ARCHIMEDEAN CLOSURES IN LATTICE-ORDERED GROUPS

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1. Introduction. Conrad (**10**) and Wolfenstein (**15**; **16**) have introduced the notion of an archimedean extension (a-extension) of a lattice-ordered group (l-group). In this note the class $\mathcal{N}$ of l-groups that possess a plenary subset of regular subgroups which are normal in the convex l-subgroups that cover them are studied. It is shown in § 3 (Corollary 3.4) that the class $\mathcal{N}$ is closed with respect to a-extensions and (Corollary 3.7) that each member of the class $\mathcal{N}$ has an a-closure. This extends (**6**, p. 324, Corollary II; **10**, Theorems 3.2 and 4.2; **15**, Theorem 1) and gives a partial answer to (**10**, p. 159, Question 1). The key to proving both of these results is Theorem 3.3, which asserts that if a regular subgroup is normal in the convex l-subgroup that covers it, then this property is preserved by a-extensions.

Theorem 4.1 of § 4 generalizes (**8**, Theorem 6.3) and (**14**, Theorem 6.2) by showing that the only members of $\mathcal{N}$ which are topological groups in their interval topology are the totally ordered groups (o-groups).

My thanks are due to my advisor, Professor Paul F. Conrad.

2. Preliminaries. In this section, some definitions and notation are given. Throughout this note G will denote an l-group. The reader is referred to (**1**; **12**) for the standard results concerning l-groups.

A convex l-subgroup that is maximal with respect to not containing some g in G is called a *regular subgroup*. Let $\Gamma(G)$ be an index set for the collection of all regular subgroups G_γ of G. For each $\gamma \in \Gamma(G)$ there exists a unique convex l-subgroup G^γ of G that covers G_γ. If g belongs to G^γ but not G_γ, then γ (or G_γ) is said to be a *value* of g. A regular subgroup G_γ is called *special* if there exists an element g in G such that G_γ is the unique value of g. If this is the case, then g is also called special. For $\gamma, \lambda \in \Gamma(G)$ we define $\gamma \leqq \lambda$ if $G_\gamma \subseteq G_\lambda$. With this order, $\Gamma(G)$ is a *root system* (**9**, Theorem 3.3), that is, a partially ordered set in which no two incomparable elements have a common lower bound. A subset Δ of $\Gamma(G)$ is said to be *plenary* if

(i) each $0 \neq g$ in G has at least one value in Δ;

(ii) if $g \notin G_\delta$ ($\delta \in \Delta$), then there exists $\lambda \geqq \delta$ ($\lambda \in \Delta$) such that λ is a value for g.

Δ_g will denote the set of all values of g which are members of Δ.

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If g and h are positive elements in an l-group H such that $g \leqq nh$ and $h \leqq mg$ for some positive integers m and n, then g and h are said to be a-*equivalent*. H is said to be an a-*extension* of an l-subgroup G of H if for each $0 < h \in H$ there exists $0 < g \in G$ such that g and h are a-equivalent. G is said to be a-*closed* if there does not exist a proper a-extension of G. An a-extension of G which is itself a-closed is called an a-*closure* of G.

Let G be an l-subgroup of an l-group H and let $\mathscr{C}(G)$ ($\mathscr{C}(H)$) denote the lattice of convex l-subgroups of G (H). We define a mapping σ from $\mathscr{C}(G)$ into $\mathscr{C}(H)$ by

$$M\sigma = \bigcap\{J \in \mathscr{C}(H) : M \subseteq J\} \qquad (M \in \mathscr{C}(G)).$$

Then σ is a lattice isomorphism of $\mathscr{C}(G)$ onto $\mathscr{C}(H)$ if and only if H is an a-extension of G (**10**, Theorem 2.1). If this is the case, then

$$J\sigma^{-1} = J \cap G \qquad (J \in \mathscr{C}(H)).$$

Suppose that H is an a-extension of G and for $\gamma \in \Gamma(G)$ let $G_\gamma\sigma = H_\gamma$ and let $G^\gamma\sigma = H^\gamma$. It was shown (**10**, p. 137) that $\{H_\gamma : \gamma \in \Gamma(G)\}$ is the collection of all regular subgroups of H, and hence the same index set may be chosen for the regular subgroups of G and H. In particular, σ maps a plenary subset onto a plenary subset.

Let Λ be a root system and for each λ in Λ let R_λ be a subgroup of the naturally ordered additive group of real numbers. Let Π denote the unrestricted direct sum of the R_λ's and for $v = (\ldots, v_\lambda, \ldots) \in \Pi$, let

$$S_v = \{\lambda \in \Lambda : v_\lambda \neq 0\}.$$

Let $V(\Lambda, R_\lambda) = \{v \in \Pi : S_v \text{ satisfies the maximum condition}\}$. For v in $V(\Lambda, R_\lambda)$, let $\Lambda_v = \{\lambda \in S_v : v_\alpha = 0 \text{ for all } \alpha > \lambda\}$. If $\lambda \in \Lambda_v$, then λ is said to be a *maximal component* of v. Then $v \in V(\Lambda, R_\lambda)$ is defined to be positive if $v_\lambda > 0$ for each $\lambda \in \Lambda_v$. With this order, $V(\Lambda, R_\lambda)$ is an abelian l-group (**11**, Theorems 2.1 and 2.2). The main embedding theorem in (**11**) asserts that every abelian l-group can be embedded as an l-subgroup in an l-group of this form. For each $\lambda \in \Lambda$ let $V_\lambda = \{v \in V(\Lambda, R_\lambda) : v_\alpha = 0 \text{ for all } \alpha \geqq \lambda\}$. Then it is shown in (**11**) that V_λ is a regular subgroup of $V(\Lambda, R_\lambda)$, Λ is a plenary subset of $\Gamma(V(\Lambda, R_\lambda))$, and if $\lambda \in \Lambda_v$, then λ is a value of v.

If $M \in \mathscr{C}(G)$, then $r(M)$ will denote the collection of right cosets of M in G. This collection is partially ordered by the relation $M + x \leqq M + y$ if and only if $m + x \leqq y$ for some $m \in M$. With respect to this order, $r(M)$ is a distributive lattice in which $(M + x) \vee (M + y) = M + x \vee y$ and dually. In particular, if M is a regular subgroup, then $r(M)$ is a totally ordered set (**9**, Theorem 3.2).

If $T \subseteq G$, then $[T]$ will denote the subgroup of G that is generated by T and if A and B are sets, then $A\backslash B$ will denote the set of elements in A but not in B.

3. Archimedean extensions. Throughout this section we shall assume that H is an a-extension of G, that σ is defined as in § 2, that $\Gamma(G) = \Gamma(H)$, and that for $\gamma \in \Gamma(G)$, $G_\gamma\sigma = H_\gamma$ and $G^\gamma\sigma = H^\gamma$. If $M \in \mathscr{C}(G)$, let $N_G(M)$ ($N(M\sigma)$) denote the normalizer of $M(M\sigma)$ in $G(H)$.

LEMMA 3.1. *If $M \in \mathscr{C}(G)$, then $N_G(M) = G \cap N(M\sigma)$.*

Proof. If $x \in N_G(M)$, then $M = x + M - x \subseteq x + M\sigma - x$. Thus, $M \subseteq (x + M\sigma - x) \cap G = x + (M\sigma \cap G) - x = x + M - x = M$. Since σ^{-1} is one-to-one, it follows that $M\sigma = x + M\sigma - x$. Conversely, if $x \in G \cap N(M\sigma)$, then $x + M - x = x + (M\sigma \cap G) - x = (x + M\sigma - x) \cap G = M\sigma \cap G = M$.

LEMMA 3.2. *If M is a maximal convex l-subgroup of G, then $x \in N_G(M)$ if and only if $M + x + g = M + x$ for all g in M.*

Proof. If for each g in M, $M + x + g = M + x$, then $M = M + x + g - x$ and it follows that $x + M - x \subseteq M$. Since $x + M - x$ is also a maximal convex l-subgroup of G, $x + M - x = M$. The converse is immediate.

THEOREM 3.3. *For each $\gamma \in \Gamma(G)$, G_γ is normal in G^γ if and only if H_γ is normal in H^γ.*

Proof. Since H^γ is an a-extension of G^γ (**10**, p. 135, Corollary I), it suffices to take G_γ maximal in G. Assume that G_γ is normal in G and suppose (by way of contradiction) that there exists $0 < y \in H\backslash N(H_\gamma)$. By Lemma 3.2 there exists $h \in H_\gamma$ such that $H_\gamma + y < H_\gamma + y + h \leqq H_\gamma + y + |h|$, where $|h| = h \vee -h$. Hence, it may be assumed that $h > 0$. By induction it follows that

$$H_\gamma < H_\gamma + y < H_\gamma + y + h < H_\gamma + y + 2h < \ldots .$$

Now $0 < y + h - y \in H$, and since H is an a-extension of G, there exists $0 < x \in G$ such that

$$y + h - y < x < n(y + h - y) = y + nh - y < y + nh$$

for some positive integer n. Thus, for all positive integers m,

$$mx < mn(y + h - y) < y + mnh.$$

Since $y + h - y \notin H_\gamma$, it follows that $x \notin G_\gamma$. Again, since H is an a-extension of G, $y < z$ for some $z \in G$.

Now G/G_γ is an archimedean o-group, hence there is a positive integer m such that

$$G_\gamma + z < m(G_\gamma + x) = G_\gamma + mx.$$

Therefore $H_\gamma + z \leqq H_\gamma + mx$. Since $y < z$, it follows that $H_\gamma + y \leqq H_\gamma + z$. Thus

$$H_\gamma + y \leqq H_\gamma + mx \leqq H_\gamma + y + mnh.$$

Since G_γ is normal in G, it follows by Lemma 3.1 that $mx \in N(H_\gamma)$. If $H_\gamma + mx = H_\gamma + y + mnh$, then by Lemma 3.2,

$$H_\gamma + mx = H_\gamma + mx + h = H_\gamma + y + (mn + 1)h > H_\gamma + y + mnh = H_\gamma + mx,$$

a contradiction. Hence

$$H_\gamma + y \leqq H_\gamma + mx < H_\gamma + y + mnh.$$

But then

$$H_\gamma + mx < H_\gamma + y + mnh \leqq H_\gamma + mx + mnh = H_\gamma + mx,$$

a contradiction. Therefore $N(H_\gamma) = H$.

Conversely, if H_γ is normal in H, then

$$G = G \cap H = G \cap N(H_\gamma) = G \cap N(G_\gamma\sigma) = N_G(G_\gamma)$$

by Lemma 3.1.

An l-group G is said to be *representable* if there exists an l-isomorphism of G into an unrestricted cardinal sum of o-groups. Let $\mathscr{N} = \{G\colon G$ is an l-group and there exists a plenary subset $\Delta(G) \subseteq \Gamma(G)$ such that G_δ is normal in G^δ for each $\delta \in \Delta\}$. By (**3**, Corollary 3.2), if G is a representable l-group, then $G \in \mathscr{N}$.

COROLLARY 3.4. *$\mathscr{N}$ is closed with respect to* a-*extensions.*

Let Δ be a plenary subset of $\Gamma(G)$ such that G_δ is normal in G^δ for each δ in Δ. It is well known that G^δ/G_δ is o-isomorphic to a subgroup of the real numbers. Form $V(\Delta, G^\delta/G_\delta)$. If v and w are elements of $V(\Delta, G^\delta/G_\delta)$, then w is said to be a *βth head of v* if $\beta \in \Delta_v$ and $w = \theta$, the identity of $V(\Delta, G^\delta/G_\delta)$, or if:

(i) $\beta < \delta$ for some $\delta \in \Delta_v$,

(ii) $v_\alpha \neq G_\alpha$ for some $\alpha \leqq \beta, \alpha \in \Delta$,

(iii) $w_\gamma = v_\gamma$ for all $\gamma > \beta$,

(iv) $w_\gamma = G_\gamma$ for all $\gamma \leqq \beta$.

Thus v has a βth head if and only if $\beta \in \Delta_v$ or (i) and (ii) hold. A mapping π of G into $V(\Delta, G^\delta/G_\delta)$ is said to be *value-preserving* if whenever $\delta \in \Delta_g$, then $\delta \in \Delta_{g\pi}$ and $(g\pi)_\delta = G_\delta + g$.

The proof of the next theorem, which is patterned after the proof of (**6**, Lemma 1.1), is exceedingly long and will be omitted. The proof may be found in (**2**, pp. 65–71).

THEOREM 3.5. *If $G \in \mathscr{N}$ and if Δ is a plenary subset of $\Gamma(G)$ such that G_δ is normal in G^δ for each δ in Δ, then there exists a one-to-one order and value-preserving mapping π of the set G into $V(\Delta, G^\delta/G_\delta)$ such that for each g in G:*

(a) *$\Delta_g = \Delta_{g\pi}$ and $(g\pi)_\delta = G_\delta + g$ if $\delta \in \Delta_g$. $0\pi = \theta$;*

(b) *If $g\pi$ has a βth head, then there exists k in G such that $k\pi$ is a βth head of $g\pi$ and $(g\pi)_\beta = G_\beta + g - k$;*

(c) *If $k\pi$ is a βth head of $g\pi$, $k \in G$, then $(g\pi)_\beta = G_\beta + g - k$;*

(d) *If $\alpha \in \Delta_{g-k}$, $k \in G$, then $(g\pi - k\pi)_\alpha = G_\alpha + g - k$.*

COROLLARY 3.6. *Let $G \in \mathscr{N}$ and let Δ be a plenary subset of $\Gamma(G)$ such that G_δ is normal in G^δ for each $\delta \in \Delta$. Then an upper bound for the number of elements in any* a-*extension of G is c^d, where c and d are the cardinal numbers of the set of real numbers and the set Δ, respectively.*

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Proof. If H is any a-extension of G, then as observed earlier, Δ is a plenary subset of $\Gamma(H) = \Gamma(G)$. By Theorem 3.3, H_δ is normal in H^δ for each δ in Δ. By Theorem 3.5, there exists a one-to-one mapping of H into $V(\Delta, H^\delta/H_\delta)$ and by (**10**, Theorem 4.1) there exists an isomorphism of $V(\Delta, H^\delta/H_\delta)$ into $V(\Delta, R_\delta)$, where R_δ is the group of real numbers for each $\delta \in \Delta$. The cardinality of $V(\Delta, R_\delta)$ is less than or equal to c^d.

COROLLARY 3.7. *If $G \in \mathcal{N}$, then G has an* a-*closure and this* a-*closure belongs to $\mathcal{N}$.*

Proof. By Corollary 3.4, any a-extension of G belongs to $\mathcal{N}$. That G has an a-closure follows from the preceding corollary and (**10**, Lemma 2.1).

The proof of the next theorem is similar to that of (**7**, Theorem 4.1).

THEOREM 3.8. *For an l-group G and for $\gamma \in \Gamma(G)$, the following are equivalent:*
(1) *G_γ is normal in G^γ;*
(2) *If $0 < x, y \in G^\gamma \backslash G_\gamma$, then there exists a positive integer n such that $G_\gamma + y < G_\gamma + nx$;*
(3) *If $0 < x \in G^\gamma \backslash G_\gamma$ and $0 \leqq g \in G_\gamma$, then there exists $h \in G_\gamma$ and a positive integer n such that $h + nx - g > x$.*

Proof. (1) implies (2) as G^γ/G_γ is an archimedean o-group.
(2) implies (3). Let $0 < x \in G^\gamma \backslash G_\gamma$ and let $0 \leqq g \in G_\gamma$. Then

$$g + x - g \in G^\gamma \backslash G_\gamma,$$

and hence by (2) there exists a positive integer n such that

$$G_\gamma + x < G_\gamma + n(g + x - g) = G_\gamma + g + nx - g = G_\gamma + nx - g.$$

Thus $x < h + nx - g$ for some $h \in G_\gamma$.
(3) implies (1). The proof is divided into three parts. Let $0 < x \in G^\gamma \backslash G_\gamma$.
(i) If $0 < a, b \in G$ such that $G_\gamma + a \leqq G_\gamma + mx$ and $G_\gamma + b \leqq G_\gamma + nx$ for some positive integer m and n, then there exists a positive integer p such that $G_\gamma + a + b \leqq G_\gamma + px$.

It follows from the definition of the order in $r(G_\gamma)$ that $a \leqq g_1 + mx$ and $b \leqq g_2 + nx \leqq |g_2| + nx$, where $g_1, g_2 \in G_\gamma$, and it may be assumed that $g_2 \geqq 0$. Thus, $a + b \leqq g_1 + mx + g_2 + nx$. Now $0 < mx \in G^\gamma \backslash G_\gamma$ and $0 \leqq g_2 \in G_\gamma$, and hence by (3), there exists h in G_γ and a positive integer q such that $mx \leqq h + qmx - g_2$, and therefore $mx + g_2 \leqq h + qmx$. Therefore $a + b \leqq g_1 + h + qmx + nx$.
(ii) Next we show that (2) is true.
Let $0 < y \in G^\gamma \backslash G_\gamma$ and suppose (by way of contradiction) that

$$G_\gamma + nx \leqq G_\gamma + y$$

for all positive integers n. Since $x \in G^\gamma \backslash G_\gamma$, it follows that $G_\gamma + nx < G_\gamma + y$ for all n. Let $S = \{z \in G: z \geqq 0$ and $G_\gamma + z \leqq G_\gamma + mx$ for some positive

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integer $m\}$. Clearly, S is a convex set that contains 0, and by (i), S is a semigroup. Moreover, $x \in [S]\backslash G_\gamma$, $y \in G^\gamma\backslash[S]$, and $G_\gamma \subseteq [S]$. By (**8**, Theorem 2.1), $[S]$ is a convex l-subgroup of G. Thus $[S]$ properly contains G_γ and is properly contained in G^γ, which is a contradiction. Therefore (ii) holds.

(iii) G_γ is normal in G^γ.

Suppose (by way of contradiction) that there exists $0 < y \in G^\gamma\backslash G_\gamma$ such that $-y + G_\gamma + y \neq G_\gamma$. Thus there exists $0 < h \in G_\gamma$ such that $G_\gamma < G_\gamma - y + h + y$. By (2), it follows that there exists a positive integer n such that

$$G_\gamma + y < G_\gamma + n(-y + h + y) = G_\gamma - y + nh + y.$$

Therefore $G_\gamma < G_\gamma - y + nh$ or $G_\gamma = G_\gamma - nh < G_\gamma - y < G_\gamma$, a contradiction.

For g in G let $G(g)$ denote the convex l-subgroup of G generated by g. Then, as well as the conditions given in Lemma 3.2 and Theorem 3.3, it is shown in (**2**, Theorem 2.7) that Theorem 3.8 (1) is equivalent to each of the following.

(4) *$G_\gamma \cap G(g)$ is normal in $G(g)$ for each $0 < g \in G^\gamma\backslash G_\gamma$.*
(5) *$G_\gamma \cap G(g)$ is normal in $G(g)$ for some $0 < g \in G^\gamma\backslash G_\gamma$.*

A convex l-subgroup M of G is said to be *closed* if whenever $\{g_\alpha | \alpha \in A\} \subseteq M$ and $\bigvee_{\alpha\in A} g_\alpha$ exists, then $\bigvee_{\alpha \in A} g_\alpha \in M$. The next theorem and its corollary were proven by J. T. Lloyd and me.

THEOREM 3.9. *If $M \in \mathscr{C}(G)$, then M is a closed subgroup of G if and only if $M\sigma$ is a closed subgroup of H.*

Proof. If $M \in \mathscr{C}(G)$ and if $J = \{g \in G\colon g = \bigvee_G g_\alpha\ (\alpha \in A)$ for some subset $\{g_\alpha| \alpha \in A\} \subseteq M^+\}$, then $[J]$ is the smallest closed subgroup of G that contains M (**4**, Lemma 3.2). We shall denote the subgroup $[J]$ by M^*.

Assume that M is closed and let $0 \leqq h \in M\sigma^*$. Since H is an a-extension of G, there exists g in G such that $h \leqq g \leqq nh$ for some positive integer n. Then $g \in M\sigma^*$, and hence $g = \bigvee_H h_\alpha\ (\alpha \in A)$, where $\{h_\alpha\colon \alpha \in A\} \subseteq M\sigma^+$. For each α in A pick g_α in G so that $h_\alpha \leqq g_\alpha \leqq n_\alpha h_\alpha$ for some positive integer n_α. Then $g \wedge g_\alpha \in M$ and $h_\alpha \leqq g \wedge g_\alpha$ for each α. Since $g = \bigvee_H h_\alpha\ (\alpha \in A)$, it follows that $g = \bigvee_G (g \wedge g_\alpha)\ (\alpha \in A)$. Thus $g \in M$, and hence $h \in M\sigma$. Therefore $M\sigma = M\sigma^*$.

Conversely, suppose that $M\sigma$ is a closed subgroup of H and let

$$g = \bigvee_G g_\alpha \qquad (\alpha \in A),$$

where $\{g_\alpha\colon \alpha \in A\} \subseteq M^+$. Suppose (by way of contradiction) that $g \notin M$. Then $g \notin M\sigma$, and hence $M\sigma < M\sigma + g$. Let $T = \{x \in H^+\colon x \in M\sigma + g\}$. By (**4**, Lemma 3.1) there exists $0 < h_1 \in H$ such that $h_1 \leqq x$ for all x in T. In particular, $h_1 \leqq -g_\alpha + g$ for all α in A. Let $0 < g_1 \in G$ such that $h_1 \leqq g_1 \leqq nh_1$ for some positive integer n. If $h_1 \wedge g_\alpha = 0$ for all α in A, then

$$0 = nh_1 \wedge g_\alpha \geqq g_1 \wedge g_\alpha \geqq 0$$

for all α, but then $0 = \bigvee_G (g_1 \wedge g_\alpha) = g_1 \wedge (\bigvee_G g_\alpha) = g_1 \wedge g \geqq h_1 > 0$, a contradiction. Thus there exists β in A such that $h_2 = h_1 \wedge g_\beta > 0$. Clearly,

$h_2 \in M\sigma$ and $h_2 < x$ for all x in T. Let $0 < g_2 \in G$ such that $h_2 \leqq g_2 \leqq mh_2$ for some positive integer m. Then $g_2 \in M$. For each α in A we have that $-h_2 - g_\alpha + g \in T$, and thus $-h_2 - g_\alpha + g > h_2$. Hence $-g_\alpha + g > 2h_2$. It follows by induction that $-g_\alpha + g > qh_2$ for all α in A and all positive integers q. In particular, $-g_\alpha + g > mh_2 \geqq g_2$ for all α, but then $g > g - g_2 \geqq \bigvee_G g_\alpha = g$, a contradiction.

In (**4**, p. 126) a distributive radical for an l-group G was defined and was denoted by $D(G)$. The main result in (**4**) was that for an l-group G, $D(G) = 0$ if and only if G is completely distributive. The following corollary shows that an a-extension of a completely distributive l-group is completely distributive.

COROLLARY 3.10. $D(G) = G \cap D(H)$.

Proof. It was shown in (**4**, Theorem 3.4) that $D(G)$ is the intersection of all closed regular subgroups $\{G_\lambda : \lambda \in \Lambda\}$ of G. Thus

$$D(G) = \bigcap_{\lambda \in \Lambda} G_\lambda = \bigcap_{\lambda \in \Lambda} (G \cap G_\lambda \sigma) = G \cap \left(\bigcap_{\lambda \in \Lambda} G_\lambda \sigma \right) = G \cap D(H).$$

4. Interval topology. The interval topology of an l-group is defined by taking as a sub-basis for the closed sets the sets of the form $\{g \in G : g \geqq a\}$ and $\{g \in G : g \leqq a\}$ $(a \in G)$. It is well known that:

(i) if G is an o-group, then G is a topological group in its interval topology, and

(ii) if G is a topological group in its interval topology, then this topology is Hausdorff.

Choe (**5**), Conrad (**8**), and Wolk (**17**) found classes of l-groups such that if an l-group belonged to the class and was Hausdorff in its interval topology, then it was an o-group. Jakubik (**14**, Theorem 6.2) showed that a representable l-group which was Hausdorff in its interval topology must be an o-group. The class of representable l-groups contains the class of abelian l-groups, and hence the classes of Choe and Wolk. As observed in § 3, if G is a representable l-group, then $G \in \mathscr{N}$. Holland (**13**) has given an example of a non-ordered l-group that is a topological group and a topological lattice in its interval topology.

Conrad's class was the class of l-groups G that satisfy the following property:

(F) *each strictly positive element of G exceeds at most a finite number of disjoint elements.*

It can be deduced from the material in (**8**; **9**) that an l-group that satisfies property (F) is generated by its special elements, and hence there is a plenary subset $\Delta \subseteq \Gamma(G)$ such that G_δ is special for each $\delta \in \Delta$. A regular subgroup which is special is normal in the convex l-subgroup that covers it (**10**, Proposition 2.4). Therefore, if G is an l-group that satisfies property (F), then $G \in \mathscr{N}$.

An element g in G is said to be a *non-unit* if $g > 0$ and if $g \wedge h = 0$ for some $0 < h \in G$. A strictly positive element of G which is not a non-unit is said to be a *unit*. Let N be the set of all non-units of G. In (**14**, p. 68) Jakubik gave the following condition:

(v_n) *there exists* $b_1, \ldots, b_n \in N$ *such that for any* $c \in N$ *the relation*

$$c \leqq b_1 + \ldots + b_n$$

holds.

Clearly, $b_1 + \ldots + b_n$ is a unit. It is shown in (**14**, Proposition 3.3) that if G is a non-ordered l-group which is Hausdorff in its interval topology, then there exists a positive integer n ($n \geqq 2$) such that the condition (v_n) is fulfilled.

Let $\mathscr{T}$ be the class of all l-groups G that possess a plenary subset Δ of $\Gamma(G)$ such that if δ is a value of a unit in G, then G_δ is normal in G^δ ($\delta \in \Delta$). Then $\mathscr{N} \subseteq \mathscr{T}$.

THEOREM 4.1. *If* $G \in \mathscr{T}$ *and if* G *is Hausdorff in its interval topology, then* G *is an* o-*group.*

Proof. Suppose (by way of contradiction) that G is not an o-group. Then there exists a positive integer n such that the condition (v_n) is satisfied. Let $b = b_1 + \ldots + b_n$ be the element given in the condition (v_n). Since N is not void, we may choose $g \in N$. Then $b \geqq g$. If $\delta \in \Delta_g$, then $G_\delta < G_\delta + g \leqq G_\delta + b$. Thus there exists an i such that $b_i \notin G_\delta$. Let $\lambda = \max\{\gamma \in \Delta: \gamma \geqq \delta$ and γ is a value of some b_j $(1 \leqq j \leqq n)\}$. Then λ is a value of some b_j, say b_k, and λ is a value of b. Hence $G_\lambda < G_\lambda + b_k$ and G_λ is normal in G^λ. By Theorem 3.8, there exists a positive integer m such that $G_\lambda + b_i < G_\lambda + mb_k$ for $i = 1, \ldots, n$. Since G_λ is normal in G^λ, it follows that $G_\lambda + b = G_\lambda + b_1 + \ldots + b_n < G_\lambda + nmb_k$. This implies that $b \ngeqq nmb_k$ (**9**, p. 114), but this is contradictory, as $nmb_k \in N$. Therefore G must be an o-group.

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