 4$, $[a_1, \ldots, a_n]$ is central in N_i, giving $c = 1$. This shows that G has class at most n and hence G can have Wielandt length at most n. To see that this bound is best possible, we take the Sylow 2-subgroups of $GL(n+1, 2)$. □

THEOREM 4. *Let G be a soluble $\mathscr{K}^*$ group generated by normal T-subgroups N_i, $i = 1, \ldots, n$. Then G has Wielandt length at most $n+1$ and this bound is best possible.*

Proof. We note first that, by Theorem 2, G is supersoluble. Moreover, a Sylow p-subgroup of G is generated by n Dedekind normal subgroups. Thus if p is an odd

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prime, a Sylow p-subgroup of G has class at most n ([5, Theorem 10.3.2]). A Sylow 2-subgroup P is a product of normal subgroups P_i where P_i is a Sylow 2-subgroup of N_i. Since P_i is a T-group, Theorem 3 gives us that P has class at most $n+1$. That G has Wielandt length at most $n+1$ is now an immediate corollary of Theorem 1.

To see that this bound is best possible, we construct a group of Wielandt length n which can be generated by $n-1$ normal T-subgroups. The construction is based on the analysis of the Sylow p-subgroups of $GL(n,p)$ given in section 3.16 of [7]. We will use the notation of that section, so that the construction given here needs to be read in conjunction with that section. We observe that if p is an odd prime the Sylow p-subgroup G of $GL(n,p)$ has class $n-1$ ([7, Satz 3.16.3]) and can be generated by the elements $E+e_{j,j-1}$ of order p, $j=2,\dots,n$ ([7, Satz 3.16.2 and Satz 3.16.4]). Since the Wielandt subgroup must centralise any element of order p in a p-group, it follows that the Wielandt series of G coincides with the upper central series of G and thus G has Wielandt length $n-1$. We now consider the abelian normal subgroups A_i ([7, satz 3.16.6]). It is easy to check that $E+e_{j,j-1}\in A_{j-1}$ and so G is certainly generated by the $n-1$ normal abelian subgroups A_i. Note that A_{n-1} is generated by the elements $E+e_{n,i}$ for $i=1,\dots,n-1$. Now define a map θ on G by

$$\theta(E+\Sigma a_{jk}e_{jk})=E+\Sigma a_{jk}e_{jk}-\Sigma a_{nk}e_{nk}.$$

It is easy to check that θ is an automorphism of G of order 2 and that each A_i is invariant under the action of θ. Moreover θ inverts every element of A_{n-1} and $[A_i,\theta]\le A_{n-1}$. Thus $A_{n-1}\langle\theta\rangle$ is a T-group and it follows that $H=G\langle\theta\rangle$ is the product of the $n-1$ normal T-subgroups $A_1,\dots,A_{n-2},A_{n-1}\langle\theta\rangle$. To determine the Wielandt length of H, we note that for $i<n-1$ we have $\zeta_i(G)=\gamma_i(G)\le\omega_i(H)$ by Lemma 2. Since $\omega_i(H)\cap G\le\omega_i(G)=\zeta_i(G)$ we have $\omega_i(H)\cap G=\zeta_i(G)$ for $i<n-1$. If $\omega_i(H)$ is not contained in G for some $i<n-1$, we would have some conjugate of θ contained in $\omega_i(H)$ and hence A_{n-1} contained in $\omega_i(H)\cap G=\zeta_i(G)$, a contradiction. Thus $\omega_{n-2}(H)=G'$. Since H/G' contains both central and noncentral p-chief factors, it is not a T-group and so $\omega_{n-1}(H)\ne H$. Thus H has Wielandt length precisely n, as required. □

5. Insoluble $\mathscr{X}$-groups. We will say that a group is single headed if it has a unique maximal normal subgroup. A component of a group G is a subnormal single headed perfect subgroup and the layer of G, denoted by $E(G)$, is the subgroup of G generated by all the components of G. (For the basic properties of components and $E(G)$, see [1, section 31] or [8, section 10.13].)

Suppose that G is generated by subnormal T-subgroups N_i, $i=1,\dots,n$. Clearly any component of N_i is a component of G and so $E(N_i)\le E(G)$ for $i=1,\dots,n$. On the other hand suppose that C is a component of G that is not a component of any N_i. By [1, 31.4] we have that C commutes with each N_i. But then C would be central in G, a contradiction. Thus $E(G)$ is generated by the $E(N_i)$, $i=1,\dots,n$. We also have that a component of G is normal in G, since it is normalised by any N_i that contains it and centralised by the others. Thus any insoluble chief factor of G is simple. Since moreover $N_i/E(N_i)$ is soluble by [3], $G/E(G)$ is generated by the subnormal soluble T-subgroups $N_iE(G)/E(G)$ and so is soluble and has the structure given by Theorem 2.

Note also that it is an immediate consequence of [1, 31.4] that $E(G)\le\omega(G)$ for any finite group G and hence by Theorem 4 if G is generated by normal T-subgroups N_i,

$i=1,\ldots,n$, G has Wielandt length at most $n+2$. Whether this bound is best possible or not seems to be a difficult question. We have only been able to find the best possible bound in one very special case, namely when $G=MN$, with M,N normal T-subgroups with $M/E(M)$ and $N/E(N)$ both of odd order. We record it here to indicate the difficulties involved.

We now suppose that $G=MN$ with M,N normal T-subgroups of G and $M/E(M)$ and $N/E(N)$ both of odd order. We will show that G has Wielandt length at most 3. If G is soluble, then G has Wielandt length at most 3 by Theorem 4 and so we may suppose that G is insoluble. Moreover if $M\cap N=1$, we have $G=M\times N$, giving $G/E(G)$ has Wielandt length at most 2 and hence G has Wielandt length at most 3. Thus we may also suppose that $M\cap N\neq 1$. The main step in the proof will be to establish the following result.

Let $P/E(G)$ be a Sylow p-subgroup of G, p a prime. If p is odd and $G/E(G)$ has noncentral p chief factors, $(P/E(G))'\leq \omega(G)/E(G)$.

We begin by assuming that G is a group of minimal order satisfying the hypotheses above such that $(P/E(G))'$ is not contained in $\omega(G)/E(G)$. Thus we can find $z=[x,y]\notin\omega(G)$, with $x\in G_p\cap M$, $y\in G_p\cap N$, with G_p a Sylow p-subgroup of G. We can then choose S minimal subnormal in G such that $S^z\neq S$. We now show that the structure of S is very restricted.

First we note if U were a nontrivial normal subgroup of G contained in S the quotient group G/U would satisfy the hypotheses given and hence we would have $(S/U)^{zU}=S/U$ and so $S^z=S$, a contradiction. Since a component of S would be a component of G and hence normal in G, we have S soluble, giving $S\leq S(G)$, the soluble radical of G. Moreover any subnormal subgroup of $M\cap N$ is then normal in M and in N and hence in G, so that we have $S\cap(M\cap N)=1$. Next we observe that S is single headed. For if $S=TU$, with T,U normal subgroups of S, then $T^z=T$ and $U^z=U$ by the minimality of S and hence $S^z=S$, a contradiction. Thus S/S' is a cyclic q-group for some prime q. Further, $SE(G)/E(G)$ is supersoluble by Theorem 2 and $S\cap E(G)=S\cap\zeta(E(G))\leq\zeta(S(G))$ and so S is supersoluble. It then follows that S' is a nilpotent q'-group.

Next suppose that G contains a nonabelian minimal normal subgroup U say. Then in G/U we have $(US/U)^{zU}=US/U$. But since S is a subnormal soluble subgroup of SU it is contained in the soluble radical of SU and then since the soluble radical of SU clearly cannot have order larger than S we must have S is the soluble radical of SU. Thus S is characteristic in SU and so $S^z=S$, giving G cannot contain a nonabelian minimal normal subgroup. Suppose then that G contains a minimal normal subgroup U of q' order. Again we have $(SU)^z=SU$. Then we have $S=O^{q'}(SU)$ and S is characteristic in SU. It follows that if U is a minimal normal subgroup of G, U is a q-group. This in turn gives $S'=1$. For if not, S' is a nontrivial subnormal q'-subgroup of G and so $O_{q'}(G)\neq 1$, and G would have a minimal normal q'-subgroup, a contradiction. Hence S is a cyclic q-group.

Now suppose that $q\neq p$. Let $V=\zeta(E(G))$ (note that $V\neq 1$). Then $SE(G)/E(G)$ is a subnormal q-subgroup of $G/E(G)$, while $(G_pE(G)/E(G))'$ is a normal p-subgroup of $G/E(G)$ and so these two subgroups commute. We have $S^z\leq SE(G)\cap S(G)=SV$ and so if s generates S we have $s^z=sv$ with $v\in V$. Thus z centralises SV/V. If C is a component of G with $C\leq M\cap N$, then we have from above that both x and y act as power automorphisms on $\zeta(C)$ and hence z centralises $\zeta(C)$. If C is a component of G not contained in M, then $[M,C]=1$ and y acts as a power automorphism of $\zeta(C)$, giving z

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centralises $\zeta(C)$. Similarly if C is not contained in N z centralises $\zeta(C)$. Thus z centralises V. But then z centralises SV and so any subgroup of SV, in particular S, a contradiction. Thus we must have $q = p$. Note also that the argument gives S not contained in V.

We also must have that distinct cyclic minimal normal subgroups are nonisomorphic as G-modules. For if X and Y are distinct cyclic minimal normal subgroups of G, isomorphic as G-modules, and S is generated by s, we would have $s^z = s^\alpha x = s^\beta y$, with α, β integers and $x \in X$, $y \in Y$. Thus $s^{\alpha-\beta} = yx^{-1}$. If $\alpha \neq \beta$, then $\langle s^{\alpha-\beta}\rangle$ is a nontrivial normal subgroup of G, a contradiction. If $\alpha = \beta$ then $x = y$ and $X = Y$, also a contradiction. If then follows from the fact that G acts on $M \cap N$ as a group of power automorphisms that $M \cap N$ contains a unique minimal normal subgroup of G.

Note that so far we have not needed any restriction on the prime p or on the p chief factors. For the remainder of the proof the hypotheses will be necessary.

Since at least one of $M/E(M)$, $N/E(N)$ must contain noncentral p chief factors, we assume that $M/E(M)$ does. We then have $M/E(M)$ contains only noncentral p chief factors. Put $K = S(M)S(N)$ and suppose that S is not contained in K. Then all chief factors in $S(G)/K$ are central (since $[S(G), M] \leq S(G) \cap M = S(M) \leq K$ and similarly $[S(G), N] \leq K$). Since G/NK is a quotient of $M/S(M)$ and thus contains no central p chief factors we have $S(G) \leq NK$. But then $S(G)/K$ is a soluble normal subgroup of NK/K which has no normal soluble subgroups, being isomorphic to $N/(N \cap K) = N/S(N)$. It follows that $S(G) \leq K$ and so $S(G) = K$. Moreover since K is supersoluble we have that a Sylow p-subgroup L of K is normal and hence $L = XY$ where $X = L \cap M_p$, $Y = L \cap N_p$, and M_p, N_p are Sylow p-subgroups of M, N respectively. But now X is an abelian normal subgroup of the T-group M and hence M acts on X as a group of power automorphisms. Thus x acts on X as a power automorphism and then $z = [x, y]$ centralises X since power automorphisms are central in the automorphism group of X (this is an immediate consequence of [**12**, 13.4.3(ii)]). Similarly z centralises Y and hence $L = XY$. But $S \leq L$ and so $S^z = S$, a contradiction.

To see that G has Wielandt length at most 3, observe that $G/\omega(G)$ is supersoluble and that by the result just proved the Sylow subgroups of $(G/\omega(G))^{\mathcal{N}}$ are abelian. Since $G/\omega(G)$ is of odd order, the Sylow subgroups of $G/\omega(G)$ are of class at most two and so, by Theorem 1, $G/\omega(G)$ has Wielandt length at most 2 and the result follows.

6. Remarks and examples. It is natural to ask if Theorem 4 can be extended and in particular if the Wielandt length of a group G which is the product of normal subgroups N, M has its Wielandt length bounded in terms of the Wielandt lengths of N and M. The construction below, due to C. Casolo, shows this is not true. The construction gives for each positive integer n a group of Wielandt length at least n which can be written as the product of two normal subgroups of Wielandt length 2. This leaves only the case where G is the product of a normal T-subgroup and another normal subgroup, of Wielandt length n say. In this case I expect that a soluble G will have Wielandt length at most $n+1$, but have been unable to prove it.

Let N be the direct product of two cyclic groups of order 2^n and let $H = \mathrm{Sym}(3)$ act on N as group of automorphisms (we can take the first of the representations in Example 1, page 505, Curtis and Reiner [**4**] of H on $\mathbb{Z} \times \mathbb{Z}$ and factor out the subgroup $(\mathbb{Z} \times \mathbb{Z})^{2^n}$ for example). Let t be the automorphism of N which inverts every element, so that t commutes with H and $\langle H, t\rangle = H \times \langle t\rangle$, with $\langle t\rangle$ of order 2. Let G be the semidirect

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product of N and $H \times \langle t \rangle$. If a is an element of H of order 2, then H and $K = H'\langle at \rangle$ are normal subgroups of $H \times \langle t \rangle$ both isomorphic to Sym(3). In HN the subnormal subgroups are the subgroups of N, NH' and NH. It is then clear that $N \leq \omega(NH)$ (in fact it is not difficult to check that $N = \omega(NH)$) and so $NH/\omega(NH)$ is a quotient of H and so is a T-group. Since NH is not a T-group we have NH of Wielandt length 2. Similarly we have NK of Wielandt length 2. If G has Wielandt length w, any subnormal normal subgroup of G has Wielandt length at most w. If $y \in N$ has order 2 the subgroup $\langle y, t \rangle$ is subnormal in G and dihedral of order 2^{n+1}. But then $\langle y, t \rangle$ has Wielandt length n by [**11**, Corollary 4]. Thus $w \geq n$.

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