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AUTOMORPHISMS OF KLEIN SURFACES WITH FIXED POINTS

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1. Introduction. It was proved by Harvey [**8**] that the order $\#(\varphi)$ of an automorphism φ of a compact Riemann surface of genus $g \geq 2$ is not bigger than $4g+2$. This bound is sharp for all values of g, and it follows from the proof that if φ attains this bound, it fixes exactly one point. After that, many authors contributed to the study of the relationship between the order and the number of fixed points of an automorphism, and we should mention here the papers of Macbeath [**12**], Moore [**15**] and Szemberg [**18**]. The latter, who also studied these questions for automorphisms of domains in the complex plane, proved that if φ has at least two fixed points, then Harvey's bound can be strengthened to $4g$, and again this bound is attained for all $g \geq 2$.

The starting point of this paper is the following result from [**7**, p. 245], which admits an elementary proof.

Theorem 1.1. *Let φ be an analytic automorphism of a compact Riemann surface of genus $g \geq 2$, having $q \geq 3$ fixed points. Then $\#(\varphi) \leqslant 2g/(q-2)+1$.*

This theorem follows also from the first formula on page 106 of the paper [**12**] of Macbeath. We shall prove in the second section that this bound is sharp. Again this almost follows from the result of Macbeath. We devote the third section to obtain analogous results for automorphisms of compact, bordered Klein surfaces having a finite number of fixed points. In particular all such points are interior points and so Macbeath's arguments can be applied in this more general setting.

Some results concerning the number of fixed points of automorphisms of odd order of nonorientable compact Klein surfaces with or without boundary are due to Etayo Gordejuela [**6**].

We employ combinatorial methods in the proofs, and for convenience, we refer the reader to the book [**4**], although the original results concerning normal subgroups of non-euclidean crystallographic (NEC in short) groups are due to E. Bujalance [**2**], [**3**] and J. A. Bujalance [**5**] (see also the papers [**9**] of A. H. M. Hoare and [**10**] of A. H. M. Hoare and D. Singerman). The reader is also referred to the papers [**11**], [**19**] and [**20**] for the notations and basic properties of signatures of Fuchsian and NEC groups.

2. The classical case. We prove that the bound $2g/(q-2)+1$ of Theorem 1.1 is sharp when it is an integer. Explicit constructions of X and φ in case $q=2g+2$ or $q=g+2$ are given in [**7**, pp. 246–247].

PROPOSITION 2.1. *Let $g \geq 2$ and $q \geq 3$ be integers such that $M = 2g/(q-2)+1$ is an integer. Then there exists a compact Riemann surface of genus g and an automorphism φ of X having q fixed points and order M.*

Proof. Macbeath proved in [**12**, p. 106], that the number of fixed points of the automorphism φ of the Riemann surface X uniformized by the Fuchsian surface group Γ equals the number of proper periods which are equal to the order $\#(\varphi)$ of φ in the signature of the Fuchsian group Λ for which $\langle\varphi\rangle = \Lambda/\Gamma$. So let Λ be a Fuchsian group with signature $(0; +; [M, \overset{q}{\ldots}, M]; \{-\})$, whose canonical elliptic generators of order M are denoted by $\lambda_1, \ldots, \lambda_q$. Let $\theta : \Lambda \to Z_M$ be the epimorphism onto the cyclic group $Z_M = \langle a\rangle$ induced by the assignment:

$$\theta(\lambda_i) = a^{(-1)^i} \text{ for } i = 1, \ldots, q,$$

if q is even and

$$\theta(\lambda_i) = a^{(-1)^i}, \text{for } i = 1, \ldots, q-3, \quad \theta(\lambda_{q-1}) = \theta(\lambda_{q-2}) = a, \text{ and } \theta(\lambda_q) = a^{-2},$$

if q is odd.

In either case, each $\theta(\lambda_i)$ has order M, because M is odd if q is, and this implies, by [**4**, Theorem 2.3.3], that $\Gamma = \mathrm{Ker}\theta$ is a Fuchsian surface group. Hence, if $\mathcal{H}$ denotes the open complex upper half plane, then $X = \mathcal{H}/\Gamma$ is a compact Riemann surface on which $Z_M \cong \Lambda/\Gamma$ acts as a group of automorphisms. Thus any generator φ of Λ/Γ fixes exactly q points of X, which by the Riemann-Hurwitz formula has genus g. This completes the proof.

3. The real case. Theorem 1.1 and Proposition 2.1 above concern birational automorphisms of complex (irreducible and nonsingular) algebraic curves, and we shall extend them now to real algebraic curves. One of the main differences is that while complex algebraic curves are connected, the number of connected components of a real algebraic curve of algebraic genus p ranges, by Harnack's theorem, between 1 and $p+1$. As in the above proof, we will employ combinatorial methods, and so we prefer the language of compact bordered Klein surfaces. First of all, we obtain an immediate consequence of Theorem 1.1.

PROPOSITION 3.1. *Let X be a compact bordered Klein surface of algebraic genus $p \geq 2$, and let φ be an automorphism of X with a finite number $q \geq 2$ of fixed points. Then $\#(\varphi) \leq p/(q-1)+1$.*

Proof. As was observed in the introduction, the fixed points of φ are in the interior of X. Hence the canonical double cover Y of X is a compact Riemann surface of genus p and from [**1**, Proposition 1.6.2] and the construction of the double cover, it follows immediately that φ can be lifted to an automorphism ψ of Y with $r \geq 2q$ fixed points. Hence by Theorem 1.1,

$$\#(\varphi) \leq \#(\psi) \leq 2p/(r-2)+1 \leq 2p/(2q-2)+1 = p/(q-1)+1.$$

As in the preceding section, we will now study under what condition this bound is sharp. Assume that this is so. Then, of course, $N = p/(q-1)+1$ is an integer, and it was proved by Preston [**16**] (see also [**4**, Theorem 1.2.3]) that X can be uniformized as $X = \mathcal{H}/\Gamma$, where Γ is a non-euclidean crystallographic group. Then, since the cyclic group Z_N generated by φ acts as an automorphism group on X, there exist a certain NEC group Λ_0 and a group epimorphism $\theta : \Lambda_0 \to Z_N$ whose kernel is Γ, as can be seen in [**14**]. We can determine the signature σ_0 of Λ_0. In fact, if it is

$$(g'; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{l1}, \ldots, n_{ls_l})\}),$$

the arguments used in the quoted paper of Macbeath show that $m_1 = \ldots = m_q = N$ because φ has q interior fixed points. Then the hyperbolic area of Λ_0 is $\mu(\Lambda_0) = 2\pi[\delta + q(1 - 1/N)]$, where

$$\delta = \alpha g' + l - 2 + \sum_{i=q+1}^{r}(1 - 1/m_i) + \frac{1}{2}\sum_{i=1}^{l}\sum_{j=1}^{s_i}(1 - 1/n_{ij})$$

and $\alpha = 2$ if the sign of σ_0 is $+$ and $\alpha = 1$ otherwise: see [**17**] and [**4**, Theorem 0.2.8].

By the Riemann-Hurwitz formula,

$$2\pi(p-1) = \mu(\Gamma) = N\mu(\Lambda_0) = 2\pi N(\delta + q(1 - (1/N))).$$

Then, since $N = p/(q+1) + 1$, one gets $\delta = -1$. But, as X is bordered, we must have $l \geq 1$ and so $l = 1$, $g' = 0$, $q = r$ and $s_1 = 0$, i.e.,

$$\sigma_0 = (0; +; [N, \overset{q}{\ldots}, N]; \{(-)\}).$$

The group Λ_0 is generated by a complete elliptic system $\{\lambda_1, \ldots, \lambda_q\}$ of elements of order N and a reflection c, which must belong to Γ since X is bordered. Hence, it follows from [**4**, Theorem 2.1.3. and Corollary 3.2.3] that X is orientable and φ preserves its orientation. Moreover, by [**4**, Theorems 2.3.1 and 2.3.2] the number k of boundary components of X is $k = N/\#(\theta(e))$, where $e = (\lambda_1 \ldots \lambda_q)^{-1}$, and in particular k divides N. Also, since Γ is a surface group, each $\theta(\lambda_i)$ has order N. Consequently, there exist a generator a of Z_N and integers l_i, $i = 1, \ldots, q$ such that $(l_i, N) = 1$ and $\theta(\lambda_i) = a^{l_i}$, $1 \leq i \leq q$; $\theta(c) = 1$; $\theta(e) = a^{-k}$. Then $\sum_{i=1}^{q} l_i \equiv k \pmod{N}$. In particular, $q \equiv k \pmod 2$ if N is even. We summarize the discussion above as follows.

Proposition 3.2. *Suppose that the bound $N = p/(q-1) + 1$ of Proposition 3.1 is attained. Then the corresponding surface X is orientable, the automorphism φ preserves the orientation of X, the number k of connected components of ∂X is a divisor of N, and $q \equiv k \pmod 2$ if N is even.*

Our next goal is to prove that these necessary conditions for attaining the bound are also sufficient. For this we state first an arithmetical lemma.

Lemma 3.3. *Let N and k be positive integers. If either N is odd or both N and k are even, there exist integers u and v such that $(u, N) = (v, N) = 1$ and $k = u + v$.*

Proof. Since the case $N = 1$ is trivial, we assume that $N > 1$ and we let $p_1, \ldots, p_r$ be the distinct prime divisors of N. There exist integers $k_1, \ldots, k_r$ such that each $k_i \not\equiv 0, k \pmod{p_i}$. This is clear if $p_i \neq 2$. If $p_i = 2$, then k is even and we choose $k_i = 1$. By the Chinese Remainder Theorem there exists an integer x such that

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$x \equiv k_i \,(\mathrm{mod}\, p_i)$ for $1 \leq i \leq r$. In particular, $x \not\equiv 0 \,(\mathrm{mod}\, p_i)$ and so x and N are coprime. Hence by Dirichlet's Theorem, the arithmetic progression $\mathcal{P} = \{x + nN : n \in \mathbb{N}\}$ contains infinitely many primes, and so we can choose a prime $p = x + nN \in \mathcal{P}$ with $p > N$. Of course, $(p, N) = 1$ and it suffices to check that $(k - p, N) = 1$. If this were not the case, then there would exist p_i dividing $k - p = k - x - nN$, and so p_i would also divide $k - x$, i.e., $k_i \equiv k \,(\mathrm{mod}\, p_i)$, a contradiction.

We are now ready to prove the following converse of Proposition 3.2.

Proposition 3.4. *Let $p \geq 2$ and $q \geq 2$ be integers such that $N = p/(q-1) + 1$ is an integer. Let k be a positive divisor of N such that $q \equiv k \,(\mathrm{mod}\, 2)$ if N is even. Then there exists an orientable compact Klein surface X of algebraic genus p whose boundary has k connected components, and an orientation-preserving automorphism φ of X of order N which fixes q points.*

Proof. Let Λ_0 be an NEC group with signature $\sigma_0 = (0; +; [N, \overset{q}{\ldots}, N]; \{(-)\})$. Let $\{\lambda_1, \ldots, \lambda_q\}$ be a complete elliptic system of Λ_0. The preceding discussion shows that all we need is to construct an epimorphism $\theta : \Lambda_0 \to Z_N = \langle a \rangle$ such that $\theta(c) = 1$, each $\theta(\lambda_i)$ has order N and if $e = (\lambda_1 \ldots \lambda_q)^{-1}$, then $\theta(e)$ has order N/k.

If q is even then, either N is odd or both N and k are even. In either case, by the Lemma 3.3, we can write $k = u + v$ for some integers u and v which are coprime with N. We can then choose

$$\theta(\lambda_1) = a^u, \theta(\lambda_2) = a^v, \theta(\lambda_i) = a^{(-1)^i}, \text{for } \ i = 3, \ldots, q.$$

So let q be odd. Now if N is even then $k+1$ is also even since $k \equiv q \,(\mathrm{mod}\, 2)$. Thus, by the Lemma 3.3, both for even and odd N, there exist u and v such that $k + 1 = u + v$ and $(u, N) = (v, N) = 1$. In this case we define

$$\theta(\lambda_1) = a^u, \theta(\lambda_2) = a^v, \theta(\lambda_i) = a^{(-1)^i}, \text{for } \ i = 3, \ldots, q.$$

In both cases, $X = \mathcal{H}/\mathrm{Ker}\theta$ is the required surface and any generator φ of $\Lambda_0/\mathrm{Ker}\theta$ is the automorphism we are looking for.

Proposition 3.2 shows in particular that the bound $p/(q-1) + 1$ can be strengthened if X is a nonorientable surface. This is equivalent to minimizing the hyperbolic area $\mu(\Lambda)$ of those NEC groups containing a surface NEC group with sign "-" as a normal subgroup with a cyclic quotient of order R_1 we are looking for and whose signature contains q proper periods equal to R_1. A straightforward computation shows that the signature σ_1 of the group Λ_1 with minimizing area is

$$\sigma_1 = (0; +; [R_1, \overset{q}{\ldots}, R_1]; \{(2, 2)\}).$$

Let g be the topological genus and k the number of boundary components of X. The algebraic genus p of X equals $g + k - 1$, and the Riemann-Hurwitz formula applied to the groups with signatures σ_1 and

$$\tau = (g; -; [-]; \{(-), \overset{k}{\ldots}, (-)\})$$

gives us: $p - 1 = R_1(-1 + q(1 - 1/R_1) + 1/2)$ and so $R_1 = 2(p + q - 1)/(2q - 1)$. This way we have proved:

PROPOSITION 3.5. *Let X be a nonorientable, compact, bordered Klein surface of algebraic genus $p \geq 2$, and let φ be a dianalytic automorphism of X leaving $q \geq 2$ fixed points. Then $\#(\varphi) \leq 2(p+q-1)/(2q-1)$.*

We will now see that this bound $R_1 = 2(p+q-1)/(2q-1)$ is sharp if it is an integer, but in contrast with the orientable case, the number of boundary components of X is completely determined. In fact, let Λ_1 be an NEC group with signature σ_1 generated by the complete elliptic system $\{\lambda_1, \ldots, \lambda_q\}$, whose elements have order R_1, and the canonical reflections c_0, c_1, c_2. Let $\theta : \Lambda_1 \to Z_{R_1} = \langle a \rangle$ be an epimorphism whose kernel Γ has signature τ. Since X is bordered, and R_1 is even, it follows from [**4**, Theorems 2.3.2 and 2.3.3] that $\theta(c_0) = \theta(c_2) = a^{R_1/2}$ and $k = R_1/2$. As in the proof of Proposition 3.1, the existence of the epimorphism θ implies that $q \equiv k \pmod 2$, i.e., p must be odd, and in fact, repeating the proof of Proposition 3.4, it is easily seen that these conditions suffice to construct such an epimorphism θ mapping the λ_i onto elements of order R_1. Consequently we get:

PROPOSITION 3.6. *Let* $p \geq 2$, $q \geq 2$ and $k \geq 1$ be *integers such that* $R_1 = 2(p+q-1)/(2q-1)$ *is an integer. Then there exists a nonorientable compact Klein surface X of algebraic genus p with k boundary components, and a dianalytic automorphism of X of order R_1 with q fixed points if and only if p is odd and $k = R_1/2$.*

To finish, we deal with the case in which either R_1 is not an integer or $k \neq R_1/2$. Then the signatures σ_2 and σ_3 providing groups Λ_2 and Λ_3 with the minimum hyperbolic area among those containing a group with signature τ as a normal subgroup with cyclic quotient are

$$\sigma_2 = (1; -; [R_2, \overset{q}{\ldots}, R_2]; \{(-)\} \text{ and } \sigma_3 = (0; +; [R_2, \overset{q}{\ldots}, R_2]; \{(2,2,2,2)\}),$$

for a suitable value of R_2. In both cases, $\mu(\Lambda_i) = 2\pi q(1 - 1/R_2)$, and so, if $\theta : \Lambda_i \to Z_{R_2}$ is an epimorphism whose kernel Γ has signature τ, then $2\pi(p-1) = 2\pi q R_2(1 - 1/R_2)$, i.e., $R_2 = (p-1)/q + 1$. Let us study under what conditions this bound is sharp. Of course, R_2 must be an integer, and we suppose first that it is odd. Let $\{d, \lambda_1, \ldots, \lambda_q, e, c\}$ be a set of canonical generators of the group Λ_2 with signature σ_2, where d is a glide-reflection, $\lambda_1, \ldots, \lambda_q$ are elliptic elements of order R_2, c is a reflection and $\lambda_1 \ldots \lambda_q \cdot e \cdot d^2 = 1$. Let $\theta : \Lambda_2 \to Z_{R_2} = \langle a \rangle$ be the epimorphism induced by the assignment

$$\theta(c) = 1; \theta(\lambda_i) = a, 1 \leq i \leq q; \theta(e) = a^k; \theta(d) = a^u,$$

where k is an arbitrary positive divisor of R_2 and $2u + k + q \equiv 0 \pmod{R_2}$. Then $\Gamma = \mathrm{Ker}\theta$ has signature τ from [**4**, Theorems 2.1.2, 2.2.3 and 2.3.1] and so $X = \mathcal{H}/\Gamma$ is a nonorientable compact surface of algebraic genus p with k boundary components, and $Z_{R_2} = \Lambda_2/\Gamma$ acts as a group of automorphisms on X in such a way that any of its generators fixes exactly q points. If R_2 is even, a necessary and sufficient condition to produce an epimorphism with suitable kernel is that $q \equiv k \pmod 2$. In such a case we can find $u \not\equiv 0 \pmod{R_2}$ such that $2u + k + q \equiv 0 \pmod{R_2}$, and the same epimorphism above produces, by virtue of [**4**, Theorems 2.1.3, 2.2.4 and 2.3.2], the required surface X and automorphism φ. In case R_2 is even and $q \not\equiv k \pmod 2$, one could try to find an epimorphism $\Lambda_3 \to Z_{R_2}$ whose kernel has signature τ but as in the proof of Proposition 3.2, the inequality $q \not\equiv k \,(\mathrm{mod}\, 2)$ is an obstruction for the existence of such an epimorphism. We have thus proved the following:

PROPOSITION 3.7. *Let $p \geq 2$, $q \geq 2$ and $k \geq 1$ be integers and assume that either $R_1 = 2(p+q-1)/(2q-1)$ is not an integer or $k \neq R_1/2$ or p is even. If X is a non-orientable, compact, Klein surface of algebraic genus p, having k boundary components, and φ is a dianalytic automorphism of X which fixes q points, then $\#(\varphi) \leq R_2 = (p-1)/q + 1$. Moreover, this bound is sharp if and only if either R_2 is an odd integer or R_2 is even and $q \equiv k \,(\mathrm{mod}\, 2)$.*

FINAL REMARKS. (1) Throughout the paper we have considered as given the algebraic genus of our surface. But, since we have studied the values of its number k of boundary components for which an automorphism of maximal order and prescribed number of fixed points exists, we actually have control on its topological genus. For example, if the bound N of Proposition 3.1 is attained for a sphere, i.e., $g = 0$, then $p = k - 1$ and since $k \leq N$ we get $p + 1 = k \leq N = p/(q-1) + 1$, i.e., $k = N$ and $q = 2$. It follows from Proposition 3.4 that this possibility actually occurs for all values of p.

(2) In the nonorientable case, the bound R_1 of Proposition 3.5 is attained just for $k = R_1/2$, and so $g = p + (q-p)/(2q-1)$. On the other hand if the bound $R_2 = (p-1)/q + 1$ is attained, then $g \geq 3$, unless $g = k = q = 2$, $p = 3$.

(3) By a theorem of Maskit [**13**], given an open Riemann surface X of genus g and an automorphism φ of X, there exists a compact Riemann surface $\tilde{X}$ of genus g, an automorphism $\tilde{\varphi}$ of $\tilde{X}$ and a conformal embedding of X into $\tilde{X}$ such that $\tilde{\varphi}_{|X} = \varphi$. Hence, also in this context, the bound $2g/(q-2) + 1$ for the order of φ holds, and this bound is sharp as stated in Proposition 2.1.

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