Calculation of π with a needle

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Introduction

The number π is perhaps the most famous irrational number. This constant is equal to the ratio of the circumference of a circle to its diameter. One of the most well-known mathematical problems of antiquity, which is related to π, is how to construct by using a ruler and compasses a square which has the same area as a circle. This particular problem cannot be solved, due to the fact that π is a transcendental number, which means that it cannot be obtained as the root of a polynomial equation with rational coefficients. It was Euler in the 18th century who established the notation π.

It is noteworthy that π is related to a famous problem in probability theory, the Buffon needle. In particular, suppose that a needle is thrown on a plane which has parallel lines drawn at equal distance from each other. The probability p of the needle intersecting one parallel line is provided by an expression which involves π, the length of the needle and the distance of the parallel lines [1]. If p were known, it would be possible to solve for π.

The probability p can be approximated by conducting the experiment of throwing the needle many times and calculating the frequency of hits of the needle on any parallel line. There is however, an alternative way to proceed: the random experiment of the throw of the needle can be mathematically simulated. Whether or not the simulated needle intersects a parallel line is recorded. By running a large number of simulations through a computer program such as *Excel*, p, and hence π, can be fairly well approximated.

A short history of π

A large number of mathematicians have tried to calculate π, as described in [2]. In ancient Egypt, the Rhind papyrus, which dates from 1650 BC, stated that if we take 8/9 of the diameter of a circle and form a square, then the area of the square is equal to the area of the circle. This is equivalent to taking π to be $4 \times \frac{8}{9}^2 \approx 3.16049$. In the Old Testament (I Kings) the value of π is considered to be approximately 3. The first systematic approximation of π dates to the ancient Greek mathematician Archimedes. Archimedes in his study *Measurement of Circle* (Κὺκλου Μὲτρησις) used circumscribed and inscribed regular polygons (96-polygons) in order to approximate the area of a circle [3]. Based on this approach, he calculated π to be between $3\frac{10}{71}$ and $3\frac{1}{7}$. In China at the end of 5th century AD, Zu Chongzhi showed that π is between the numbers 3.1415926 and 3.1415927. The Indian mathematician Aryabhata provided the approximate value $\frac{62\,832}{20\,000}$, which yields 3.1416.

In continuation, during the 16 th century in France, Viette employed

https://doi.org/10.1017/mag.2019.13 Published online by Cambridge University Press

regular polygons with 392 216 sides in order to show that

$$3.1415926535 < \pi < 3.1415926537.$$

In 1751 the Swiss mathematician Lambert showed that π is irrational and, sometime later, in 1759, Legendre provided another stricter proof of this fact. During the 18 th century, a number of mathematicians, such as the Englishman Machin, were able to express π as an expansion of an infinite series based on trigonometric functions. In 1882, Lindemann proved that π is transcendental. In our time, using computers, π has been calculated with great precision. For example, in 1999, Kanada and Takahashi calculated the first 68 719 470 000 digits of π. Unfortunately no rule exists which fully describes the digits of π, as they appear not to follow any particular pattern.

The problem of Buffon's needle

Georges Louis Leclere, Count of Buffon (1707-1788), posed a problem [4], which is the first application of probability theory in a geometric setting, and can be described as follows: On a table, we draw parallel lines at equal distances c among them. Then we randomly throw on the table a needle of length l (less than c) and seek to find the probability p that the needle will touch any of the parallel lines. This probability is found to be $\frac{2l}{\pi c}$. The derivation can be found in [1].

Mathematical simulation of the throw of the needle

We will attempt an approximation of π by throwing a needle virtually. We consider an orthogonal Cartesian system on which we draw a square taking x from 0 to 1 and y from 0 to 1. This square represents the surface of the table on which the needle is thrown. The parallel lines are drawn perpendicularly to the x-axis at distance $c = 0.1$. Figure 1 shows how the needle could appear after being thrown.

The length of the needle is 0.08. For the simulation, we randomly take a point (x_1, y_1) on the plane to be the location of the sharp end of the needle. To determine the orientation of the needle, we randomly generate the angle θ of the needle with respect to the x-axis. Figure 2 depicts the location of the needle.

Approximation of π

In *Excel*, the coordinates x_1 and y_1 are defined by the function = RAND(). To determine the line of the needle completely, the angle θ of the needle with the x-axis is generated (in degrees) by the formula = 360* RAND(). The z-coordinate of the blunt end of the needle is given by $x_2 = x_1 + l \cos\theta$.

https://doi.org/10.1017/mag.2019.13 Published online by Cambridge University Press

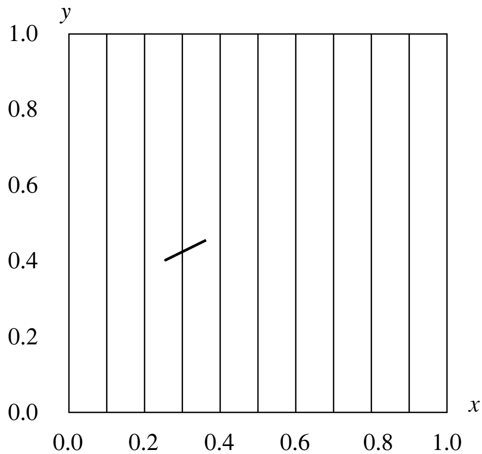

FIGURE 1: The needle thrown on the square

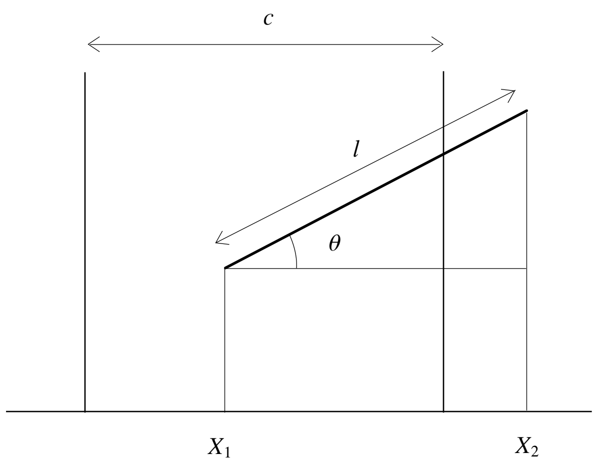

FIGURE 2: Determination of the location of the needle

We now divide x_1 and x_2 by the distance between the parallel lines c. In the case that x_1/c or x_2/c is integer, one end of the needle is exactly located on a parallel line. The integer part of x_1/c, denoted by $[x_1/c]$ provides the rank number of the parallel line at the left of x_1. Similarly, $[x_2/c]$ corresponds to the rank number of the parallel line to the left of x_2. Note that, if x_2 is negative, $[x_2/c] = -1$.

We define $D = [x_1/c] - [x_2/c]$. When $|D| = 1$, there is a parallel line lying between x_1 and x_2 and as a result, the needle intersects this particular parallel line. Similarly, when $D = 0$, no internal point of the needle intersects any parallel line.

Finally, we need to account for the event that the needle intersects a parallel line. We define the binary variable X, which takes the value of 1

https://doi.org/10.1017/mag.2019.13 Published online by Cambridge University Press

when the needle intersects a parallel line, and 0 otherwise. We observe that $X = 1$ in the following cases:

(a) An internal point of the needle intersects a parallel line, which holds when $|D| = 1$;

(b) Only the sharp end of the needle is on a parallel line, so that x_1 / c is an integer but x_2 / c is not an integer;

(c) Only the non-pointed end of the needle is on a parallel line, so that x_2 / c is an integer but x_1 / c is not an integer.

(d) The complete needle is on a parallel line, in which case both x_1 / c and x_2 / c are equal integers.

It is noteworthy that since the length of the needle is less than the distance between the parallel lines, only one of the above alternatives is possible.

Simulation results

In our simulation, 10 cycles of 3 000 and of 10 000 iterations of needle throws were generated, respectively. The probability p that the needle intersects a parallel line can be approximated by the sum of the X variable over all the iterations divided by the total number of iterations of each cycle. Since $p = \dfrac{2l}{\pi c}$, the value of π is given by $\dfrac{2l}{\pi c}$. In our study, $l = 0.08$ and $c = 0.1$. For the exact value of π the theoretical probability p is almost 50.9296%. The average value of π for the 10 cycles was also calculated. The results of the simulation are summarised in Table 1.

	Cycles of 3 000 iterations		Cycles of 10 000 iterations	
For $l = 0.08$ and $c = 0.1$	Estimated probability p	π	Estimated probability p	π
1st cycle	51.8%	3.0888	50.12%	3.1923
2nd cycle	50.6%	3.1621	51.23%	3.1232
3rd cycle	50.067%	3.1957	51.81%	3.0882
4th cycle	49.967%	3.2021	50.45%	3.1715
5th cycle	50.967%	3.1393	51.11%	3.1305
6th cycle	50.233%	3.1851	50.39%	3.1752
7th cycle	50.9%	3.1434	50.65%	3.1589
8th cycle	50.133%	3.1915	51.22%	3.1238
9th cycle	49.867%	3.2086	51.43%	3.1110
10th cycle	52.133%	3.0691	50.38%	3.1759
Average of 10 cycles	50.667%	3.1586	50.879%	3.1451

TABLE 1: Results of the simulation for the calculation of π in 10 simulation cycles

Evaluation of the accuracy of the approximation

Even though the number of iterations in our simulation was relatively large (3 000 and 10 000 iterations respectively), considerable variability was present in the approximation of the probability p and consequently in the calculation of π. By increasing the number of iterations from 3 000 to 10 000, it is expected that the variability will be reduced. Indeed, each iteration was produced by generating random numbers independently from the other iterations. As a result, for each cycle, our estimated probability p is the average of independent realisations of the X variable. Therefore, the variability of p should be inversely proportional to the square root of the number of iterations. In particular, since $\sqrt{\frac{10\,000}{3\,000}} \approx 1.826$, cycles of 3 000 iterations are expected to present 82.6% more variability in the estimation of p compared to cycles of 10 000 iterations.

From Table 1 it is evident, that the variability of the approximation of π was reduced when the number of iterations increased to 10 000, although the second decimal digit of π continued to be fluctuating.

The average of the findings of the 10 cycles, with 3000 repetitions each, provides the approximate value of 3.1586 while the average of the cycles, with 10 000 iterations each, was 3.1451. This should be compared against the theoretical value 3.1415926. Therefore, together the 10 cycles of simulation allow accuracy in the second decimal digit of π. In general, we conclude that the random throw of the needle exhibits great variability. Thus the accuracy in the approximation of π can be enhanced only by conducting a large number of simulated or actual needle throws.

Comparison with other approximations

The first person to calculate π using Buffon's method was the Swiss mathematician Wolf in 1850, who ran 5000 throws of a needle of length 36 mm with parallel lines at 45 mm distance from each other. In this way, the probability p was found to be approximately 50.64%, which yields the value of 3.1596 to π. Similar experiments were conducted by Ambrose Smith in 1855 with 3204 throws deriving the value of 3.1553 to π [5]. In 1864, Captain O. C. Fox, while recovering from a wound, experimented with different needles and distances of parallel lines [6]. After performing 590 tosses, he was able to approximate π by the value 3.1416.

Our calculation of π, with 3 000 simulation iterations, is certainly comparable with the results of Smith as the number of throws is similar and the precision in the calculation of π does not extend further than the second decimal digit. Likewise, our findings are similar to the approximation of Wolf. The approximation of Fox, on the other hand, raises a number of questions for its incredible precision as it is based on 590 throws and achieves a precision up to the third decimal digit. Given the high variability which is observed from the simulation study, we conclude that Fox was particularly lucky in the experiment that he conducted.

Conclusion

We developed a computer simulation of the experiment of the throw of Buffon's needle problem. The probability that the needle touches one of the parallel lines drawn on the plane was calculated, providing an approximation to π. Our findings are comparable with previous approximations of π, conducted by various researchers. We established that the experiment of tossing the needle presents considerable variability in the measurement of π.

It would be of interest to develop a similar simulation for the extension to the problem of Buffon needle given by Laplace, with equal orthogonal rectangles rather than parallel lines drawn on the plane. The probability that the needle intersects the sides of any of these rectangles, as calculated by Arrow [7], depends on π. Therefore a similar simulation can be developed to approximate π.

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10.1017/mag.2019.13

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