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ON CHAIN CONDITIONS IN INTEGRAL DOMAINS

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ABSTRACT. The following two theorems are proved. If R is an Archimedean conducive integral domain, then R is quasilocal and $\dim(R) \leq 1$. If each overring of an integral domain R has ascending chain condition on divisorial ideals, then the integral closure of R is a Dedekind domain. Both theorems sharpen results already known in the Noetherian case. The second theorem leads to a strengthened converse of the Krull–Akizuki Theorem. We also investigate the effect of restricting the hypothesis in the second theorem to the proper overrings of R.

1. **Introduction.** The two theorems in this note generalize some known facts concerning Noetherian integral domains by replacing "Noetherian" hypotheses with appropriate chain conditions. The first of these concerns the recent result of Dobbs–Fedder [4, Corollary 2.7] that a conducive Noetherian integral domain must be local and of (Krull) dimension at most 1. (All relevant definitions will be recalled below, as needed.) In Theorem 2.2, this result is sharpened by replacing "Noetherian" with the weaker "Archimedean" condition.

Our second result concerns the converse of the Krull–Akizuki Theorem (cf. [10, Exercise 20, page 64]), in the version stating that if all overrings of an integral domain R are Noetherian then R', the integral closure of R, is a Dedekind domain. Several recent papers have shown that by imposing various weak finiteness or divisibility conditions on the overrings of an integral domain R, one can ensure that R' is at least a Prüfer domain. (See [1, Theorem 2.1] and Proposition 3.1 for some results in this direction.) Theorem 3.4 establishes that R' must in fact still be a Dedekind domain if one assumes only that the overrings of R each satisfy the ascending chain condition on divisorial ideals. One consequence, Proposition 3.7(b), is a further sharpening of the converse of the Krull–Akizuki Theorem. The concept of complete integral closure plays a central role in the proofs of both our theorems, producing results (Propositions

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2.1 and 3.3) of some independent interest. Section 3 also sharpens some classification results in [11] and [1]; and treats related classes of integral domains, including i.a., those for which the ascending chain condition on divisorial ideals holds in all proper overrings.

Throughout, R will denote an integral domain, with integral closure R', complete integral closure R^*, and quotient field K. Any unexplained material may be found in [1], [6], [10].

2. **Archimedean conducive domains.** Following [14], we say that an integral domain R is *Archimedean* in case $\bigcap Rr^n = 0$ for each nonunit $r \in R$. The most natural examples of Archimedean integral domains are arbitrary completely integrally closed integral domains; arbitrary one-dimensional integral domains (by [12, Corollary 1.4]); and, as a consequence of the Krull Intersection Theorem, arbitrary Noetherian integral domains. We next give a helpful characterization of such rings. As usual, $U(A)$ will denote the group of units of a ring A.

PROPOSITION 2.1. *The following conditions on R are equivalent:*
(1) *$U(S) \cap R = U(R)$ for each overring S of R which is contained in R^*;*
(2) *$U(R^*) \cap R = U(R)$;*
(3) *R is Archimedean.*

Proof. (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): Let r, d be nonzero elements of R such that $d \in \bigcap Rr^n$. As $d(r^{-1})^n \in R$ for each $n \geq 0$, r^{-1} is almost integral over R; that is, $r^{-1} \in R^*$. Then $r \in U(R^*) \cap R$; so, by (2), $r \in U(R)$, as desired.

(3) $\Rightarrow$ (1): One inclusion is trivial. For the other inclusion, consider any element $r \in U(S) \cap R$. As $S \subset R^*$, r^{-1} is almost integral over R; that is, there exists a nonzero element $d \in R$ such that $d(r^{-1})^n \in R$ for each $n \geq 0$. Since $d \in \bigcap Rr^n$, (3) implies $r \in U(R)$, completing the proof.

As in [4], we shall say that R is *conducive* if, for each overring T of R other than K, the conductor $(R:T) = \{u \in K : uT \subset R\}$ is nonzero. Familiar examples of conducive integral domains are arbitrary valuation domains and arbitrary $D+M$ constructions. In [4, Corollary 2.7], it was shown that a Noetherian conducive integral domain must be local and of dimension at most 1. The proof used the main theorem in [4, section 2], the principal ideal theorem, and a result of Chevalley, itself a consequence of the Cohen structure theory for complete local rings. An alternative proof, sketched in [4], appealed to the main theorem in [4, section 2] and the Mori–Nagata Theorem. We next generalize [4, Corollary 2.7] with a result whose elementary proof avoids any appeal to deep facts about Noetherian rings. (For another instance where "Archimedean" reduces to "dimension at most 1", see [3, Proposition 3.5].)

THEOREM 2.2. *Let R be conducive. Then R is Archimedean if and only if R is quasilocal and* $\dim(R) \leq 1$.

Proof. By the above-cited result from [12], we need only prove the "only if" assertion. Assume R is Archimedean. It is enough to show that if P is a nonzero prime ideal of R, then $R \backslash P = U(R)$. Note first that $(R : R_P) \neq 0$, since R is conducive. Thus, by [6, Lemma 26.5], $R_P \subset R^*$. As R is Archimedean, Proposition 2.1 yields $U(R_P) \cap R = U(R)$. Since $U(R_P) \cap R = R \backslash P$, the proof is complete.

REMARK 2.3. It is interesting to note that Theorem 2.2 admits a proof in the spirit of that given for [4, Corollary 2.7]. Indeed, it is enough to show that $M \subset N$ for any nonzero prime ideals M, N of R. If this fails, choose $r \in M \backslash N$ and consider $I = (R : R[r^{-1}])$. Since R is (simply) conducive, it follows that $I \neq 0$. As $I = IR[r^{-1}]$, we see that $I \subset rI$, whence $I \subset \bigcap Ir^n$. Since R is Archimedean and r is a nonunit, $I = 0$, the desired contradiction, completing the proof.

In view of the pullback characterization of seminormal conducive domains [4, Proposition 2.12(i)], Theorem 2.2 leads easily to a result which simultaneously generalizes [4, Corollaries 2.6 and 2.9]. We leave its formulation to the reader.

3. **Mori overrings.** As usual, we shall take a *Mori domain* to be an integral domain satisfying the ascending chain condition on divisorial ideals. We shall be interested in knowing what happens if each overring of R is a Mori domain. Initially, one can assume less than "Mori", just the ascending chain condition on principal ideals, in short, accp. Of course, the accp condition is still strong enough to have useful consequences: Archimedean, for instance.

Mott–Gilmer [11] characterized the integral domains whose proper overrings are all Noetherian; and Anderson–Anderson–Dobbs–Houston [1] classified the integral domains whose proper overrings are Krull domains. We next give an analogous Archimedean-theoretic classification result, and follow it with a contribution to the accp-theoretic situation.

PROPOSITION 3.1. (a) *Each overring of R is Archimedean if and only if R' is a Prüfer domain and* $\dim(R) \leq 1$.

(b) *Each proper overring of R is Archimedean if and only if one of the following two conditions holds*:

(i) *R' is a Prüfer domain and* $\dim(R) \leq 1$;

(ii) *R is a valuation domain and* $\dim(R) = 2$.

Proof. (a) A valuation domain V is Archimedean if and only if $\dim(V) \leq 1$ (cf. [3, Remark 3.2]). If each overring of R is Archimedean then $\dim_v(R)$, the valuative dimension of R, is therefore at most 1, and [9, Corollaire 3, page 61]

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(cf. also [7, Theorem 6]) then implies that R' is a Prüfer domain (and $\dim(R) \leq 1$).

Conversely, if R' is a Prüfer domain and $\dim(R) \leq 1$, then [6, page 363, lines 2–4] gives (the second equation in)

$$\dim_v(R) = \dim_v(R') = \dim(R') = \dim(R) \leq 1,$$

so that [12, Corollary 1.4] implies that each overring of R is Archimedean.

(b) If R is a valuation domain, then the proper (valuation) overrings of R are the rings R_P, P ranging over the nonmaximal primes of R [6, Theorem 26.1(1)]. Therefore, (b) follows from (a), to complete the proof.

The reader will have noticed from the above proof that the Archimedean condition in Proposition 3.1(a) (resp., (b)) need only be imposed on the valuation (resp., proper valuation) overrings of R. Such a statement is given for the next result, for which we give a proof independent of Proposition 3.1.

LEMMA 3.2. *Assume that each proper valuation overring of R satisfies accp. Then R' is a Prüfer domain. In fact, (at least) one of the following three conditions holds:*

(i) *R is a valuation domain and* $\dim(R) \leq 1$;

(ii) *R is a valuation domain,* $\dim(R) = 2$, *and R_P is a DVR, where P denotes the height* 1 *prime ideal of R*;

(iii) *R' is an almost Dedekind domain.*

Note that (i), (ii) each imply that the proper overrings of R are all Noetherian. A classification result in the spirit of [11], [1] would require deeper analysis of case (iii).

Proof of Lemma 3.2. A valuation domain V satisfies accp if and only if V is either a DVR or a field. So, if R is not a valuation domain, [6, Theorem 36.2] yields (iii). Without loss of generality, R can thus be assumed a valuation domain. Since the proper (valuation) overrings of R are the rings R_P, P ranging over the nonmaximal primes of R [6, Theorem 26.1(1)], it is easy to see then that either (i) or (ii) holds.

PROPOSITION 3.3. *Assume that each proper simple overring of R satisfies accp. Then:*

(a) *Either R is integrally closed or $R^* = R'$.*

(b) *Assume, in addition, that R satisfies accp. (For instance, assume that R is not quasilocal.) Then $R^* = R'$.*

Proof. (a) Suppose that R is not integrally closed. Of course, $R' \subset R^*$. If the reverse inclusion fails, consider $s \in R^* \backslash R'$, and set $T = R[s^{-1}]$. We claim that T is a proper overring of R. Otherwise, select $u \in R' \backslash R$, set $S = R[u]$, observe that S is Archimedean (since it satisfies accp by hypothesis), and conclude via Proposition 2.1 that $s^{-1} \in U(S^*) \cap S = U(S)$, whence $s \in R'$, a contradiction.

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Note next that $s \notin R'$ forces $s \notin T$ (cf. [10, Theorem 15]). Moreover, since $s \in R^*$, there exists a nonzero element $d \in R$ such that $ds^n \in R$ for each $n \geq 0$. Then $\{T ds^n\}$ forms a strictly ascending chain of principal ideals of T, contrary to hypothesis. Thus, as desired, no such s exists.

(b) If R satisfies accp, one can argue as in (a), ignoring u and S, since it is immaterial for present purposes whether the overring T is proper.

As for the parenthetical assertion, if R is not quasilocal, there exist nonunits f, g in R such that $(f, g) = R$. Then $R = R[f^{-1}] \cap R[g^{-1}]$ (cf. [1, Lemma 2.10]), an intersection of two integral domains each satisfying accp. Hence R also satisfies accp, and the proof is complete.

To place the next result in perspective, recall that the integral closure of a Mori domain need not even be a Krull domain [2, Examples 3.8(b)].

THEOREM 3.4. *If each overring of R is a Mori domain, then R' is a Dedekind domain and* $\dim(R) \leq 1$.

Proof. Since R' inherits the hypotheses from R, applying Proposition 3.3 to R' reveals that R' is completely integrally closed. As R' is a Mori domain by hypothesis, R' is therefore a Krull domain. However, by Proposition 3.1 or Lemma 3.2, R' is also a Prüfer domain. An application of [6, Theorem 43.16] now establishes the first assertion, and the second follows by integrality (cf. [10, Theorem 48]).

Since each overring of a Dedekind domain is also a Dedekind domain, hence Noetherian and integrally closed, we immediately infer

COROLLARY 3.5. *R is a Dedekind domain if and only if R is integrally closed and each overring of R is a Mori domain.*

We recall the next useful result of Querré, as a companion for Corollary 3.5. In conjunction with Lemma 3.2, it can lead to an alternate proof of Theorem 3.4 which avoids mentioning R^*.

PROPOSITION 3.6 (cf. [13, Corollaire 2]). *R is a Dedekind domain if and only if R is both a Prüfer domain and a Mori domain.*

We next give a strengthened version of the converse of the Krull–Akizuki Theorem. Recall first that an integral domain S is called *coherent* in case $I \cap J$ is a finitely generated ideal of S for each choice of finitely generated ideals I, J of S. Natural examples of coherent integral domains include arbitrary Noetherian domains and arbitrary Prüfer domains.

PROPOSITION 3.7. (a) *If R is a coherent Mori domain, then each divisorial ideal of R is finitely generated.*

(b) *Each overring of R is a coherent Mori domain if and only if R is Noetherian and* $\dim(R) \leq 1$.

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Proof. (a) Let I be a divisorial ideal of R. Since R is a Mori domain, $(R:I)=(R:J)$ for some finitely generated ideal J of R which is contained in I [13, Théorème 1]. Write $J=\sum Ra_i$ for some finite generating set, $\{a_i\}$. Since R is coherent, $(R:J)=\bigcap(R:a_i)$ is a finitely generated R-submodule of K, hence of the form $b^{-1}D$ for some $b\in R$ and finitely generated ideal D of R. Then

$$I=(R:(R:I))=(R:b^{-1}D)=b(R:D)\cong(R:D)$$

is indeed finitely generated.

(b) We need only tend to the "only if" half. As each overring of R is a Mori domain, Theorem 3.4 (or Proposition 3.1(a)) assures that $\dim(R)=\dim(R')\leq 1$. By the criterion of Cohen (cf. [10, Theorem 8]), it is therefore enough to prove that each height 1 prime ideal P of R is finitely generated. However, since R is a Mori domain, [13, Proposition 1] shows that any such P is divisorial, and so an appeal to (a) completes the proof.

A principal consequence of [11] is the following classification result. Each integral domain whose proper overrings are all Noetherian must be (at least) one of the following three types: Noetherian of dimension at most one, valuation domain of dimension one, or two-dimensional valuation domain with a DVR overring. By Proposition 3.7(b), this result can be sharpened, for we now see that the same catalogue classifies the integral domains whose proper overrings are all coherent Mori domains.

It seems worthwhile to indicate next that the hypotheses of Theorem 3.4 accommodate more rings R than the union of the earlier-mentioned catalogues in [11] and [1]. First, recall from [8] that a *pseudo-valuation domain* is a (necessarily quasilocal) integral domain which has a (uniquely determined) valuation overring with the same set of prime ideals.

Remark 3.8. (a) Let F/k be an infinite-dimensional algebraic field extension and let $V=F+M$ be a DVR with maximal ideal M. (For instance, $V=F[[X]]$, $M=XV$.) Set $S=k+M$. Then each overring of S is a Mori domain. Moreover, if $u\in F\backslash k$, then $T=k(u)+M$ is an overring of S which is neither Noetherian nor a Krull domain.

Being Noetherian, V and the quotient field of S are certainly Mori domains. The only other overrings of S take the form $A=L+M$, where L ranges over the fields intermediate between k and F. As such a ring A is a pseudo-valuation domain, [8, Corollary 2.15] shows that the nonprincipal divisorial ideals of A are just the nonzero ideals of its canonically associated valuation overring V. It is easy to show that A satisfies accp: cf. [3, page 373, lines 5–8]. Thus A is a Mori domain. (Another way to see this is by appeal to [2, Theorem 3.2]: A inherits the "Mori" property from V since $\mathrm{Spec}(A)=\mathrm{Spec}(V)$ as sets.) Finally, T is not Noetherian since $[F:k(u)]=\infty$ (cf. [6, Exercise 8(3), page 270]); and T is not a Krull domain since $T'=V\neq T$.

(b) As surveyed in [5], arbitrary pseudo-valuation domains admit a tractable, though not necessarily $D+M$, pull back structure. One may therefore adapt the argument in (a), to prove the following generalization. Let S be a pseudo-valuation domain, with maximal ideal M and canonically associated valuation domain V. Assume that V/M is an infinite-dimensional algebraic extension of S/M. Then each overring of S is a Mori domain, and S has a proper simple overring which is neither Noetherian nor a Krull domain.

COROLLARY 3.9. *If each proper overring of R is a Mori domain, then* (*at least*) *one of the following three conditions holds:*

(i) *R is a valuation domain and* $\dim(R)\leq 1$;

(ii) *R is a valuation domain,* $\dim(R)=2$, *and R_P is a DVR, where P denotes the height* 1 *prime ideal of R*;

(iii) *R' is a Dedekind domain.*

Proof. Suppose first that R is not integrally closed. Then R' is a Mori domain by hypothesis and, by Proposition 3.1 or Lemma 3.2, R' is also a Prüfer domain. An application of Proposition 3.6 yields (iii). (An alternate proof for this case proceeds by applying Theorem 3.4 to R'.)

Thus we may assume that R is integrally closed and, by Lemma 3.2, actually an almost Dedekind domain. Let T be any proper overring of R other than K. By [6, Corollary 36.3], T is also an almost Dedekind domain and, *a fortiori*, integrally closed. Then by applying Theorem 3.4 to T, we conclude that T is a Dedekind domain. In other words, each proper overring of R is a Dedekind domain. Mott–Gilmer have also catalogued this situation, and an application of their result [11, Theorem 2] completes the proof.

Let P be one of the following six properties: Krull domain, Dedekind domain, PID, UFD, GCD-domain, Bézout domain. In [1, Theorem 3.4, Proposition 3.5, Corollary 3.6], it was proved that if each proper simple overring of R has P, then so does each proper overring of R. We next show that the property considered in Corollary 3.9 behaves differently.

REMARK 3.10. There exists an integral domain S such that each proper simple overring of S is a (coherent) Mori domain but not every proper overring of S is a Mori domain.

To see this, it is enough to let S be any Noetherian UFD of dimension $n\geq 2$, for instance the regular local ring $k[[X_1,\ldots,X_n]]$. The assertion regarding proper simple overrings follows from Hilbert Basis Theorem. Moreover, Corollary 3.9 shows that not all proper overrings of S are Mori domains, since $S=S'$ is not a Prüfer domain.

Our final result sharpens some work done in [1]. It is a classification result, in the spirit of [11] and [1]. Recall first that a proper overring T of R is said to be *the unique minimal overring of* R in case T is contained in each proper overring of R.

PROPOSITION 3.11. *The following four conditions on* R *are equivalent:*
(1) *Each proper simple overring of* R *is an integrally closed Mori domain;*
(2) *Each proper overring of* R *is an integrally closed Mori domain;*
(3) *Each proper overring of* R *is a Dedekind domain;*
(4) *At least one of the following four conditions holds:*
(i) R *is a valuation domain and* $\dim(R) \leq 1$;
(ii) R *is a valuation domain,* $\dim(R) = 2$, *and* R_P *is a DVR where* P *denotes the height* 1 *prime ideal of* R;
(iii) R *is a Dedekind domain;*
(iv) R *is quasilocal and* R' *is a Dedekind domain which is the unique minimal overring of* R. (*Moreover,* R *is a one-dimensional Noetherian integral domain and* R' *is a PID*).

Proof. The reader may easily verify that $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$.

For the parenthetical assertion in (4, iv), note that R inherits the Noetherian property from R', by virtue of Eakin's Theorem; and R' is a semilocal [10, Exercise 21, page 64] Dedekind domain, hence a PID.

It remains to show $(1) \Rightarrow (4)$. Assume (1). Since each proper simple overring of R is integrally closed, [1, Corollary 3.2] shows that each proper overring of R is integrally closed and R' is a Prüfer domain. By Proposition 3.6, each proper simple overring of R (must contain R' and so) is a Dedekind domain. We easily infer (3). The proof may now be completed in either of two ways. The first is by appeal to [1, Proposition 2.11, Theorems 2.12 and 3.4]. Alternatively, use Corollary 3.9, noting in case $R \neq R'$ that R is quasilocal, lest R be the intersection of integrally closed (localization) overrings.

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