XI
Phenomenological models

QCD has turned out to be a theory of such subtlety and difficulty that a concerted effort over an extended period has not yielded a practical procedure for obtaining analytic solutions. At the same time, vast amounts of hadronic data which require theoretical analysis and interpretation have been collected. This has spurred the development of accessible phenomenological methods. We devote this chapter to a discussion of three dynamical models (potential, bag, and Skyrme) along with a methodology based on sum rules. Although the dynamical models are constructed to mimic aspects of *QCD*, none of them *is QCD*. That is, none contains a rigorous program of successive approximations which, for arbitrary quark mass, can be carried out to arbitrary accuracy. Therefore, our treatment will emphasize issues of basic structure rather than details of numerical fits. By using all of these methods, one hopes to gain physical insight into the nature of hadron dynamics. Despite its inherent limitations the program of model building, fortified by the use of sum rules, has been generally successful, and there is now a reasonable understanding of many aspects of hadron spectroscopy.

XI–1 Quantum numbers of $Q\overline{Q}$ and Q^3 states

Among the states conjectured to lie in the spectrum of the *QCD* hamiltonian are mesons, baryons, glueballs, hybrids, dibaryons, etc. However, since practically all currently known hadrons can be classified as either $Q\overline{Q}$ states (*mesons*) or Q^3 states (*baryons*), it makes sense to focus on just these systems. We shall begin by determining the quark model construction of the light-hadron ground states. Much of the material will be valid for heavy-quark systems as well.

Hadronic flavor–spin state vectors

In many respects, the language of quantum field theory provides a simple and flexible format for implementing the quark model. Let us assume that for any given

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

dynamical model, it is possible to solve the field equations of motion and obtain a complete set of spatial wavefunctions, $\{\psi_\alpha(x)\}$ for quarks and $\{\psi_{\overline{\alpha}}(x)\}$ for antiquarks, where the labels α and $\overline{\alpha}$ refer to a complete set of observables. A quark field operator can then be expanded in terms of these wavefunctions,

$$\psi(x) = \sum_\alpha \left[\psi_\alpha(x)e^{-i\omega_\alpha t}b(\alpha) + \psi_{\overline{\alpha}}(x)e^{i\omega_{\overline{\alpha}}t}d^\dagger(\overline{\alpha})\right], \tag{1.1}$$

where $\omega_\alpha, \omega_{\overline{\alpha}}$ are the energy eigenvalues, $b(\alpha)$ destroys a quark and $d^\dagger(\overline{\alpha})$ creates the corresponding antiquark. The quark creation and annihilation operators obey

$$\begin{aligned} \left\{b(\alpha), b^\dagger(\alpha')\right\} &= \delta_{\alpha\alpha'}, & \left\{d(\overline{\alpha}), d^\dagger(\overline{\alpha}')\right\} &= \delta_{\bar{\alpha}\bar{\alpha}'}, \\ \left\{b(\alpha), b(\alpha')\right\} &= 0, & \left\{d(\overline{\alpha}), d(\overline{\alpha}')\right\} &= 0, \\ \left\{b(\alpha), d^\dagger(\overline{\alpha}')\right\} &= 0, \end{aligned} \tag{1.2}$$

which are the usual anticommutation relations for fermions.

In all practical quark models, an assumption is made which greatly simplifies subsequent steps in the analysis, that *the spatial, spin, and color degrees of freedom factorize*, at least in lowest-order approximation. This is true provided the zeroth-order hamiltonian is spin-independent and color-independent. Spin-dependent interactions are then taken into account as perturbations. This assumption allows us to write the sets $\{\alpha\}$ and $\{\overline{\alpha}\}$ in terms of the spatial (n), spin $(s,\ m_s)$, flavor (q), and color (k) degrees of freedom respectively, i.e., $\alpha = (n, s, m_s, q, k)$. If we are concerned with just the ground state, we can suppress the quantum number n, and for simplicity replace the symbols b, $d^\dagger$, etc., for annihilation and creation operators with the flavor symbol q $(q = u, d, s$ for the light hadrons),

$$\begin{aligned} b^\dagger(n = 0, q, m_s, k) &\to\ q^\dagger_{k,m_s}, \\ d^\dagger(n = 0, \overline{q}, m_s, k) &\to\ \overline{q}^\dagger_{k,m_s}. \end{aligned} \tag{1.3}$$

Hadrons are constructed in the Fock space defined by the creation operators for quarks and antiquarks. Light hadrons are labeled by the spin $(\mathbf{S}^2, S_3)$, isospin $(\mathbf{T}^2, T_3)$, and hypercharge (Y) operators as well as by the baryon number (B). Other observables like the electric charge $Q_{\rm el}$ and strangeness S are related to these,

$$Q_{\rm el} = T_3 + Y/2, \qquad S = Y - B. \tag{1.4}$$

Since quarks have spin one-half, the baryon (Q^3) and meson $(Q\overline{Q})$ configurations can carry the spin quantum numbers $S = 1/2, 3/2$ and $S = 0, 1$, respectively. If we neglect the mass difference between strange and nonstrange quarks, then flavor $SU(3)$ is a symmetry of the theory, and both quarks and hadrons occupy $SU(3)$

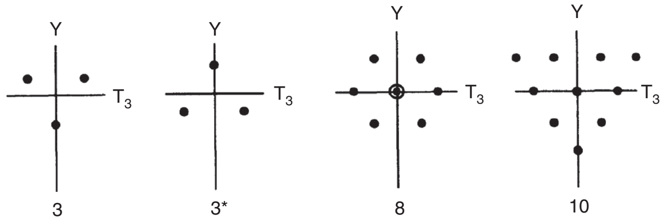

Fig. XI–1 Some $SU(3)$-flavor representations.

multiplets. The quarks are assigned to the triplet representation **3** and the antiquarks to $\mathbf{3}^*$. The $Q\overline{Q}$ and Q^3 constructions then involve the group products

$$\begin{aligned} \mathbf{3} \times \mathbf{3}^* &= \mathbf{8} \oplus \mathbf{1}, \\ (\mathbf{3} \times \mathbf{3}) \times \mathbf{3} = (\mathbf{6} \oplus \mathbf{3}^*) \times \mathbf{3} &= \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}, \end{aligned} \tag{1.5}$$

so that baryons appear as decuplets, octets, and singlets whereas mesons appear as octets and singlets. The $SU(3)$-flavor representations **3**, $\mathbf{3}^*$, **8**, **10** are depicted in Y vs. T_3 plots in Fig. XI–1. The circle around the origin for the eight-dimensional representation denotes the presence of *two* states with identical Y, I_3 values. Finally, quarks and antiquarks transform as triplets and antitriplets of the *color* $SU(3)$ gauge group, and all baryons and mesons are color singlets.

Two simple states to construct are the ρ_1^+ meson and the $\Delta_{3/2}^{++}$ baryon,

$$|\rho_1^+\rangle = \frac{1}{\sqrt{3}} u^\dagger_{i\uparrow} \overline{d}^\dagger_{i\uparrow}|0\rangle, \qquad |\Delta^{++}_{3/2}\rangle = \frac{1}{6}\epsilon_{ijk} u^\dagger_{i\uparrow} u^\dagger_{j\uparrow} u^\dagger_{k\uparrow}|0\rangle, \tag{1.6}$$

where the superscript and subscript on the hadrons denote electric charge and spin component, and a summation over color indices for the creation operators is implied. The normalization constants are fixed by requiring that the hadrons $\{H_n\}$ form an orthonormal set, $\langle H_m|H_n\rangle = \delta_{mn}$. The other ground-state hadrons can be reached from those in Eq. (1.6) by means of ladder operations in the spin and flavor variables. In this manner, one can construct the flavor–spin–color state vectors of the 0^- octet and singlet mesons and the $\frac{1}{2}^+$ octet baryons displayed in Tables XI–1 and XI–2.

A convenient notation for fields which transform as $SU(3)$ octets involves the use of a cartesian basis rather than the 'spherical' basis of Tables XI–1,2. In fact, we have already encountered this description in Sect. VII–1 during our discussion of $SU(3)$ Goldstone bosons where the quantity $U = \exp(i\boldsymbol{\varphi} \cdot \boldsymbol{\lambda})$ played a central role. The eight cartesian fields $\{\varphi_a\}$ are related to the usual pseudoscalar fields by

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

Table XI–1. *State vectors of the pseudoscalar octet and singlet mesons.*

$\lvert\pi^+\rangle = \frac{1}{\sqrt{6}}\,[u^\dagger_{i\uparrow}\overline{d}^\dagger_{i\downarrow} - u^\dagger_{i\downarrow}\overline{d}^\dagger_{i\uparrow}]\,\lvert 0\rangle$
$\lvert\pi^-\rangle = \frac{1}{\sqrt{6}}\,[d^\dagger_{i\uparrow}\overline{u}^\dagger_{i\downarrow} - d^\dagger_{i\downarrow}\overline{u}^\dagger_{i\uparrow}]\,\lvert 0\rangle$
$\lvert\pi^0\rangle = \frac{1}{\sqrt{12}}\,[-u^\dagger_{i\uparrow}\overline{u}^\dagger_{i\downarrow} + u^\dagger_{i\downarrow}\overline{u}^\dagger_{i\uparrow} + d^\dagger_{i\uparrow}\overline{d}^\dagger_{i\downarrow} - d^\dagger_{i\downarrow}\overline{d}^\dagger_{i\uparrow}]\,\lvert 0\rangle$
$\lvert K^+\rangle = \frac{1}{\sqrt{6}}\,[u^\dagger_{i\uparrow}\overline{s}^\dagger_{i\downarrow} - u^\dagger_{i\downarrow}\overline{s}^\dagger_{i\uparrow}]\,\lvert 0\rangle$
$\lvert K^0\rangle = \frac{1}{\sqrt{6}}\,[s^\dagger_{i\uparrow}\overline{d}^\dagger_{i\downarrow} - s^\dagger_{i\downarrow}\overline{d}^\dagger_{i\uparrow}]\,\lvert 0\rangle$
$\lvert\overline{K}^0\rangle = \frac{1}{\sqrt{6}}\,[s^\dagger_{i\uparrow}\overline{u}^\dagger_{i\downarrow} - s^\dagger_{i\downarrow}\overline{u}^\dagger_{i\uparrow}]\,\lvert 0\rangle$
$\lvert K^-\rangle = \frac{1}{\sqrt{6}}\,[d^\dagger_{i\uparrow}\overline{s}^\dagger_{i\downarrow} - d^\dagger_{i\downarrow}\overline{s}^\dagger_{i\uparrow}]\,\lvert 0\rangle$
$\lvert\eta_8\rangle = \frac{1}{\sqrt{36}}\,[u^\dagger_{i\uparrow}\overline{u}^\dagger_{i\downarrow} - u^\dagger_{i\downarrow}\overline{u}^\dagger_{i\uparrow} + d^\dagger_{i\uparrow}\overline{d}^\dagger_{i\downarrow} - d^\dagger_{i\downarrow}\overline{d}^\dagger_{i\uparrow} - 2s^\dagger_{i\uparrow}\overline{s}^\dagger_{i\downarrow} + 2s^\dagger_{i\downarrow}\overline{s}^\dagger_{i\uparrow}]\,\lvert 0\rangle$
$\lvert\eta_1\rangle = \frac{1}{\sqrt{18}}\,[u^\dagger_{i\uparrow}\overline{u}^\dagger_{i\downarrow} - u^\dagger_{i\downarrow}\overline{u}^\dagger_{i\uparrow} + d^\dagger_{i\uparrow}\overline{d}^\dagger_{i\downarrow} - d^\dagger_{i\downarrow}\overline{d}^\dagger_{i\uparrow} + s^\dagger_{i\uparrow}\overline{s}^\dagger_{i\downarrow} - s^\dagger_{i\downarrow}\overline{s}^\dagger_{i\uparrow}]\,\lvert 0\rangle$

$$
\begin{aligned}
\pi^\pm &= \frac{1}{\sqrt{2}}(\varphi_1 \mp i\varphi_2), \qquad \pi^0 = \varphi_3, \qquad \eta_8 = \varphi_8, \\
K^\pm &= \frac{1}{\sqrt{2}}(\varphi_4 \mp i\varphi_5), \\
K^0 &= \frac{1}{\sqrt{2}}(\varphi_6 - i\varphi_7), \qquad \overline{K}^0 = \frac{1}{\sqrt{2}}(\varphi_6 + i\varphi_7),
\end{aligned} \tag{1.7}
$$

which is an alternative way of stating the content of Eq. (VIII–1.12). The physical spin one-half baryons $p, n, \ldots$ can likewise be expressed in terms of an octet of states $\{B_i\}$ $(i = 1, \ldots, 8)$ in cartesian basis as

$$
\begin{aligned}
\Sigma^\pm &= \frac{1}{\sqrt{2}}(B_1 \mp iB_2), \qquad \Sigma^0 = B_3, \qquad \Lambda = B_8, \\
p &= \frac{1}{\sqrt{2}}(B_4 - iB_5), \qquad n = \frac{1}{\sqrt{2}}(B_6 - iB_7), \\
\Xi^0 &= \frac{1}{\sqrt{2}}(B_6 + iB_7), \qquad \Xi^- = \frac{1}{\sqrt{2}}(B_4 + iB_5).
\end{aligned} \tag{1.8}
$$

In the quark model, hadron observables have simple interpretations, e.g., the baryon number is simply one-third the difference in the number of quarks and antiquarks, etc. Thus, writing quark and antiquark number operators as $N(q)$ and $N(\bar{q})$ for a quark flavor q, we have

$$
\begin{aligned}
B &= [N(u) + N(d) + N(s) - N(\overline{u}) - N(\overline{d}) - N(\overline{s})]/3, \\
T_3 &= [N(u) - N(d) - N(\overline{u}) + N(\overline{d})]/2, \\
Y &= [N(u) + N(d) - 2N(s) - N(\overline{u}) - N(\overline{d}) + 2N(\overline{s})]/3, \\
Q_{\text{el}} &= [2N(u) - N(d) - N(s) - 2N(\overline{u}) + N(\overline{d}) + N(\overline{s})]/3,
\end{aligned} \tag{1.9}
$$

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

Table XI–2. *State vectors of baryon spin-one-half octet.*

$\vert p_\uparrow\rangle = \frac{\epsilon_{ijk}}{\sqrt{18}}\ [u^\dagger_{i\downarrow}d^\dagger_{j\uparrow} - u^\dagger_{i\uparrow}d^\dagger_{j\downarrow}]\ u^\dagger_{k\uparrow}\ \vert 0\rangle$
$\vert n_\uparrow\rangle = \frac{\epsilon_{ijk}}{\sqrt{18}}\ [d^\dagger_{i\uparrow}u^\dagger_{j\downarrow} - d^\dagger_{i\downarrow}u^\dagger_{j\uparrow}]\ d^\dagger_{k\uparrow}\ \vert 0\rangle$
$\vert \Lambda_\uparrow\rangle = \frac{\epsilon_{ijk}}{\sqrt{12}}\ [u^\dagger_{i\uparrow}d^\dagger_{j\downarrow} - u^\dagger_{i\downarrow}d^\dagger_{j\uparrow}]\ s^\dagger_{k\uparrow}\ \vert 0\rangle$
$\vert \Sigma^+_\uparrow\rangle = \frac{\epsilon_{ijk}}{\sqrt{18}}\ [s^\dagger_{i\downarrow}u^\dagger_{j\uparrow} - s^\dagger_{i\uparrow}u^\dagger_{j\downarrow}]\ u^\dagger_{k\uparrow}\ \vert 0\rangle$
$\vert \Sigma^0_\uparrow\rangle = \frac{\epsilon_{ijk}}{6}\ [s^\dagger_{i\uparrow}d^\dagger_{j\downarrow}u^\dagger_{k\uparrow} + s^\dagger_{i\uparrow}d^\dagger_{j\uparrow}u^\dagger_{k\downarrow}\text{-}2s^\dagger_{i\downarrow}d^\dagger_{j\uparrow}u^\dagger_{k\uparrow}]\ \vert 0\rangle$
$\vert \Sigma^-_\uparrow\rangle = \frac{\epsilon_{ijk}}{\sqrt{18}}\ [s^\dagger_{i\uparrow}d^\dagger_{j\downarrow} - s^\dagger_{i\downarrow}d^\dagger_{j\uparrow}]\ d^\dagger_{k\uparrow}\ \vert 0\rangle$
$\vert \Xi^0_\uparrow\rangle = \frac{\epsilon_{ijk}}{\sqrt{18}}\ [s^\dagger_{i\downarrow}u^\dagger_{j\uparrow} - s^\dagger_{i\uparrow}u^\dagger_{j\downarrow}]\ s^\dagger_{k\uparrow}\ \vert 0\rangle$
$\vert \Xi^-_\uparrow\rangle = \frac{\epsilon_{ijk}}{\sqrt{18}}\ [s^\dagger_{i\uparrow}d^\dagger_{j\downarrow} - s^\dagger_{i\downarrow}d^\dagger_{j\uparrow}]\ s^\dagger_{k\uparrow}\ \vert 0\rangle$

and the hadronic spin operator is

$$\mathbf{S} = \sum_q q^\dagger_{i,m_s'} \frac{(\boldsymbol{\sigma})_{m_s' m_s}}{2} q_{i,m_s}. \tag{1.10}$$

Quark spatial wavefunctions

Many applications of the quark model require the knowledge of the quark spatial wavefunctions within hadrons. It is here that the greatest variation in the different models can occur, but several general features still remain. Indeed, in many instances it is the general features that are primarily tested.

For example, the ground state in all models is a spatially symmetric S state in which the wavefunction peaks at $r = 0$. The normalization condition of the quark spatial wavefunction,

$$\int d^3x\ \psi^\dagger(x)\psi(x) = 1, \tag{1.11}$$

ensures that the magnitude of ψ will be similar in those models having wavefunctions of comparable spatial extent. This accounts for the agreement which can be found among diverse quark models in specific applications. How does one fix the spatial extent? One approach is to use an observable like the hadronic electromagnetic charge radius, e.g.,

$$\langle r^2\rangle^{1/2}_{\text{proton}} = 0.87 \pm 0.02\ \text{fm}, \qquad \langle r^2\rangle^{1/2}_{\text{pion}} = 0.66 \pm 0.02\ \text{fm}. \tag{1.12}$$

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

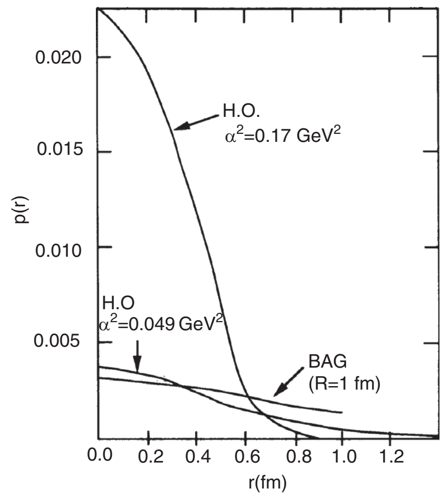

Fig. XI–2 Quark probability density in the bag and oscillator models.

Viewed this way, the bound states are seen to define a scale of order 1 fm. For example, we display two models in Fig. XI–2, the oscillator result with $\alpha^2 = 0.17$ GeV^2 and the bag profile, which are each obtained by fitting to ground-state baryon observables like the charge radius. Not surprisingly, their behaviors are quite similar. Also shown in Fig. XI–2 is an oscillator model wavefunction whose parameter ($\alpha^2 = 0.049$ GeV^2) was determined by using data from decays of excited hadrons. The difference is rather striking, and serves to demonstrate that the most important general feature in setting the scale in quark model predictions of dimensional matrix elements is the spatial extent of the wavefunction.[1]

Another aspect of quark wavefunctions involves the issue of relativistic motion. A relativistic quark moving in a spin-independent central potential has a ground-state wavefunction of the form

$$\psi_{\rm gnd}(x) = \begin{pmatrix} i\, u(r)\chi \\ \ell(r)\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\chi \end{pmatrix} e^{-iEt}, \tag{1.13}$$

where u, ℓ signify ‘upper’ and ‘lower’ components. As we shall see, in the bag model these radial wavefunctions are just spherical Bessel functions. The above form also appears in some relativized harmonic-oscillator models, which use a central potential. If we allow for relativistic motion, then the major remaining difference in the quark wavefunctions concerns the lower two components of the Dirac wavefunction. Nonrelativistic models automatically set these equal to zero, while relativistic models can have sizeable lower components. Which description is the

[1] We could obtain a bag result which behaves similarly by employing a charge radius of 0.5 fm rather than the 1 fm value shown.

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

correct one? Quark motion in light hadrons must be at least somewhat relativistic since quarks confined to a region of radius R have a momentum given by the uncertainty principle,[2]

$$p \geq \sqrt{3}R^{-1} \simeq 342 \text{ MeV} \qquad (\text{for } R \simeq 1 \text{ fm}). \tag{1.14}$$

Since this momentum is comparable to or larger than all the light-quark masses, relativistic effects are unavoidable. A more direct indication of the relativistic nature of quark motion comes from the hadron spectrum. Nonrelativistic systems are characterized by excitation energies which are small compared to the constituent masses. In the hadron spectrum, typical excitation energies lie in the range 300–500 MeV, again comparable to or larger than light-quark masses. Such considerations have motivated relativistic formulations of the quark model.

Interpolating fields

In the LSZ procedure (App. B–3) for analyzing scattering amplitudes the central role is played by interpolating fields. These are the quantities which experience the dynamics of the theory in the course of evolving between the asymptotic *in*-states and *out*-states. They turn out to be also useful as a kind of bookkeeping device. That is, one way to characterize the spectrum of observed states is to use operators made of appropriate combinations of quark fields $\psi(x)$. For example, corresponding to the meson sector of $\overline{Q}Q$ states, one could employ a sequence of quark bilinears, the simplest of which are

$$\overline{\psi}\psi, \ \overline{\psi}\gamma_5\psi, \ \overline{\psi}\gamma_\mu\psi, \ \overline{\psi}\gamma_\mu\gamma_5\psi, \ \overline{\psi}\sigma_{\mu\nu}\psi, \ldots . \tag{1.15}$$

Any of these operators acting on the vacuum creates states with its own quantum numbers. The lightest states in the quark spectrum will be associated with those operators which remain nonzero for static quarks, i.e., with creation operators and Dirac spinors of the form

$$\psi \sim \begin{pmatrix} 0 \\ \chi_{\overline{m}} \end{pmatrix} d^\dagger_{\overline{m}}, \qquad \overline{\psi} \sim \begin{pmatrix} \chi_m \\ 0 \end{pmatrix} b^\dagger_m. \tag{1.16}$$

Only the pseudoscalar operators $\overline{\psi}\gamma_5\psi$, $\overline{\psi}\gamma_0\gamma_5\psi$ and the vector operators $\overline{\psi}\gamma_i\psi$, $\overline{\psi}\sigma_{0i}\psi$ are nonvanishing in this limit. All the other operators have a nonrelativistic reduction proportional to spatial momentum, indicating the need for a unit of orbital angular momentum in forming a state.

The interpolating-field approach is particularly useful in situations where the imposition of gauge invariance determines whether a given field configuration can occur in the physical spectrum. We shall return to this point in Sect. XIII–4 in the

[2] The $\sqrt{3}$ factor is associated with the fact that there are *three* dimensions.

course of discussing glueball states. We now turn to a summary, carried throughout the next three sections, of various attempts to *model* the dynamics of light-hadronic states.

XI–2 Potential model

The potential model posits that there is a relatively simple effective theory in which the quarks move nonrelativistically within hadrons. In the light of our previous comments on relativistic motion, this would seem to be acceptable only for truly massive quarks like the b quark and certainly questionable for the light quarks u, d, s. However, in the potential model it is assumed that *QCD* interactions dress each quark with a cloud of virtual gluons and quark–antiquark pairs, and that the resulting dynamical mass contribution is so large that quarks move nonrelativistically. These 'dressed' degrees of freedom are called *constituent quarks*, and their masses are called 'constituent masses'. Constituent masses are not to be directly identified with the mass parameters occurring in the *QCD* lagrangian.[3] Energy levels and wavefunctions are then obtained by solving the nonrelativistic Schrödinger equation in terms of the constituent masses and some assumed potential energy function.

The potential model is not without flaws. For light-quark dynamics, it is far from clear that a static potential can adequately describe the *QCD* interaction. Even with the use of constituent masses, one finds from fits to the mass spectrum and/or the charge radius that quark velocity is nevertheless near the speed of light (cf. Prob. XI–1). Also, although it is possible [LeOPR 85] to make a connection between the lightest pseudoscalar mesons as Goldstone bosons on the one hand and $Q\bar{Q}$ composites on the other, this is not ordinarily done. Such criticisms notwithstanding, the nonrelativistic quark model does provide a framework for describing both ground and excited hadronic states, and brings a measure of order to a spectrum containing hundreds of observed levels. Besides, virtually all physicists are familiar with the Schrödinger equation, and find the potential model to be an understandable and intuitive language.

Basic ingredients

One begins by expressing the mass M_α of a hadronic state α as

$$M_\alpha = \sum_i M_i + E_\alpha, \tag{2.1}$$

[3] We shall continue to denote the *QCD* mass parameter of quark q_i as m_i, and shall write the corresponding constituent mass as M_i.

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

where the sum is over the constituent quarks and antiquarks in α. The internal energy E_α is an eigenvalue of the Schrödinger equation

$$H\psi_\alpha = E_\alpha \psi_\alpha, \tag{2.2}$$

with hamiltonian

$$H = \sum_i \frac{1}{2M_i}\mathbf{p}_i^{\,2} + \sum_{i<j} V_{\rm color}(\mathbf{r}_{ij}), \tag{2.3}$$

where $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$, and the subscript 'color' on the potential energy indicates that the dynamics of quarks necessarily involves the color degree of freedom in some manner. It is standard to assume that the potential energy is a sum of two-body interactions. Although there exists no unique specification of the interquark potential $V_{\rm color}$ from *QCD*, the following features are often adopted:

(1) a spin-and flavor-independent long-range confining potential,
(2) a spin-and flavor-dependent short-range potential,
(3) basis mixing in the baryon and meson sectors, and
(4) relativistic corrections.

We shall discuss specific models of the potential energy function in Sect. XIII–1. They all have in common the color dependence in which *the two-particle potential is twice as strong in mesons as it is in baryons*,

$$V_{\rm color}(\mathbf{r}_{ij}) = \begin{cases} V(\mathbf{r}_{ij}) & \text{(mesons)}, \\ \frac{1}{2}\, V(\mathbf{r}_{ij}) & \text{(baryons)}. \end{cases} \tag{2.4}$$

We shall describe a simple empirical test for such behavior at the end of this section. To appreciate its theoretical basis, note that the quark–antiquark pair in a meson must occur in the **1** representation of color, whereas any two quarks in a baryon must be in a $\mathbf{3}^*$ representation (in order that the three-quark composite be a color singlet),

$$\begin{aligned} V_{\rm color} &\propto \begin{cases} F(\mathbf{3})\cdot F(\mathbf{3}^*) & \text{(mesons)}, \\ F(\mathbf{3})\cdot F(\mathbf{3}) & \text{(baryons)}, \end{cases} \\ &\propto \begin{cases} (F^2(\mathbf{1}) - F^2(\mathbf{3}) - F^2(\mathbf{3}^*))/2 = -4/3 & \text{(mesons)}, \\ (F^2(\mathbf{3}^*) - 2F^2(\mathbf{3}))/2 = -2/3 & \text{(baryons)}, \end{cases} \end{aligned} \tag{2.5}$$

where $F^a(\mathbf{R})$ is a color generator for $SU(3)$ representation $\mathbf{R}$. Thus, the color dependence in Eq. (2.4) is that which one would naturally associate with the interaction between two quarks or a quark–antiquark pair.

Table XI–3. *Quantum numbers of $Q\bar{Q}$ composites.*

L	Singlet	Triplet
0	${}^1S_0(0^{-+})$	${}^3S_1(1^{--})$
1	${}^1P_1(1^{+-})$	${}^3P_{0,1,2}(0^{++}, 1^{++}, 2^{++})$
2	${}^1D_2(2^{-+})$	${}^3D_{1,2,3}(1^{--}, 2^{--}, 3^{--})$
3	${}^1F_3(3^{+-})$	${}^3F_{2,3,4}(2^{++}, 3^{++}, 4^{++})$

Mesons

For the two-particle $Q\bar{Q}$ system, it is straightforward to remove the center-of-mass dependence. In the center-of-mass frame the Schrödinger equation becomes

$$\left(\frac{\mathbf{p}^2}{2M} + V(\mathbf{r})\right)\psi_\alpha(\mathbf{r}) = E_\alpha \psi_\alpha(\mathbf{r}), \tag{2.6}$$

where $\mathbf{r} = \mathbf{r}_Q - \mathbf{r}_{\bar{Q}}$ and $M^{-1} = M_Q^{-1} + M_{\bar{Q}}^{-1}$ is the inverse reduced mass. The LS coupling scheme is typically employed to classify the eigenfunctions of this problem. One constructs the total $Q\bar{Q}$ spin, $\mathbf{S} = \mathbf{s}_Q + \mathbf{s}_{\bar{Q}}$, and adds the orbital angular momentum $\mathbf{L}$ to form the total angular momentum $\mathbf{J} = \mathbf{S} + \mathbf{L}$. There is an infinite tower of eigenstates, each labeled by the radial quantum number n and the angular momentum quantum numbers J, J_z, L, S.

The $Q\bar{Q}$ states are sometimes described in terms of spectroscopic notation ${}^{2S+1}L_J(J^{PC})$, where P is the parity and C is the charge conjugation,

$$P = (-)^{L+1}, \qquad C = (-)^{L+S}. \tag{2.7}$$

Strictly speaking, although only electrically neutral particles like π^0 can be eigenstates of the charge conjugation operation, C is often employed as a label for an entire isomultiplet, like $\boldsymbol{\pi} = (\pi^+, \pi^0, \pi^-)$. The lowest $Q\bar{Q}$ orbital configurations, expressed in ${}^{2S+1}L_J(J^{PC})$ notation, are displayed in Table XI–3. The $0^+, 1^-, 2^+, \ldots$ series of J^P states is called *natural*, and has the same quantum numbers as would occur for two spinless mesons of a common intrinsic parity. The alternate sequence, $0^-, 1^+, 2^-, \ldots$ is referred to as *unnatural*. There are a number of J^{PC} configurations, called *exotic* states, which cannot be accommodated within the $Q\bar{Q}$ construction. For example, the 0^{--} state is exotic because any state with $J = 0$ must have $L = S$, and according to the $Q\bar{Q}$ constraint of Eq. (2.7) must therefore carry $C = +$. Likewise, the $CP = -1$ sequence $0^{+-}, 1^{-+}, 2^{+-}, \ldots$ is forbidden because the $Q\bar{Q}$ model requires $CP = (-)^{S+1}$, implying $S = 0$ and hence $J = L$. Thus, one would obtain $P = (-)^{J+1}$ in the $Q\bar{Q}$ model and not $P = (-)^J$.

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

Baryons

Most applications of the quark model for Q^3 baryons involve the light quarks. If, for simplicity, we assume degenerate constituent mass M, the Schrödinger equation is

$$H_0 = \frac{1}{2M}\sum_{i=1}^{3}\mathbf{p}_i^2 + \frac{1}{2}\sum_{i<j} V(\mathbf{r}_{ij}), \tag{2.8}$$

where the prefactor of $1/2$ in the potential energy term follows from Eq. (2.4). It is convenient to define a center-of-mass coordinate $\mathbf{R}$ and internal coordinates $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ by

$$\begin{aligned}\mathbf{R} &= (\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3)/3,\\ \boldsymbol{\rho} &= (\mathbf{r}_1 - \mathbf{r}_2)/\sqrt{2},\\ \boldsymbol{\lambda} &= (\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3)/\sqrt{6}.\end{aligned} \tag{2.9}$$

Because it is not possible to remove the three-particle center-of-mass dependence for an arbitrary potential, the following approach is often followed [IsK 78]. The potential $V(\mathbf{r}_{ij})$ is rewritten as

$$V(\mathbf{r}_{ij}) = V_{\text{osc}}(r_{ij}) + U(\mathbf{r}_{ij}), \tag{2.10}$$

where

$$V_{\text{osc}} = \frac{k}{2}\mathbf{r}_{ij}^2, \qquad U \equiv V - V_{\text{osc}}. \tag{2.11}$$

The Schrödinger equation is solved in terms of the oscillator potential and U is evaluated perturbatively in the oscillator basis. Having removed the center-of-mass coordinate, we are left with the following hamiltonian for the internal energy:

$$H_{\text{int}} = \left(\frac{\mathbf{p}_\rho^2}{2m} + \frac{3k}{2}\boldsymbol{\rho}^2\right) + \left(\frac{\mathbf{p}_\lambda^2}{2m} + \frac{3k}{2}\boldsymbol{\lambda}^2\right), \tag{2.12}$$

which is just that of two independent quantum oscillators each with spring constant $3k$. For later purposes, we write the number of excitation quanta for the two oscillators as N_ρ and N_λ ($N_{\rho,\lambda} = 0, 1, 2, \ldots$) and let $N \equiv N_\rho + N_\lambda$. The angular momentum for the three-quark system is found in a similar manner as for the $Q\bar{Q}$ mesons, $\mathbf{J} = \mathbf{L} + \mathbf{S}$. The total quark spin is $\mathbf{S} = \sum \mathbf{s}_i$, the orbital angular momentum is given by $\mathbf{L} = \mathbf{L}_\rho + \mathbf{L}_\lambda$, and the parity is $P = (-)^{\ell_\rho + \ell_\lambda}$. The ground-state wavefunction has the form

$$\psi_{\text{gnd}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \left(\frac{\alpha^2}{\pi}\right)^{3/2} e^{i\mathbf{P}\cdot\mathbf{R}} e^{-\alpha^2(\varrho^2 + \boldsymbol{\lambda}^2)/2}, \tag{2.13}$$

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

where $\alpha^2 = (3km)^{1/2}$. A cautionary remark is in order. One should not misinterpret the use of an oscillator potential – it is *not* the intent to model the observed baryon spectrum as that of a system of quantum oscillators because such a picture would fail. For example, the oscillator spectrum has $E_N \sim N$, whereas the baryon spectrum obeys the law of linear Regge trajectories (cf. Sect. XIII–2), $E_N^2 \sim N$. The oscillator potential provides a convenient basis for structuring the calculation and nothing more.

Color dependence of the interquark potential

Short of doing a complete spectroscopic analysis, we can find experimental support in the following simple example for the assertion that the two-particle interquark potential is twice as strong in mesons as it is in baryons.

A potential model description for the meson and baryon mass splittings $\rho(770)-\pi(138)$ and $\Delta(1232)-N(939)$ is given by a *QCD* hyperfine interaction, $H_{\rm hyp}$, akin to the delta function contribution in the *QED* hyperfine potential of Eq. (V–1.16),

$$H_{\rm hyp} = k_\alpha \sum_{i<j} \bar{\mathcal{H}}_{ij}\ \mathbf{s}_i \cdot \mathbf{s}_j\ \delta^{(3)}(\mathbf{r}) \qquad (\alpha = M, B), \tag{2.14}$$

where the $\{\bar{\mathcal{H}}_{ij}\}$ are constants and, assuming the color dependence is that given by Eqs. (2.4), (2.5), $k_M = 1$ for mesons and $k_B = 1/2$ for baryons. We shall discuss in Sect. XIII–2 how this effect could arise from gluon exchange. Although there is ordinarily dependence on quark mass in the $\{\bar{\mathcal{H}}_{ij}\}$, it suffices to treat the $\{\bar{\mathcal{H}}_{ij}\}$ as an overall constant since the hadrons in this example contain only light nonstrange quarks. The point is then to see whether the condition $k_M = 2k_B$ is in accord with phenomenology. Noting that for mesons the spin factors yield

$$\mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{2\mathbf{S}^2 - 3}{4} = \begin{cases} 1/4 & (S = 1), \\ -3/4 & (S = 0), \end{cases} \tag{2.15a}$$

whereas for baryons one has

$$\sum_{i<j} \mathbf{s}_i \cdot \mathbf{s}_j = \frac{4\mathbf{S}^2 - 9}{8} = \begin{cases} 3/4 & (S = 3/2), \\ -3/4 & (S = 1/2), \end{cases} \tag{2.15b}$$

we find after taking expectation values that

$$\frac{m_\rho - m_\pi}{m_\Delta - m_N} = \frac{2k_M}{3k_B}\frac{|\psi_M(0)|^2}{|\psi_B(0)|^2} \simeq \frac{2k_M}{3k_B}\frac{(\text{Volume})_B}{(\text{Volume})_M} \simeq \frac{2k_M}{3k_B}\left[\frac{\langle r^2\rangle_B}{\langle r^2\rangle_M}\right]^{3/2}. \tag{2.16}$$

The measured values (cf. Eq. (1.13)) of the proton and pion charge radii imply that $k_M/k_B \simeq 2$. This example, along with others, lends credence to the assumed color dependence of Eq. (2.4).

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

At this point we shall temporarily leave our discussion of the potential model to consider other descriptions of hadronic structure. We shall return to the potential model for the discussion of hadron spectroscopy in Chaps. XII–XIII.

XI–3 Bag model

A superconductor has an ordered quantum mechanical ground state which does not support a magnetic field (*Meissner effect*) and which is brought about by a condensation of dynamically paired electrons (*Cooper pairs*). An order parameter for this medium is provided by the Landau–Ginzburg wavefunction of a Cooper pair. Even at zero temperature, a sufficiently strong magnetic field, $B_{\rm cr}$, can induce a transition from the superconducting phase to the normal phase. For example, in tin the critical field is $B_{\rm cr}(\text{tin}) \simeq 3.06 \times 10^{-2}$ tesla, and the energy density of superconducting pairing (*condensation* energy) is $U_{\rm super}/V \simeq 373\ \text{J/m}^3$.

Chromodynamics exhibits similar behavior, and this is the basis for the bag model [ChJJTW 74]. The *QCD* ground state evidently does not support a chromoelectric field, and is thus analogous to the superconducting state, although a compelling description of the *QCD* pairing mechanism has not yet been provided. In the bag model, the analog of the normal conducting ground state is called the *perturbative vacuum*. The vacuum expectation value of the quark bilinear $\bar{q}q\,(q = u, d, s)$ plays the role of an order parameter by distinguishing between the two vacua,

$$
{}_{QCD}\langle 0|\bar{q}q|0\rangle_{QCD} < 0, \qquad {}_{\rm pert}\langle 0|\bar{q}q|0\rangle_{\rm pert} = 0. \tag{3.1}
$$

Hadrons are represented as color-singlet ‘bags’ of perturbative vacuum occupied by quarks and gluons. The bag model employs as its starting point the lagrange density [Jo 78]

$$
\mathcal{L}_{\rm bag} = (\mathcal{L}_{QCD} - B)\,\theta(\bar{q}q), \tag{3.2}
$$

where the θ function (which vanishes for negative argument) defines the spatial volume encompassed by the perturbative vacuum. B is called the *bag constant*, and is often expressed in units of $(\text{MeV})^4$. Physically, it represents the difference in energy density between the *QCD* and perturbative vacua. Phenomenological determinations of B yield $B^{1/4} \simeq 150$ MeV, which translates to a *QCD* condensation energy of $U_{QCD}/V \simeq 1.0 \times 10^{34}\ \text{J m}^{-3}$. Although huge on the scale of the condensation energy for superconductivity, this value appears less remarkable in more natural units, $B \simeq 66\ \text{MeV fm}^{-3}$.

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

Static cavity

To obtain the equations of motion and boundary conditions for the bag model, we must minimize the action functional of the theory. We shall consider at first a simplified model consisting of a bag which contains only quarks of a given flavor q and mass m. The equations of motion that follow from the lagrangian of Eq. (3.2) are

$$(i\partial\!\!\!/ - m)q = 0, \tag{3.3}$$

within the bag volume V and

$$i n^\mu \gamma_\mu q = q, \tag{3.4a}$$

$$n_\mu \partial^\mu (\bar{q}q) = 2B \tag{3.4b}$$

on the bag surface S, where n_μ is the covariant inward normal to S. Eq. (3.3) describes a Dirac particle of mass m moving freely within the cavity defined by volume V. Since the order parameter $\bar{q}q$ vanishes at the surface of the bag, the linear boundary condition in Eq. (3.4a) amounts to requiring that the normal component of the quark vector current also vanish at the surface. Thus, quarks are confined within the bag. The nonlinear boundary condition represents a balance between the outward pressure of the quark field and the inward pressure of B.

Spherical-cavity approximation

In principle, the bag surface should be determined dynamically. However, the only manageable approximation for light-quark dynamics is one in which the shape of the bag is taken as spherical with some radius R. For such a static configuration, the nonlinear boundary condition becomes equivalent to requiring that the energy be minimized as a function of R. The static-cavity hamiltonian is

$$H = \int_V d^3x \left[q^\dagger (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla}) q + q^\dagger \beta m q + B \right]. \tag{3.5}$$

Observe that B plays the role of a constant energy density at all points within the bag. As in Eq. (1.1), the normal modes of the cavity-confined quarks and antiquarks provide a basis for expanding quantum fields. They are determined by solving the Dirac equation Eq. (3.3) in a spherical cavity. We characterize each mode in terms of a radial quantum number n, an orbital angular momentum quantum number ℓ (as would appear in the nonrelativistic limit), and a total angular momentum, j. Only $j = 1/2$ modes are consistent with the nonlinear boundary condition since the rigid spherical cavity cannot accommodate the angular variation of $j > 1/2$ modes. Such nonspherical orbitals can be treated only approximately, by implementing the nonlinear boundary condition as an angular average or by minimizing the solution with respect to the energy. In addition, since neither $p_{1/2}$ modes nor radially excited

$s_{1/2}$ modes are orthogonal to a translation of the ground state, they must be admixed with some of the $j = 3/2$ modes to construct physically acceptable excitations. For these reasons, the bag model has been most widely applied in modeling properties of the ground-state hadrons rather than their excited states.

Let us consider the $s_{1/2}$ case in some detail. Even with the restriction to a single spin-parity state, there are still an infinity of eigenfrequencies ω_n. Each ω_n is fixed by the linear boundary condition, expressible as the transcendental equation

$$\tan\, p_n = -\frac{p_n}{\omega_n + mR - 1} \qquad (n = 1, 2, \ldots), \tag{3.6}$$

where $p_n \equiv \sqrt{\omega_n^2 - m^2R^2}$. For zero quark mass, the lowest eigenfrequencies are $\omega = 2.043,\ 4.611, \ldots$. For light-quark mass ($mR \leq 1$) the lowest mode frequency is approximated by $\omega_1 \simeq 2.043 + 0.493mR$, and in the limit of heavy-quark mass ($mR \gg 1$) becomes $\omega_1 \to \sqrt{m^2R^2 + \pi^2}$. The spatial wavefunction which accompanies destruction of an $s_{1/2}$ quark with spin alignment λ and mode n is

$$\psi_n(\mathbf{x}) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} i j_0(p_n r/R)\chi_\lambda \\ -\epsilon j_1(p_n r/R)\sigma \cdot \hat{r}\chi_\lambda \end{pmatrix}, \tag{3.7}$$

while for creation of an $s_{1/2}$ antiquark we have

$$\psi_{\bar{n}}(\mathbf{x}) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} -i\epsilon j_1(p_n r/R)\sigma \cdot \hat{r}\bar{\chi}_\lambda \\ j_0(p_n r/R)\bar{\chi}_\lambda \end{pmatrix}, \tag{3.8}$$

where $\epsilon \equiv ((\omega_n - mR)/(\omega_n + mR))^{1/2}$, χ_λ is a two-component spinor, and $\bar{\chi}_\lambda \equiv i\sigma_2\chi_\lambda$. The full quark field $q(x)$, expanded in terms of the $s_{1/2}$ modes, is given by

$$q(x) = \sum_n N(\omega_n)\left[\psi_n(\mathbf{x})e^{-i\omega_n t/R}b(n) + \psi_{\bar{n}}(\mathbf{x})e^{i\omega_n t/R}d^\dagger(\bar{n})\right], \tag{3.9}$$

where

$$N(\omega_n) = \left(\frac{p_n^4}{R^3(2\omega_n^2 - 2\omega_n + mR)\sin^2 p_n}\right)^{1/2} \tag{3.10}$$

is a normalization factor which is fixed by demanding that the number operator $N_q = \int_{\text{bag}} d^3x\, q^\dagger(x)q(x)$ for quark flavor q have integer eigenvalues.

By computing the expectation value of the hamiltonian in a state of N quarks and/or antiquarks of a given flavor, one obtains

$$\langle H\rangle = N\omega/R + 4\pi BR^3/3 - Z_0/R. \tag{3.11}$$

In the final term, Z_0 is a phenomenological constant that has been used in the literature to summarize effects having a $1/R$ dimension, most notably the effect of zero-point energies, which for an infinite-volume system would be unobservable. However, just as the Casimir effect is present for a finite-volume system

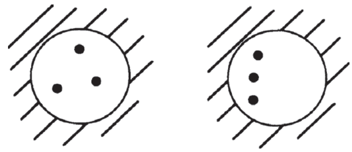

Fig. XI–3 Quarks in a bag.

with fixed boundaries, such a term must be present in the static cavity bag model [DeJJK 75]. Unfortunately, a precise calculation of this effect has proven to be rather formidable, and so one treats Z_0 as a phenomenological parameter.

Upon solving the condition $\partial\langle H\rangle/\partial R = 0$, we obtain expressions for the bag radius

$$R^4 = \frac{1}{4\pi B}(N\omega - Z_0), \tag{3.12}$$

and the bag energy

$$E = \frac{4}{3}(4\pi B)^{1/4}(N\omega - Z_0)^{3/4}. \tag{3.13}$$

The bag energy E is *not* precisely the hadron mass. Although the bag surface remains fixed in the cavity approximation, the quarks within move freely as independent particles. Thus, at one instant, the configuration of quarks might appear as in Fig. XI–3(a), whereas at another time, the quarks occupy the positions of Fig. XI–3(b). As a result, there are unavoidable fluctuations in the bag center-of-mass position. The bag energy is thus $E = \langle\sqrt{\mathbf{p}^2 + M^2}\rangle$, where M is the hadron mass and $\mathbf{p}$ represents the instantaneous hadron momentum. Although the average momentum vanishes ($\langle\mathbf{p}\rangle = 0$), the fluctuations do not, ($\langle\mathbf{p}^2\rangle \neq 0$). For all hadrons but the pion, it is reasonable to expand the bag energy in inverse powers of the hadron mass,

$$E = M + \langle\mathbf{p}^2\rangle/2M + \cdots . \tag{3.14}$$

For the pion, one should instead expand as

$$E = \langle|\mathbf{p}|\rangle + M_\pi^2\langle|\mathbf{p}|^{-1}\rangle/2 + \cdots . \tag{3.15}$$

One can employ the method of wave packets, to be explained in Sect. XII–1, to estimate that $\langle|\mathbf{p}|\rangle \simeq 2.3R^{-1}$, $\langle|\mathbf{p}|^{-1}\rangle \simeq 0.7R$ for the pion bag, and $\langle\mathbf{p}^2\rangle \simeq N\omega_1^2R^{-2}$ for a bag containing N quarks and/or antiquarks in the $s_{1/2}$ mode.

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

Gluons in a bag

Any detailed phenomenological fit of the bag model to hadrons must include the spin–spin interaction between quarks. One way to incorporate this effect is to posit that gluons, as well as quarks, can exist within a bag. With only gluons present, the lagrangian is taken to be [Jo 78]

$$\mathcal{L}_{\text{bag}}^{\text{gluon}} = \left[-\frac{1}{4}F_{\mu\nu}^{a}F^{a\mu\nu} - B\right]\theta\left(-F_{\mu\nu}^{a}F^{a\mu\nu}/4 - B\right), \tag{3.16}$$

and the Euler–Lagrange equations are

$$\partial^{\mu}F_{\mu\nu}^{a} = 0 \tag{3.17}$$

in the bag volume V, and

$$n^{\mu}F_{\mu\nu}^{a} = 0 \tag{3.18a}$$

$$F_{\mu\nu}^{a}F^{a\mu\nu} = -4B \tag{3.18b}$$

on the bag surface S. In the limit of zero coupling, the gluon field strength becomes $F_{\mu\nu}^{a} = \partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a}$. The field equations in V are sourceless Maxwell equations with boundary conditions $\mathbf{x}\cdot\mathbf{E}^{a} = 0$ and $\mathbf{x}\times\mathbf{B}^{a} = 0$ on S, where $\mathbf{E}^{a}$ and $\mathbf{B}^{a}$ are the color electric and magnetic fields, respectively. It is convenient to work directly with the gluon field $\mathbf{A}^{a}(x)$, and with a gauge choice to restrict the dynamic degrees of freedom to the spatial components. In mode n, these obey

$$\left[\nabla^{2} + (k_n/R)^{2}\right]\mathbf{A}_{n}^{a} = 0, \tag{3.19}$$

and

$$\nabla\cdot\mathbf{A}_{n}^{a} = 0 \tag{3.20}$$

within the bag. The gluon eigenfrequencies k_n are determined by the linear boundary condition

$$\mathbf{r}\times\left(\nabla\times\mathbf{A}_{n}^{a}\right) = 0. \tag{3.21}$$

Restricting our attention to modes of positive parity, we have for the gluon field operator

$$\mathbf{A}^{a}(x) = \sum_{n,\sigma} N_G(k_n)\left(j_1(k_n r/R)\mathbf{X}_{1\sigma}(\Omega)a_{n,\sigma}^{a} + \text{h.c.}\right), \tag{3.22}$$

where $\mathbf{X}_{1\sigma}$ is a vector spherical harmonic. The gluon normalization factor is obtained, analogously to $N(\omega_n)$ for quarks, by constraining the gluon number operator to be integer-valued and we find

$$[N_G(k_n)]^{-2} = \left[3(1 - \sin(2k_n)/2k_n) - 2(1 + k_n^{2})\sin^{2}(k_n)\right]R^{2}. \tag{3.23}$$

The quark–gluon interaction

In the following, we shall work with the lowest positive parity mode, for which $k_1 = 2.744$. The quark hyperfine interaction in hadron H can be computed from the second-order perturbation theory formula,

$$E_{\text{hyp}} = \langle H | H_{\text{q-g}} (E_0 - H_0 + i\epsilon)^{-1} H_{\text{q-g}} | H \rangle, \tag{3.24}$$

where the unperturbed hamiltonian H_0 is given in Eq. (3.5) and $H_{\text{q-g}}$ is the quark-gluon interaction

$$H_{\text{q-g}} = -g_3 \int_V d^3x \, \mathbf{J}^a(x) \cdot \mathbf{A}^a(x), \tag{3.25}$$

defined in terms of the quark color current density

$$\mathbf{J}^a(x) = \frac{1}{2} \bar{q}_i(x) \boldsymbol{\gamma} \lambda^a_{ij} q_j(x). \tag{3.26}$$

Implicit in Eq. (3.24) is an infinite sum over all intermediate states. In practice, the sum can be well approximated by the lowest-energy intermediate state, and we find for hadron H

$$E_{\text{hyp}} = \langle H | H_{\text{hyp}} | H \rangle = \alpha_s h_H R^{-1}, \tag{3.27}$$

where

$$h_H = -0.177 \langle H | \sum_{i<j} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \, \mathbf{F}_i \cdot \mathbf{F}_j \, | H \rangle. \tag{3.28}$$

The numerical factor arises from an overlap integral of quark and gluon spatial wavefunctions, and $\mathbf{F}_i, \boldsymbol{\sigma}_i$ are, respectively, the color and spin operators for quark i. It is straightforward to demonstrate that $h_\pi = 0.708$, $h_N = -h_\Delta = h_\pi/2$, and $h_\rho = -h_\pi/3$.

We have described the primary ingredients of the bag model. Fits to the masses of the ground-state hadrons can be accomplished within this framework, for example in [DeJJK 75, DoJ 80]. These reproduce many of the features of these particles, and we return to baryon properties in the next chapter.

XI–4 Skyrme model

In Chap. X, we explored the $N_c \to \infty$ limit of QCD. In some respects the world thus defined is not unlike our own. Mesons and glueballs exist with masses which are $\mathcal{O}(1)$ as $N_c \to \infty$. To lowest order, these particles are noninteracting because their coupling strength is $\mathcal{O}(N_c^{-1})$. What becomes of baryons in this world? It takes N_c quarks to form a totally antisymmetric color-singlet composite, so baryon mass is expected to be $\mathcal{O}(N_c)$. Note the inverse correlation between interparticle coupling $\mathcal{O}(N_c^{-1})$ and baryon mass $\mathcal{O}(N_c)$. This is reminiscent of soliton behavior in theories with nonlinear dynamics.

Sine–Gordon soliton

An example is afforded by the Sine–Gordon model, defined in one space and one time dimension by the lagrangian,

$$\mathcal{L}_{SG} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{\alpha}{\beta^2}(1 - \cos\ \beta\varphi), \tag{4.1}$$

where α and β are constants. For small-amplitude field excitations, an expansion in powers of φ,

$$\mathcal{L}_{SG} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{\alpha}{2}\varphi^2 + \frac{\alpha\beta^2}{4!}\varphi^4 + \mathcal{O}\left(\beta^4\varphi^6\right), \tag{4.2}$$

identifies the parameter α as the boson squared mass and β as a coupling strength. For $\beta \to 0$ we recover the free field theory. The Sine–Gordon lagrangian has also a nonperturbative static solution,

$$\varphi_0(x) = \frac{4}{\beta}\ \tan^{-1}\left(\exp\left(\sqrt{\alpha}x\right)\right), \tag{4.3a}$$

with energy

$$E_0 = 8\sqrt{\alpha}/\beta^2. \tag{4.3b}$$

This solution is a Sine–Gordon soliton. The natural unit of length for the soliton is $\alpha^{-1/2}$, and the energy E_0 diverges as the coupling is turned off ($\beta \to 0$). The potential energy in this theory has an infinity of equally spaced minima, with $\varphi^{(n)} = 2\pi n/\beta$ ($n = 0, \pm 1, \pm 2, \dots$). As the coordinate x is varied continuously from $-\infty$ to $+\infty$, the soliton amplitude $\varphi_0(x)$, starting from the minimum $\varphi^{(0)} = 0$, moves to the adjoining minimum $\varphi^{(1)} = 2\pi/\beta$. An index ΔN, the *winding number*, counts the number of minima shifted. It can be expressed as the charge associated with a current density,

$$J^\mu = \frac{\beta}{2\pi}\,\epsilon^{\mu\nu}\,\partial_\nu \varphi, \tag{4.4}$$

such that

$$\Delta N = \int_{-\infty}^{\infty} dx\ J^0(x) = \frac{\beta}{2\pi}\ [\varphi(+\infty) - \varphi(-\infty)]\,. \tag{4.5}$$

For $\varphi = \varphi_0$ as in Eq. (4.3a) we see that $\Delta N = 1$. The current density is conserved, $\partial_\mu J^\mu = 0$. Thus its charge, the winding number ΔN, does not change with time.

This is an example of a *topological* conservation law, whose origin lies in the nontrivial boundary conditions (viz. Eq. (4.5)) which a given field configuration is constrained to obey.

Chiral SU*(2) soliton*

Let us now seek a soliton solution for an $SU(2)_L \times SU(2)_R$ invariant theory in a spacetime of dimension four. It is natural to consider first the lowest-order chiral lagrangian $\mathcal{L}_2$,

$$\mathcal{L}_2 = \frac{F_\pi^2}{4} \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right), \tag{4.6}$$

where U is an $SU(2)$ matrix which transforms as $U \to LUR^{-1}$ under a chiral transformation for $L \in SU(2)_L$ and $R \in SU(2)_R$. Unfortunately, $\mathcal{L}_2$ cannot support an acceptable soliton, as the soliton would have zero size and zero energy. To see why, recall that the Sine–Gordon soliton has a natural unit of length $\alpha^{-1/2}$. Suppose there is an analogous quantity, R, for the chiral soliton. Then we can write the radial variable as $r = \tilde{r}R$, where $\tilde{r}$ is dimensionless. For a static solution, the energy becomes

$$E = \int d^3x\, \mathcal{H} = -\int d^3x\, \mathcal{L} = \frac{F_\pi^2}{4}\int d^3x\ \mathrm{Tr}\left(\nabla U \cdot \nabla U^\dagger\right). \tag{4.7}$$

Upon expressing the integral in terms of the dimensionless variable $\tilde{r}$, we find $E = aR$, where a is a nonnegative number. The energy is minimized at $R = 0$ to the value $E = 0$. This trivial solution is unacceptable, and thus the model must be extended.

The Skyrme model [Sk 61] employs, in addition to $\mathcal{L}_2$, a quartic interaction of a certain structure,

$$\mathcal{L} = \frac{F_\pi^2}{4} \mathrm{Tr}\ \left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{32e^2} \mathrm{Tr}\ \left[\partial_\mu U\, U^\dagger, \partial_\nu U\, U^\dagger\right]^2, \tag{4.8}$$

where e (not to be confused with the electric charge!) is a dimensionless real-valued parameter. The above chiral lagrangian should look familiar, since it is part of the general fourth-order chiral lagrangian used in Chap. VII. In particular, Eq. (4.8) is reproduced if $2L_1^r + 2L_2^r + L_3^r = 0$, in which case $(32e^2)^{-1} = (L_2^r - 2L_1^r - L_3^r)/4$. The comparison with the phenomenology of Chap. VII is not completely straightforward, as the pion physics was treated to one-loop order while the Skyrme lagrangian is used at tree level. We note, however, that the coefficients in Table VII–1 give

$$\frac{2L_1^r + 2L_2^r + L_3^r}{L_2^r - 2L_1^r - L_3^r} = 0.685, \qquad L_2^r - 2L_1^r - L_3^r = 0.0040. \tag{4.9}$$

The latter combination, which is independent of renormalization scale, numerically gives $e \simeq 5.6$. In the following development, we shall follow standard practice by taking the parameter e as arbitrary.

We seek a static solution of the Skyrme model. Our strategy shall be to first determine the energy functional of the theory, and then minimize it. Following the procedure leading to Eq. (4.7), we can write the energy as

$$E = \int d^3x \ \mathrm{Tr} \left[\frac{F_\pi^2}{4} X_i X_i^\dagger + \frac{1}{16e^2} (\epsilon_{ijk} X_i X_j)(\epsilon_{abk} X_a X_b)^\dagger \right], \tag{4.10}$$

where $X_\mu \equiv U \partial_\mu U^\dagger$ and $X_\mu = -X_\mu^\dagger$. It is necessary that $X_i \to 0$ as $|\mathbf{x}| \to \infty$ in order that the energy be finite. Thus, U must approach a constant element of $SU(2)$, which we are free to choose as the identity I. For the mesonic sector of the theory, the vacuum state corresponds to $U(\mathbf{x}) = I$ for *all* $\mathbf{x}$. In this state, both the field variable X_i and the energy E vanish. The form $U \simeq I + i\boldsymbol{\pi} \cdot \boldsymbol{\tau}/F_\pi$, used extensively in earlier chapters, corresponds to small-amplitude pionic excitations of the vacuum.

To see that the Skyrme model does support a nontrivial soliton, we cast the energy integrals of Eq. (4.10) in terms of a natural length scale R and find

$$E = aR + bR^{-1}, \tag{4.11}$$

where a, b are nonnegative. For $a, b \neq 0$, the energy is minimized at nonzero R and nonzero E. Thus, the quartic term of Eq. (4.8) is seen to have the desired effect of inducing soliton stability. Moreover, for arbitrary U a lower bound on the energy is provided by applying the Schwartz inequality to Eq. (4.10),

$$E \geq \frac{F_\pi}{4e} \int d^3x |\, \mathrm{Tr} \ \epsilon_{ijk} X_i X_j X_k|. \tag{4.12}$$

It is not hard to show that the integrand of Eq. (4.12) is proportional to the zeroth component of a four-vector current,

$$B^\mu = \frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2} \ \mathrm{Tr} \ X_\nu X_\alpha X_\beta, \tag{4.13}$$

which is divergenceless, $\partial_\mu B^\mu = 0$, and thus has conserved charge

$$B = \int d^3x \ B^0(\mathbf{x}). \tag{4.14}$$

It turns out that the current B^μ can be identified as the baryon current density and B is the baryon number of the theory. Note that this is consistent with our prescription $U(\mathbf{x}) = I$ for the meson vacuum, where we see from Eq. (4.13) that $B = 0$. Interestingly, B turns out to have an *additional* significance. It is the topological winding number for the Skyrme model, analogous to ΔN for the Sine–Gordon model. The point is, by having associated spatial infinity with a group element of $SU(2)$ to ensure that the field energy is finite, we have placed the elements of physical space into a correspondence with the elements of the compact group $SU(2)$. The parameter space of each set is S^3, the unit sphere in four dimensions, and it is precisely the field U which implements the mapping. The mappings from S^3 to S^3 are known to fall into classes, each labeled by an integer-valued winding number. In this context, B serves to measure the number of times that the set of space points covers the group parameters of $SU(2)$ for some solution U of the theory.

The Skyrme soliton

The Skyrme *ansatz* for a chiral soliton (*skyrmion*) has the functional form [BaNRS 83, AdNW 83]

$$U_0(\mathbf{x}) = \exp\left[iF(r)\boldsymbol{\tau}\cdot\hat{\mathbf{x}}\right]. \tag{4.15}$$

The unknown quantity is the skyrmion profile function $F(r)$. To specify it, we first determine the energy functional by substituting U_0 into Eq. (4.10),

$$\begin{aligned} E[F] = 4\pi\int_0^\infty dr\, r^2 \Bigg[& \frac{F_\pi^2}{2}\left(F'^2 + 2\frac{\sin^2 F}{r^2}\right) \\ & + \frac{1}{2e^2}\frac{\sin^2 F}{r^2}\left(\frac{\sin^2 F}{r^2} + 2F'^2\right)\Bigg], \end{aligned} \tag{4.16}$$

where a prime signifies differentiation with respect to the argument. For a static solution, the minimization of the energy generates an extremum of the action, and is hence equivalent to the equations of motion. The variation $\delta E/\delta F = 0$ generates a differential equation for F,

$$\left(\frac{\tilde{r}^2}{4} + 2\sin^2 F\right)F'' + \frac{\tilde{r}}{2}F' + F'^2\sin 2F - \frac{\sin 2F}{4} - \frac{\sin^2 F\sin 2F}{\tilde{r}^2} = 0, \tag{4.17}$$

as expressed in terms of a dimensionless variable $\tilde{r} = r/R$, with $R^{-1} \equiv 2eF_\pi$. This nonlinear equation must be solved numerically, subject to certain boundary conditions. The condition $U = I$ at spatial infinity implies $F(\infty) = 0$. The boundary condition at $r = 0$ is fixed by requiring that the soliton corresponds to baryon number 1. For the Skyrme ansatz, the baryon-number charge density is

$$B^0(r) = -\frac{1}{2\pi^2}\frac{F'\sin^2 F}{r^2}, \tag{4.18}$$

and corresponds to a baryon number

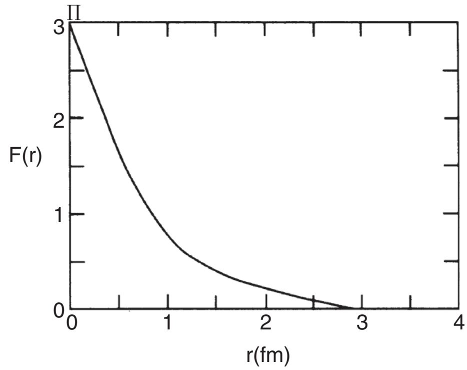

Fig. XI–4 Radial profile of the skyrmion.

$$B = \frac{1}{2\pi}\,[2F(0) - 2F(\infty) - \sin 2F(0) + \sin 2F(\infty)]. \tag{4.19}$$

This leads to the choice $F(0) = \pi$. Although the profile $F(r)$ cannot be determined analytically over its entire range, it is straightforward to show that

$$F(r) \sim \begin{cases} \pi - \text{const.}\, r & (r \to 0), \\ \text{const.}\, r^{-2} & (r \to \infty). \end{cases} \tag{4.20}$$

We display $F(r)$ in Fig. XI–4. Insertion of the solution to Eq. (4.17) into the energy functional $E[F]$ yields the mass M of the skyrmion, and from a numerical integration we obtain $M \simeq 73\ F_\pi/e$. There is an important point to be realized about the skyrmion – it represents a use of chiral lagrangians outside the region of validity of the energy expansion. Recall that the full chiral lagrangian is written as a power series, $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \cdots$ in the number of derivatives. When matrix elements of pions are taken, terms with n derivatives produce n powers of the energy. Hence, at low energy, one may consistently ignore operators with large n, as their contributions to matrix elements are highly suppressed. However, in forming the skyrmion one employs only $\mathcal{L}_2$ and a subset of $\mathcal{L}_4$. The relative effects of the two are balanced in the minimization of the energy functional, and as a result both contribute equally. In an extended model containing $\mathcal{L}_6$, one would expect the import of $\mathcal{L}_6$ to be analogously comparable to $\mathcal{L}_4$, etc. Higher-derivative lagrangians thus will contribute to skyrmion matrix elements, and the result cannot be considered a controlled approximation. However, this is not sufficient cause for abandoning the skyrmion approach. It simply becomes a phenomenological *model* rather than a rigorous method, and thus has a status similar to potential or bag models.

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

Quantization and wavefunctions

The analysis done thus far is at the classical level, and merely shows that the chiral soliton satisfies the equations of motion. To determine the quantum version of the theory, we shall follow a canonical procedure. An analogy with quantization of the *rigid rotator* may help in understanding the process. A classical solution consists of the rotator being at any fixed angular configuration $\{\theta, \varphi\}$. To obtain the quantum theory, one allows the rotator to move among these solutions, and describes its motion in terms of the angular coordinates and their conjugate momenta $\{p_\theta, p_\varphi\}$. The quantum states are those with definite angular momentum quantum numbers $\{\ell, m\}$, and have wavefunctions given by the spherical harmonics,

$$\langle \theta, \varphi | \ell, m \rangle = Y_{\ell,m}(\theta, \varphi). \tag{4.21}$$

The classical skyrmion solutions consist not only of U_0 (cf. Eq. (4.15)), but also of any constant $SU(2)$ rotation thereof, $U_0' = AU_0A^{-1}$ with $A \epsilon SU(2)$. A particularly simple approach to quantization is then to allow the soliton to rotate rigidly in the space of these solutions,

$$U = A(t)U_0A^{-1}(t), \tag{4.22}$$

where now $A(t)$ is an arbitrary time-dependent $SU(2)$ matrix. One proceeds to define a set of coordinates $\{a_k\}$, their conjugate momenta $\{\pi_k \equiv \partial\mathcal{L}/\partial a_k\}$, and a hamiltonian constructed via Legendre transformation

$$H = \pi_k \dot{a}_k - L. \tag{4.23}$$

We shall presently describe how to choose quantum numbers and determine the associated wavefunctions. Note that this approach is approximate in that it neglects the possibility of spacetime-dependent excitations such as pion emission. As such, it would be most appropriate for a weakly coupled theory (as occurs for $N_c \to \infty$) where the soliton rotates slowly, but is only approximate in the real world.

In general, an $SU(2)$ matrix like A can be written in terms of three unconstrained parameters $\{\theta_k\}$ as

$$A(t) = \exp(i\boldsymbol{\tau} \cdot \boldsymbol{\theta}) = \mathbf{I}\, \cos\theta + i\boldsymbol{\tau} \cdot \hat{\boldsymbol{\theta}}\, \sin\theta. \tag{4.24}$$

However, we can equivalently employ the four constrained parameters,

$$a_0 = \cos\theta, \qquad \mathbf{a} = \hat{\boldsymbol{\theta}}\, \sin\theta, \tag{4.25a}$$

where

$$\sum_{k=0}^{3} a_k^2 = 1. \tag{4.25b}$$

Substitution of the rotated quantity U into Eq. (4.7) and evaluation of the spatial integration yields

$$L = -M + \lambda\,\mathrm{Tr}\,(\partial_0 A^\dagger \partial_0 A) = -M + 2\lambda \sum_{k=0}^{3} \dot{a}_k^2, \tag{4.26}$$

where $\lambda = \pi\Lambda/3e^3 F_\pi$, with

$$\Lambda = \int d\tilde{r}\,\tilde{r}^2\ \sin^2 F\left[1 + 4\left(F'^2 + \sin^2 F/\tilde{r}^2\right)\right] \simeq 50.9. \tag{4.27}$$

As written in terms of the conjugate momenta $\pi_k = 4\lambda\dot{a}_k$, the hamiltonian is

$$H = M + \frac{1}{8\lambda}\sum_{k=0}^{3} \pi_k^2. \tag{4.28}$$

Adopting the canonical quantization conditions

$$[a_k,\ \pi_l] = i\delta_{kl}, \tag{4.29}$$

we see that the canonical momenta can be expressed as differential operators, $\pi_k = -i\partial/\partial a_k$. Thus, the hamiltonian has the form

$$H = M - \frac{1}{8\lambda}\boldsymbol{\nabla}_4^2, \tag{4.30}$$

where $\boldsymbol{\nabla}_4^2$ is the four-dimensional laplacian restricted to act on the three-sphere by the constraint of Eq. (4.25b).

We can determine the eigenvalues and eigenvectors of H by working in analogy with the more familiar three-dimensional laplacian,

$$\boldsymbol{\nabla}_3^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\mathbf{L}^2. \tag{4.31}$$

If constrained to the unit two-sphere by the condition $\sum_{k=1}^{3} x_k^2 = r^2 = 1$, the three-dimensional laplacian $\boldsymbol{\nabla}_3^2$ reduces to $-\mathbf{L}^2$. As is well known, the three components of $\mathbf{L}$ are operators $L_1,\ L_2,\ L_3$ which satisfy

$$[L_j, L_k] = i\epsilon_{jkl}L_l, \tag{4.32}$$

and generate rotations in the 2–3, 3–1, 1–2 planes respectively. The underlying symmetry group is $SO(3)$, and the eigenfunctions are the spherical harmonics.

The four-dimensional problem is treated by analogy. Upon adding an extra dimension labeled by the index 0, we encounter the additional operators K_1, K_2, K_3, which generate rotations in the 0–1, 0–2, 0–3 planes. The full set of six rotational generators can be represented as

$$L_k = \epsilon_{ijk}a_i\pi_j, \qquad K_k = a_0\pi_k - a_k\pi_0. \tag{4.33}$$

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

The extended symmetry group is $SO(4)$ and the commutator algebra of the rotation generators is

$$\left[L_j,\ L_k\right] = i\epsilon_{jkl}L_l, \quad \left[L_j,\ K_k\right] = i\epsilon_{jkl}K_l, \quad \left[K_j,\ K_k\right] = i\epsilon_{jkl}L_l. \tag{4.34}$$

The mathematics of this algebra is well known, underlying, for example the symmetry of the Coulomb hamiltonian in nonrelativistic quantum mechanics. By the substitutions

$$\mathbf{T} = (\mathbf{L} - \mathbf{K})/2, \qquad \mathbf{J} = (\mathbf{L} + \mathbf{K})/2, \tag{4.35}$$

we arrive at operators $\mathbf{T}$ and $\mathbf{J}$, which generate commuting $SU(2)$ algebras. We associate $\mathbf{T}$ with the isospin and $\mathbf{J}$ with the angular momentum. The explicit operator representations,

$$\begin{aligned} T_k &= i(-\epsilon_{ijk}a_i\partial_j + a_0\partial_k - a_k\partial_0), \\ J_k &= i(-\epsilon_{ijk}a_i\partial_j - a_0\partial_k + a_k\partial_0), \end{aligned} \tag{4.36}$$

follow immediately from Eq. (4.33), and the Skyrme hamiltonian becomes

$$H = M + \left(\mathbf{T}^2 + \mathbf{J}^2\right)/4\lambda. \tag{4.37}$$

It follows from the commutator algebra of Eq. (4.34) that $\mathbf{T}^2 = \mathbf{J}^2$. Thus, the quantum spectrum consists of states with equal isospin and angular momentum quantum numbers, $T = J$. This is no surprise. After all, in the Skyrme ansatz of Eq. (4.15), the isospin and spatial coordinates appear symmetrically, and we expect the quantum spectrum to respect this reciprocity. Our final form for the hamiltonian,

$$H = M + \mathbf{J}^2/2\lambda, \tag{4.38}$$

has the eigenvalue spectrum

$$E = M + J(J+1)/2\lambda, \tag{4.39}$$

where in general $J = 0, 1/2, 1, 3/2, \ldots$.

By analogy with the usual spherical harmonics, the eigenfunctions of H are seen to be traceless symmetric polynomials in the $\{a_k\}$. However, both $\{a_k\}$ and $\{-a_k\}$ describe the same solution U (cf. Eq. (4.22)). In the quantum theory, eigenfunctions thus fall into either of two classes, $\psi(\{-a_k\}) = \pm\,\psi(\{a_k\})$. Since fermions correspond to the antisymmetric choice, we select only the half-integer values in Eq. (4.39). In the Skyrme model, the N and Δ baryons will have wavefunctions which are respectively linear and cubic in the $\{a_k\}$. To construct such states, it is convenient to employ the differential representations of Eq. (4.36) to prove

$$\begin{aligned} L_3(a_1 \pm ia_2) &= \pm(a_1 \pm ia_2), \qquad L_3 a_{0,3} = 0, \\ K_3(a_0 \pm ia_3) &= \pm(a_0 \pm ia_3), \qquad K_3 a_{1,2} = 0. \end{aligned} \tag{4.40}$$

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

From these and Eq. (4.36), the $T_3 = J_3 = 1/2$ eigenstate of a proton with spin up is found to be

$$\langle A|\, p_\uparrow\rangle = \frac{1}{\pi}\,(a_1 + i a_2). \tag{4.41}$$

The normalization of this state is obtained from the angular integral over the three-sphere

$$1 = \langle p_\uparrow|\, p_\uparrow\rangle = \int d\Omega_3\, \langle p_\uparrow|A\rangle\langle A|\, p_\uparrow\rangle = \frac{1}{\pi^2}\int d\Omega_3\, \left(a_1^2 + a_2^2\right), \tag{4.42}$$

where the angular measure is

$$\int d\Omega_3 = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta \int_0^{\pi} d\chi\, \sin^2\chi, \tag{4.43}$$

and spherical coordinates in four dimensions are defined by

$$\begin{aligned} a_1 &= \sin\chi\, \sin\theta\, \cos\varphi, \qquad & a_2 &= \sin\chi\, \sin\theta\, \sin\varphi, \\ a_3 &= \sin\chi\, \cos\theta, & a_0 &= \cos\chi\,. \end{aligned} \tag{4.44}$$

The remaining nucleon states can be found by application of the spin and isospin lowering sperators

$$\begin{aligned} J_- &= [(a_1 - i a_2)\partial_3 - (a_3 + i a_0)\partial_1 + (-a_0 + i a_3)\partial_2 + (a_2 + i a_1)\partial_0]/2, \\ T_- &= [(a_1 - i a_2)\partial_3 + (-a_3 + i a_0)\partial_1 + (a_0 + i a_3)\partial_2 - (a_2 + i a_1)\partial_0]/2, \end{aligned} \tag{4.45}$$

where $\partial_k \equiv \partial/\partial a_k$. The $T = J = 3/2$ Δ states are formed by employing analogous ladder operations on

$$\langle A|\, \Delta^{++}_{3/2}\rangle = \frac{i\sqrt{2}}{\pi}\,(a_1 + i a_2)^3. \tag{4.46}$$

It is remarkable that fermions can be constructed from a chiral lagrangian which contains nominally bosonic degrees of freedom. However, the presence of a nonzero fermion quantum number can be easily verified by direct calculation.

The wavefunctions for the eigenstates (the equivalents of $Y_{\ell,m}(\theta,\varphi)$ for the rigid rotator) are given by $SU(2)$ rotation matrices with half-integer values. These are defined by the transformation properties of states under an $SU(2)$ rotation A,

$$|T, T_3'\rangle = \sum_{T_3} \mathcal{D}^{(T)}_{T_3' T_3}(A)|T, T_3\rangle. \tag{4.47}$$

The simplest case is then just the $T = 1/2$ representation, which we know is rotated by the matrix A,

$$\mathcal{D}^{(1/2)}_{ij}(A) = \mathbf{A}_{ij} = \begin{pmatrix} (a_0 + i a_3) & i(a_1 - i a_2) \\ i(a_1 + i a_2) & (a_0 - i a_3) \end{pmatrix}_{ij}. \tag{4.48}$$

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

Comparison with Eq. (4.41) and with the results of Eq. (4.47) shows that the properly normalized nucleon wavefunctions are

$$\langle A|\, N_{T_3,S_3}\rangle = \frac{1}{\pi}(-)^{T_3+1/2}\, \mathcal{D}^{(1/2)}_{-T_3,S_3}(A). \tag{4.49}$$

The general case for a nonstrange baryon B of isospin T and spin S ($S = T$) is given by

$$\langle A|\, B_{T,T_3,S_3}\rangle = \left[\frac{2T+1}{2\pi^2}\right]^{1/2} (-)^{T+T_3}\, \mathcal{D}^{(T)}_{-T_3,S_3}(A), \tag{4.50}$$

of which the Δ states are specific examples.

Finally, the N and Δ masses are

$$\begin{aligned} M_N &= M + 3/8\lambda = 73F_\pi/e + e^3F_\pi/45.2\pi, \\ M_\Delta &= M + 15/8\lambda = 73F_\pi/e + 5e^3F_\pi/45.2\pi. \end{aligned} \tag{4.51}$$

If the measured N, Δ masses are used as input, one obtains $e = 5.44$ and $F_\pi = 65$ MeV. Alternatively, from the empirical value for F_π and the determination $e \simeq 5.6$ from pion–pion scattering data, the model implies $M_N \simeq 1.27$ GeV, $M_\Delta \simeq 1.80$ GeV. In either case, agreement between theory and experiment is at about the 30% level. The next state in the spectrum would have quantum numbers $T = J = 5/2$ and is predicted by the first of the above fitting procedures to have mass $M_{5/2} = M + 35/8\lambda \simeq 1.72$ GeV. There is no experimental evidence for such a baryon.

Although the development of the skyrmion and its quantization have been motivated by large-N_c ideas, we know of no proof that requires the skyrmion to come arbitrarily close to the baryons of *QCD* in the $N_c \to \infty$ limit. An oft-cited counterexample is the existence of a one-flavor version of *QCD*. Such a theory still contains baryons, such as the Δ^{++}. However, it makes no sense to speak of a one-flavor Skyrme model, as an $SU(2)$ group is required for the underlying soliton U_0. The Skyrme model remains an interesting picture for nucleon structure because it is in many ways orthogonal to the quark model, and thus offers opportunities for new insights.

XI–5 *QCD* sum rules

Low-energy *QCD* involves a regime where the degrees of freedom are hadrons, and where it is futile to attempt perturbative calculations of hadronic masses and decay widths. Contrasted with this is the short-distance asymptotically free limit in which quarks and gluons are the appropriate degrees of freedom, and in which perturbative calculations make sense. The method of *QCD* sum rules represents an

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

attempt to bridge the gap between the perturbative and nonperturbative sectors by employing the language of dispersion relations [ShVZ 79*a*].

The existence of sum rules in *QCD* is quite general, and some might dispute the classification on these sum rules as a phenomenological method. However, in practice, to utilize the sum rules involves the introduction of various approximations and heuristic procedures. Like quark model methods, these are motivated by physical intuition but are not always rigorous consequences of *QCD*. As a result, there remains a certain degree of uncontrollable approximation in their use. Nonetheless, they have been employed in a large number of applications; some early reviews are [ReRY 85, Na 89] and for somewhat more recent entries see [Ra 98, CoK 00, Sh 10].

Correlators

It is convenient to approach the subject by considering the relatively simple two-point functions. Thus, we consider the quark bilinear,

$$J_\Gamma(x) = \overline{q}_1(x)\Gamma q_2(x), \tag{5.1}$$

where Γ is a Dirac matrix, and analyze the *correlator*,

$$i \int d^4x\, e^{iq\cdot x} \langle 0|T(J_\Gamma(x) J_\Gamma^\dagger(0))|0\rangle. \tag{5.2}$$

Such quantities can be expressed in terms of invariant functions $\Pi_\Gamma(q^2)$ and attendant kinematical factors, e.g., as for the correlators of pseudoscalar currents (J_P) and of conserved vector currents (J_V),

$$\Pi_P(q^2) = i \int d^4x\, e^{iq\cdot x} \langle 0|T\,(J_P(x) J_P(0))\,|0\rangle, \tag{5.3a}$$

$$(q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_V(q^2) = i \int d^4x\, e^{iq\cdot x} \langle 0|T\left(J_V^\mu(x) J_V^\nu(0)\right)|0\rangle. \tag{5.3b}$$

Analogous structures occur for other currents.

There are several means for analyzing a quantity like $\Pi_\Gamma(q^2)$. One is to write a dispersion relation based on its singularity structure in the complex q^2 plane. The singularities are just those imposed by unitarity. For example, by inserting a complete set of intermediate states into Eq. (5.3a) for the pseudoscalar function $\Pi_P(q^2)$ and invoking the constraints of Lorentz invariance and positivity of energy, we obtain

$$\Pi_P(q^2) = \int_{s_0}^\infty ds\, \frac{\rho_P(s)}{s - q^2 - i\epsilon},$$
$$\theta(q^0)\rho_P(q^2) = (2\pi)^3 \sum_n \delta^4(p_n - q)|\langle 0|J_P(0)|n\rangle|^2, \tag{5.4}$$

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

where s_0 is the threshold for the physical intermediate states. Such considerations, together with the application of Cauchy's theorem in the complex q^2 plane, imply a dispersion relation for $\Pi_\Gamma(q^2)$,

$$\Pi_\Gamma(q^2) = \frac{(q^2)^N}{\pi}\int_{s_0}^{\infty} ds\, \frac{\text{Im}\,\Pi_\Gamma(s)}{s^N(s-q^2-i\epsilon)} + \sum_{n=0}^{N-1}(q^2)^n a_n, \tag{5.5}$$

where the $\{a_n\}$ are N subtraction constants.[4] One attempts to introduce a phenomenological component to the dispersion relation by expressing Im $\Pi_\Gamma(s)$ in terms of measureable quantities, e.g., with cross section-data as in the case of the charm contribution $\bar{c}\gamma^\mu c$ to the vector current,

$$\text{Im}\,\Pi_V^{(\text{chm})} = \frac{1}{12\pi e_c^2}\frac{\sigma_{e^+e^-\to\ \text{charm}}}{\sigma_{e^+e^-\to\ \mu^+\mu^-}} = \frac{9s}{64\pi^2\alpha^2}\sigma_{e^+e^-\to\ \text{charm}}, \tag{5.6}$$

where e_c is the c-quark electric charge and s is the squared center-of-mass energy. If such data are not available, another means must be found for expressing Im $\Pi_\Gamma(s)$ in the range $s_0 \le s < \infty$.

To approximate the low-s part of Im $\Pi_\Gamma(s)$, one usually employs one or more single-particle states. As an illustration, let us determine the contribution to $\Pi_P(q^2)$ of a flavored pseudoscalar meson M, which is a bound state or a narrow-width resonance of the quark–antiquark pair $\bar{q}_1 q_2$. In this instance, we take the pseudoscalar current in the form of an axial-vector divergence, $J_{\text{P}} \to \partial_\mu A_-^\mu$, with

$$\begin{aligned}\partial_\mu A_-^\mu &= i(m_1+m_2)\bar{q}_1\gamma_5 q_2,\\ \langle 0|\partial_\mu A_-^\mu(0)|M\rangle &= \sqrt{2}F_M m_M^2,\end{aligned} \tag{5.7}$$

where m_M and F_M are the meson's mass and decay constant. Then Eq. (5.4) implies

$$\begin{aligned}\theta(q_0)\rho_{\text{p}}(q^2) &= (2\pi)^3\int \frac{d^3p}{(2\pi)^3 2\omega_p} 2F_M^2 m_M^4 \delta^4(p-q)\\ &= 2F_M^2 m_M^4 \delta(q^2-m_M^2)\theta(q_0),\end{aligned} \tag{5.8}$$

which yields $\rho_{\text{p}}(q^2) = 2F_M^2 m_M^4\delta(q^2-m_M^2)$ for the spectral function or

$$\text{Im}\,\Pi_{\text{p}}|_{\text{meson}} = 2F_M^2 m_M^4 \pi\delta(s-m_M^2) \tag{5.9}$$

for the dispersion kernel. Thus, bound-state or narrow-resonance contributions give rise to delta-function contributions. It is not difficult to take resonant finite-width effects into account if desired. One or more of these single-particle contributions are then used to represent the low-s part of the dispersion integral.

[4] The number of subtraction constants needed depends on the behavior of Im $\Pi_\Gamma(s)$ in the $s \to \infty$ limit, with $\Pi_\Gamma(q^2) \sim q^{2N} \ln q^2$ requiring N subtractions.

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

Table XI–4. *Local operators of low dimension.*

d:	0	4	4	6	6	6
O_n:	1	$m_q\overline{q}q$	$G^a_{\mu\nu}G^{\mu\nu}_a$	$\overline{q}\Gamma q\overline{q}\Gamma q$	$m\overline{q}\sigma_{\mu\nu}\frac{\lambda^a}{2}qG^{\mu\nu}_a$	$f_{abc}G^a_{\mu\nu}G^{\nu\lambda}_b G^{\mu c}_{\lambda}$

Proceeding to higher s values in the dispersion integral, one enters the continuum region, where multiparticle intermediate states become significant and the bound-state (or resonance) approximation breaks down. Although, as described below, one ordinarily attempts to suppress the large-s part of Im $\Pi_\Gamma(s)$ by taking moments or transforms of the dispersion integral, it has been common to add to the low-s contribution a '*QCD* continuum' approximation,

$$\text{Im}\ \Pi_\Gamma(s) \underset{\text{large-s}}{\longrightarrow} \theta(s - s_c)\text{Im}\ \Pi_{\text{cont}}(s), \tag{5.10}$$

taken from discontinuities of *QCD* loop amplitudes and their $\mathcal{O}(\alpha_s)$ corrections. In Eq. (5.10), s_c parameterizes the point where the continuum description begins and the form of Im Π_{cont} depends on the specific correlator. Experience has shown that this 'parton' description can yield reasonable agreement of scattering data even down into the resonance region, provided the resonances are averaged over (*duality*).

Operator-product expansion

A representation for correlators which is distinct from the above phenomenological approach can be obtained by employing an operator-product expansion for the product of currents,

$$i\int d^4x\ e^{iq\cdot x}\ T\left(J_\Gamma(x)J^\dagger_\Gamma(0)\right) = \sum_n \mathcal{C}^\Gamma_n(q^2)O_n. \tag{5.11}$$

The $\{O_n\}$ are local operators and the $\{\mathcal{C}^\Gamma_n(q^2)\}$ are the associated Wilson coefficients. As usual, the $\{O_n\}$ are organized according to their dimension and, aside from the unit operator I, are constructed from quark and gluon fields. Table XI–4 exhibits the operators up to dimension six which might contribute to the correlator of Eq. (5.2).

Although one may naively expect all the operators but the identity to have vanishing vacuum expectation values (as is the case for normal-ordered local operators in perturbation theory), nonperturbative long-distance effects like those discussed in Sect. III–5 generally lead to nonzero values. Most often, the operator-product approach contains vacuum expectation values like $\langle\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^{\mu\nu}_a\rangle_0 \equiv \langle\frac{\alpha_s}{\pi}G^2\rangle_0$ and

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

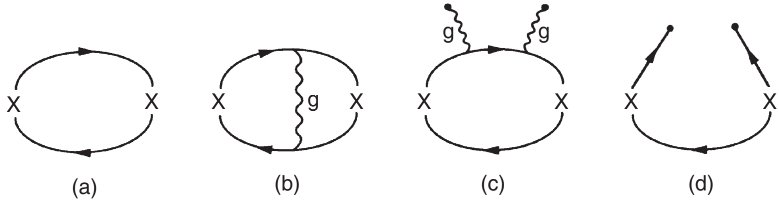

Fig. XI–5 Contributions to coefficient functions.

$\langle m_q \overline{q} q \rangle_0$ as universal parameters, 'universal' in the sense that the same few parameters appear repeatedly in applications. Calculation reveals that the quantity $\langle \frac{\alpha_s}{\pi} G^2 \rangle_0$ is divergent in perturbation theory, so the perturbative infinities must be subtracted off if one is working beyond tree-level. In principle, all the vacuum expectation values should be computable from lattice gauge theory once the renormalization prescriptions are specified. At present, the only theoretically determined combinations are the products

$$\langle \hat{m}(\overline{u}u + \overline{d}d) \rangle_0 \simeq -2F_\pi^2 m_\pi^2, \qquad \langle m_s \overline{s} s \rangle_0 \simeq -F_\pi^2 m_K^2, \tag{5.12}$$

which follow from the lowest-order chiral analysis in Chap. VII. We caution that only the *product* $m\overline{\psi}\psi$ is renormalization-group invariant (the gluon condensate $\langle \frac{\alpha_s}{\pi} G^2 \rangle_0$ does not, however, depend on scale). It is difficult to separate out the quark masses uniquely, and values for input parameters like quark masses and condensates tend to vary throughout the literature.

Use of the short-distance expansion must be justified. We have seen in previous chapters how a given hadronic system is characterized in terms of the energy scales of confinement (Λ) and quark mass ($\{m_q\}$). Given these, it is indeed often possible to choose the momentum q such that short-distance, asymptotically free kinematics obtain. Two situations which have received the most attention are the heavy-quark limit ($m_q^2 \gg \Lambda^2, q^2$) and the light-quark limit ($q^2 \gg \Lambda^2 \gg m_q^2$). Once in the asymptotically free domain, it is legitimate to apply *QCD* perturbation theory to the $\mathcal{C}_n^\Gamma(q^2)$, with the expansion being carried out to one or more powers of α_s,

$$\mathcal{C}_n^\Gamma\left(q^2\right) = A_n^\Gamma\left(q^2\right) + B_n^\Gamma\left(q^2\right)\alpha_s + \cdots . \tag{5.13}$$

Rather extensive lists of Wilson coefficients already appear in the literature. Fig. XI–5 depicts contributions to a few of the Wilson coefficients. Denoting there the action of a current by the symbol '$\times$', we display in (a)–(b) the lowest-order and an $\mathcal{O}(\alpha_s)$ correction to operator I and in (c)–(d), the lowest-order contributions to $\langle \frac{\alpha_s}{\pi} G^2 \rangle_0$ and to $\langle m\overline{q}q \rangle_0$, respectively.

Finally, as seen in Eq. (5.13), besides the vacuum expectation values, additional parameters which generally occur in the operator-product representation are the quark mass m_q and the strong coupling α_s. Since these quantities will depend on the momentum q, one must interpret them as running quantities whose renormalization is to be specified. Due to asymptotic freedom, they too can be treated perturbatively, e.g., as in the familiar expression Eq. (II–2.78) for the running coupling α_s or Eq. (XIV–1.9) for the running mass $\overline{m}$.

Master equation

The essence of the *QCD* sum rule approach is to equate the dispersion and the operator-product expressions to obtain a 'master equation',

$$\frac{(q^2)^N}{\pi} \int_{s_0}^{\infty} ds \, \frac{\mathrm{Im}\,\Pi_\Gamma(s)}{s^N \left(s - q^2 - i\epsilon\right)} + \cdots = \sum_n C_n^\Gamma \left(q^2\right) \langle O_n \rangle_0. \tag{5.14}$$

It is important to restrict use of this equation to a range of q^2 for which both the short-distance expansion and also any 'resonance + continuum' approximation to $\mathrm{Im}\,\Pi_\Gamma$ are jointly valid. To satisfy these twin constraints, it is common practice not to analyze Eq. (5.14) directly, but rather first to perform certain differential operations leading to either *moment* or *transform* representations. The nth *moment* $M_n^\Gamma(Q_0^2)$ is defined as

$$M_n^\Gamma \left(Q_0^2\right) \equiv \frac{1}{n!} \left(-\frac{d}{dQ^2}\right)^n \Pi_\Gamma \left(Q^2\right) \Big|_{Q^2=Q_0^2} = \frac{1}{\pi} \int_{s_0}^{\infty} ds \, \frac{\mathrm{Im}\,\Pi_\Gamma(s)}{\left(s + Q_0^2\right)^{n+1}}, \tag{5.15}$$

where, in the spacelike region $q^2 < 0$, one usually works with the variable $Q^2 = -q^2$. By taking sufficiently many derivatives, one can remove unknown subtraction constants from the analysis and at the same time, enhance the contribution of a single-particle state at low s in the dispersion integral.

Alternatively, one can express the dispersion integral as a kind of transform. The *Borel transform* is constructed from the moment $M_n^\Gamma(Q^2)$ as

$$n \, Q^{2n} M_n^\Gamma \left(Q^2\right) \underset{n, Q^2 \to \infty}{\longrightarrow} \frac{1}{\pi \tau} \int_{s_0}^{\infty} ds \, e^{-s/\tau} \mathrm{Im}\,\Pi_\Gamma(s), \tag{5.16}$$

where $Q^2/n \equiv \tau$ remains fixed in the limiting process and defines the transform variable τ. To obtain the factor $e^{-s/\tau}$ in the above dispersion integral, we note

$$\frac{n \, Q^{2n}}{\left(s + Q^2\right)^{n+1}} = \frac{n}{s + Q^2} \left(1 + s/Q^2\right)^{-n} \underset{n, Q^2 \to \infty}{\longrightarrow} \frac{e^{-s/\tau}}{\tau}. \tag{5.17}$$

A slightly different version of exponential transform which has appeared in the literature is defined analogously,

$$Q^{2(n+1)} M_n^{\Gamma}\left(Q^2\right) \underset{n, Q^2 \to \infty}{\longrightarrow} \frac{1}{\pi} \int_{s_0}^{\infty} ds\, e^{-s\sigma} \mathrm{Im}\, \Pi_{\Gamma}(s), \tag{5.18}$$

where the transform variable is now $\sigma = n/Q^2$. The transform method serves to remove the subtraction constants and to suppress the contributions from operators of higher dimension in the operator-product expansion.

Examples

Applications of the *QCD* sum rule approach generally proceed according to the following steps.

(1) Choose the currents and write a dispersion relation for the correlator.
(2) Model the dispersion integrals with phenomenological input, usually some combination of single-particle states and continuum.
(3) Employ the operator-product expansion, including all appropriate operators up to some dimension at which one truncates the series.
(4) Obtain the Wilson coefficients as an expansion in α_s.
(5) Use the moment or transform technique to extract information from the master equation.
(6) Vary the underlying parameters until stability of output is achieved.

Let us consider several examples, keeping the treatment on an elementary footing to better emphasize the kinds of relationships which *QCD* sum rules entail. In fact, modern calculations can be quite technical, involving issues such as optimizing the organization of input data, inclusion of ever higher orders of both perturbation theory and vacuum condensates.

(i) *Rho meson decay constant* f_ρ: This was among the first applications of the *QCD* sum rule approach [ShVZ 79*a*]. The ρ isovector current $J_\mu^{(\rho)}$ and decay constant f_ρ are

$$J_\mu^{(\rho)} = \frac{\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d}{2}, \qquad \langle \rho^0(\mathbf{p}, \lambda)| J_\mu^{(\rho)} |0\rangle = \epsilon_\mu^\dagger(\mathbf{p}, \lambda) \frac{f_\rho m_\rho}{\sqrt{2}}. \tag{5.19a}$$

The sum rule which gives f_ρ is [CoK 00]

$$f_\rho^2 = \frac{e^{m_\rho^2 \tau}}{\tau} \left[\frac{1}{4\pi^2}\left(1 - e^{-s_0\tau}\right)\left(1 + \frac{\alpha_s(\tau^{-1})}{\pi}\right) + (m_u + m_d)\tau^2 \langle \bar{q} q\rangle + \cdots \right], \tag{5.19b}$$

where τ is the Borel parameter, s_0 is threshold above which $\Pi_\rho(s)$ is to be approximated via perturbation theory and ellipses represent additional condensate contributions. The variation of f_ρ vs. $1/\tau$ turns out to display little variation in f_ρ for, say,

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

$0.6 \leq \tau^{-2}(\text{GeV}^2) \leq 1.3$; this is the Borel window of stability. The range of values $f_\rho \sim 208 \to 218$ MeV which occur within the stability window is in accord with the experimental determination cited in Eq. (V–3.14) for the equivalent quantity $g_\rho = f_\rho m_\rho/\sqrt{2}$.

(ii) *Mass of the charm quark*: We consider the correlator for the charm-quark vector current $J_\mu^{(\text{chm})} = \bar{c}\gamma_\mu c$,

$$\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right) \Pi_V^{(\text{chm})}\left(q^2\right) = i \int d^4x\, e^{iq\cdot x} \langle 0|T\left(J_\mu^{(\text{chm})}(x) J_\nu^{(\text{chm})}(0)\right)|0\rangle, \tag{5.20a}$$

and the corresponding dispersion relation,

$$\frac{\partial}{\left(-\partial Q^2\right)} \Pi_V^{(\text{chm})}\left(Q^2\right) = \frac{1}{\pi}\int_{s_0}^\infty ds\, \frac{\text{Im}\, \Pi_V^{(\text{chm})}(s)}{\left(s+Q^2\right)^2}. \tag{5.20b}$$

Following the original treatment of this system [ShVZ 79*a*], we work at $Q^2 = 0$ and employ a moment analysis of the short-distance expansion containing just the identity and gluon contributions. The experimental input is obtained from

$$M_n^{(\text{expt})} = \int \frac{ds}{s^{n+1}} \frac{\sigma_{e^+e^- \to\, c\bar{c}}(s)}{\sigma_{e^+e^- \to\, \mu^-\mu^+}(s)}, \tag{5.21a}$$

whereas the theory side involves

$$M_n^{(\text{thy})} = \frac{12\pi^2 e_c^2}{n!} \cdot \frac{d}{dq^{2n}} \Pi_V^{(\text{chm})}(q^2 = 0). \tag{5.21b}$$

The moments $M_n^{(\text{thy})}$ can be determined in terms of an operator-product expansion, which is dominated by the *QCD*-perturbative contribution provided the value of n is not too large, i.e., $m_c/n > \Lambda_{QCD}$. Perturbative contributions have long been studied and a library of exact results is now available (see [DeHMZ 11]):

(1) $\mathcal{O}(\alpha_s^0)$ and $\mathcal{O}(\alpha_s^1)$: known for all n.
(2) $\mathcal{O}(\alpha_s^2)$: known up to $n = 30$.
(3) $\mathcal{O}(\alpha_s^3)$: known up to $n = 3$.

Below, we cite two specific $\mathcal{O}(\alpha_s^3)$ determinations of the charm quark mass [KuSS 07], [DeHMZ 11]. Both adopt the gluon condensate value $\langle \frac{\alpha_s}{\pi} G^2\rangle_0 = 0.006 \pm 0.012\ \text{GeV}^4$ (each analysis obtains only a minor effect for this term). The results obtained in $\overline{\text{MS}}$ renormalization are

$$\begin{aligned} \overline{m}_c(\overline{m}_c) &= (1.286 \pm 0.013)\ \text{GeV} \qquad [\text{KuSS 07}], \\ \overline{m}_c(\overline{m}_c) &= (1.277 \pm 0.026)\ \text{GeV} \qquad [\text{DeHMZ 11}], \end{aligned} \tag{5.22}$$

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

whereas the value $\overline{m}_c(\overline{m}_c) = (1.275 \pm 0.025)$ GeV is cited in [RPP 12]. The issue of how to assign uncertainty in *QCD*-sum-rule-determinations of charm mass, especially involving the data side of the calculation, is currently a topic of some interest,

(iii) *Weak decay constant of the D^+ meson*: Consider first the axial-current divergence and correlator associated with a heavy quark Q and a light antiquark $\overline{q}$, respectively of mass m_Q and m_q, which comprise a heavy meson M_Q,

$$\begin{aligned} \partial_\mu A^\mu_- &= i(m_Q + m_q)\overline{q}\gamma_5 Q, \\ \Pi_P(q^2) &= i \int d^4x\, e^{iq\cdot x} \langle 0|T(\partial_\mu A^\mu_-(x)\partial_\nu A^{\nu\dagger}_-(0))|0\rangle. \end{aligned} \tag{5.23}$$

A transformation with Borel variable τ yields

$$\Pi_P(\tau) = \int_{(m_Q+m_q)^2}^{\infty} ds\, e^{-s\tau} \rho_{\text{pert}}(s,\mu) + \Pi_{\text{pwr}}(\tau, m_Q, \mu) \tag{5.24}$$

where $\rho = \mathcal{I}m\ \Pi/\pi$ and $\rho_{\text{pert}}(s,\mu)$ and $\Pi_{\text{pwr}}(\tau, m_Q, \mu)$ represent respectively, the perturbative and nonperturbative contributions.

Let us now consider specifically the decay constant of the D^+, where symbolically $D^+ \sim (c\bar{u})$. The experimental value, $f_D \equiv \sqrt{2}F_D = 206.7 \pm 8.9$ MeV is found from $D^+ \to \mu^+\nu_\mu$ via decay formulas akin to the tree-level Eq. (VII–1.24) or radiatively corrected Eq. (VII–1.34) for pion leptonic decay. On the theory side, it is shown in [LuMS 11] that a straightforward *QCD* sum rule approach yields

$$f^2_{D^+} M^4_{D^+} e^{-M^2_{D^+}\tau} = \int_{(m_c+m_u)^2}^{s_{\text{eff}}(\tau)} ds\, e^{-s\tau} \rho_{\text{pert}}(s,\mu) + \Pi_{\text{pwr}}(\tau, m_Q, \mu), \tag{5.25}$$

where the condensate values adopted are

$$\langle \overline{q}q\rangle(\mu) = -(267 \pm 17\ \text{MeV})^3, \qquad \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = (0.024 \pm 0.012)\,\text{GeV}^4, \tag{5.26}$$

with $\mu = 2$ GeV being the $\overline{\text{MS}}$ renormalization scale. The most novel part of the expression in Eq. (5.25) is the presence of an ‘effective continuum threshold’ $s_{\text{eff}}(\tau)$. The τ-dependence of s_{eff} supplants the traditional form of Eq. (5.10) in which a constant cut-off s_c is used to describe the onset of continuum contributions. We leave a detailed discussion of the effective threshold to [LuMS 11] and simply state the final result,

$$f_D^{(\text{thy})} = \left(206.2 \pm 7.3_{(\text{OPE})} \pm 5.1_{(\text{sys})}\right) \text{MeV},$$

which is consistent with the experimental finding shown above.

(iv) *Nucleon mass*: It is not necessary to restrict oneself to mesonic currents as in Eq. (5.1). Here, we consider a current η_N (and its correlator), which carries the quantum numbers of the nucleon,

$$\eta_N = \epsilon_{ijk} u^i C \gamma^\mu u^j \gamma_5 \gamma_\mu d^k,$$

$$\Pi(q^2) = \Pi_1(q^2) + \not{q}\,\Pi_2(q^2) = i \int d^4x \; e^{iq\cdot x} \langle 0| T\left(\eta_N(x)\overline{\eta}_N(0)\right)|0\rangle, \tag{5.27}$$

where C is the charge-conjugation matrix. The simplest approximation to the dispersion integral comes from the nucleon pole,

$$\Pi\left(q^2\right)|_{\text{pole}} = \lambda_N^2 \frac{M_N + \not{q}}{q^2 - M_N^2}, \tag{5.28}$$

where the coupling λ_N^2 is proportional to the 'nucleon decay constant', i.e., the probability of finding all three quarks within the nucleon at one point. Upon making a simple approximation to the operator-product expansion,

$$\Pi_1\left(q^2\right) \simeq -\frac{q^2}{4\pi^2} \ln\left(-q^2\right) \langle \overline{q} q\rangle_0, \qquad \Pi_2\left(q^2\right) \simeq \frac{q^4}{64\pi^2} \ln\left(-q^2\right), \tag{5.29}$$

and employing a Borel transform, one obtains an amusing relation between nucleon mass and quark condensate [Io 81],

$$M_N = \left(-8\pi^2 \langle \overline{q} q\rangle_0\right)^{1/3} + \ldots \simeq 1 \text{ GeV}, \tag{5.30}$$

and implies the vanishing of the former with the latter. However, it should be realized that this result is subject to important corrections in a more complete treatment.

Each of the above examples has involved two-point functions. It is possible to apply the method to three-point functions as well, where one can obtain coupling-constant relations. The underlying principles are the same, but some technical details are modified owing to the larger number of variables, e.g., one encounters double-moments or double-transforms.

QCD sum rules work best when there is a reliable way to estimate the dispersion integral, most often with ground-state single-particle contributions. However, the method has its limitations. It is not at its best in probing radial excitations since their dispersion effects are generally rather small. Even having a good approximation to the dispersion integral is not sufficient to guarantee success. For example, the method has trouble in dealing with high-spin ($J > 3$) mesons because, even with dispersion integrals which are dominated by ground-state contributions, power corrections in the operator-product expansion become unmanageable.

Problems

(1) **Velocity in potential models**

Truly nonrelativistic systems have excitation energies small compared to the masses of their constituents. However, fitting the observed spectrum of light

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

hadrons requires excitation energies comparable to or larger than the constituent masses.

Assuming nonrelativistic kinematics, consider a particle of reduced mass m moving in a harmonic-oscillator potential of angular frequency ω. Expressing ω in terms of the energy splitting $E_1 - E_0$ between the first-excited state and the ground state, use the virial theorem to determine the 'rms' velocities of the ground state ($v_{\rm rms}^{(0)}$) and of the first-excited state ($v_{\rm rms}^{(1)}$) in terms of $E_1 - E_0$. Compute the magnitude of $v_{\rm rms}^{(0)}/c$ and $v_{\rm rms}^{(1)}/c$ using as inputs (i) $m_{f_2} - m_\rho \simeq 500$ MeV for light hadrons and (ii) $m_{\psi(2S)} - m_{J/\psi} \simeq 590$ MeV for charmed quarks.

Your results should demonstrate that the kinematics of quarks in light hadrons is not truly nonrelativistic. However, one tends to overlook this flaw given the potential model's overall utility.

(2) **Nucleon mass and the Skyrme model**

(a) Use the Skyrme ansatz of Eq. (4.15) to derive the expression Eq. (4.16) for the nucleon energy $E[F]$.

(b) Using the simple trial function $F(r) = \pi \exp(-r/R)$, scale out the range factor R to put $E[F]$ in the form of Eq. (4.11), where $a \simeq 30.8 F_\pi^2$ and $b \simeq 44.7/e^2$ are determined via numerical integration.

(c) Minimize $E[F]$ by varying R and compare your result with the value $73 F_\pi/e$ determined with a more complex variational function.

(d) Using the numerical value of the nucleon mass, determine e and compare with the value

$$\frac{1}{32e^2} \cdot \frac{4}{F_\pi^2} \sim \frac{1}{(4\pi F_\pi)^2}$$

expected from chiral-scaling arguments.

(3) **A '*QCD* sum rule' for the isotropic harmonic oscillator**

Consider three-dimensional isotropic harmonic motion with angular frequency ω of a particle of mass m.

(a) Using ordinary quantum mechanics or more formal path-integral methods, determine the exact Green's function $G(\tau)$ for propagation from time $t = 0$ to imaginary time $t = -i\tau$ at fixed spatial point $\mathbf{x} = 0$. $G(\tau)$ is the analog of the 'correlator' for our quantum mechanical system.

(b) From the representation $G(\tau) = \langle \mathbf{0}, -i\tau | \mathbf{0}, 0 \rangle$, use completeness to express $G(\tau)$ in terms of the S-wave radial wavefunctions $\{R_n(0)\}$ evaluated at the origin and the energy eigenvalues $\{E_n\}$. What values of n contribute? This representation is the analog of the dispersion relation expression for a correlator in which one takes into account an infinity of resonances.

https://doi.org/10.1017/9781009291033.012 Published online by Cambridge University Press

(c) Plot the negative logarithmic derivative $-d[\ln G(\tau)]/d\tau$ for the range $0 \leq \omega\tau \leq 5$ and interpret the large $\omega\tau$ behavior in terms of your result in part (b).

(d) Obtain the first three terms in a power series for $-d[\ln G(\tau)]/d\tau$, expanded about $\tau = 0$. This is the analog of the series of operator-product 'power corrections' to $-d[\ln G(\tau)]/d\tau$. Assume, as is the case in *QCD*, that you know only a limited number of terms in this series, first two terms and then four terms. Is there a common range of $\omega\tau$ for which (i) your truncated series reasonably approximates the exact behavior, and (ii) the approximation for keeping just the lowest bound state in part (b) is likewise reasonable? It is this compromise between competing demands of the resonance and operator-product approximations which must be satisfied in sucessfully applying the *QCD* sum rules to physical systems.