Proceedings of the Royal Society of Edinburgh, **154**, 600–628, *2024*
DOI:10.1017/prm.2023.25

A note on the differentiability of discrete Palmer's linearization

Álvaro Castañeda and Néstor Jara
Departamento de Matemáticas, Universidad de Chile, Casilla 653, Santiago, Chile (castaneda@uchile.cl, nestor.jara@ug.uchile.cl)

(Received 11 July 2022; accepted 20 February 2023)

In the context of discrete nonautonomous dynamics, we prove that the homeomorphisms in the linearization theorem are C^2 diffeomorphisms. In contrast to other related works, our result does not involve non-resonance conditions or spectral gaps. Our approach is based on the interlacing of the properties of nonautonomous hyperbolicity of the linear part, and boundedness and Lipschitzness of the nonlinearities. Moreover, we propose a functional approach to find conditions for regularity of arbitrary degree.

Keywords: Nonautonomous hyperbolicity; nonautonomous difference equation; smooth linearization

2020 *Mathematics Subject Classification:* Primary: 37C60; 37D25

1. Introduction

The well-known Hartman–Grobman theorem [**13**, theorem I] is an essential tool in the study of local behaviour of autonomous and nonautonomous nonlinear dynamical systems. This theorem establishes a local topological conjugacy between the continuous dynamical system given by a C^1 diffeomorphism of the space $\mathbb{R}^d$ and its linearization around an hyperbolic equilibrium, i.e. the dynamics near the equilibrium points are topologically the same. This result helps to design local phase portraits around all equilibria.

The global behaviour study begins when Pugh [**17**] studied a particular case of the Hartman–Grobman's theorem focused on linear systems with bounded and Lipschitz perturbations allowing the construction of an explicit and global homeomorphism, but without considering its smoothness properties. This work inspired Palmer [**15**] to achieve the first result of global linearization in the nonautonomous framework: the Hartman–Grobman theorem for nonautonomous differential equations. This seminal article considered vector fields whose linear component inherits, in some sense, the hyperbolicity property of the autonomous case, namely uniform exponential dichotomy property (see [**8**, **9**] for more information about this property); while the nonlinear parts of the vector fields are bounded and Lipschitzian.

Although the task of finding such a homeomorphism is delicate, to show that it has a class of differentiability is an inherently more interesting challenge in dynamics, since this allows a better understanding of the information that is carried from one system to another. In this sense, in the autonomous framework, Sternberg [**20**, **21**] proved that under certain non-resonance conditions the local conjugation between two systems can actually be a C^r $(r \geqslant 0)$ map. Later, Belickiĭ [**3**, **4**] achieved the same result under assumptions related to the spectrum of the diffeomorphism that defines the dynamics.

Nevertheless, to the best of our knowledge, there are no results about the differentiability of Palmer's linearization prior to this study [**7**], which establishes that, under some technical assumptions, a orientation preserving linearization of class C^2 exists. For a recent generalization and improvement of this result, we refer the reader to [**6**].

Noteworthy is that there is also a recent study by Cuong *et al.* [**10**], which extends the Sternberg theorem [**20**, **21**] to the nonautonomous case, i.e. a nonresonance condition related to Sacker–Sell spectrum is satisfied in order to obtain smooth local linearization; for more details on spectrum mentioned above, we refer the reader to [**18**] and [**14**, chapter 5]. Additionally, there is an interesting and novel approach given by Dragičević *et al.* in [**12**], where a nonuniform spectral bound condition is given for ensuring that the linearization of the perturbed system to be simultaneously differentiable at 0 and Hölder continuous linearization in a neighbourhood of 0; we point out that in this work the authors assume that the linear system has a nonuniform strong exponential dichotomy and the spectrum of the linear part is addressed in terms of a time-discrete evolution operator.

In [**11**] Dragičević *et al.*, in a discrete nonautonomous framework, investigate C^1 smooth linearization with a nonuniform strong exponential dichotomy in the linear part. The main idea of the authors was to convert the linear part of the nonautonomous system to an autonomous one in order to obtain a spectral gap condition for C^1 linearization of first-order nonautonomous difference equation. On the contrary, in [**2**], Barreira and Valls discussed Hölder continuous linearization with nonuniform dichotomy, however the smoothness is not considered.

Our work follows the results initiated in [**7**], and is strongly based on the study of discrete framework expressed in [**5**], since we use some of their results and strategies to find a Palmer's homeomorphism. The main difference between our work and [**5**], in terms of differentiability, is that we allow the existence of nonempty unstable manifolds for the linear system; furthermore, our analysis neither involves a change from a nonautonomous environment to an autonomous one nor the use of spectral gaps in order to obtain C^1-differentiability such as employed in [**11**]. Also our results do not include the autonomous case. Another interesting fact of our work is that differentiability is reached when in the linear part a wider family of dichotomies is considered.

Our paper is organized as follows. In § 2 we establish the nonautonomous tools in order to achieve the main results in this work. In § 3 we state a couple of results, that show some class of differentiability of the homeomorphisms, by using interlacing properties of hyperbolicity of the linear system, and boundedness and Lipschitzness of the nonlinear part. Moreover, we give examples of systems in order to achieve the conditions that allow homeomorphisms to have these classes of regularity. In

§ 4 we study a generalization of the previous conditions in order to obtain C^2 differentiability. Finally, in the last section, we propose an algebraic set of functions and an operator that can be inductively applied in order to find conditions for arbitrary high-order derivatives.

2. Preliminaries

Inspired by Palmer's work [15], we studied the following nonautonomous discrete systems:

$$x(k+1) = A(k)x(k) \tag{2.1}$$

and

$$y(k+1) = A(k)y(k) + f(k, y(k)), \tag{2.2}$$

where $A : \mathbb{Z}^+ \to \mathcal{M}_{d\times d}(\mathbb{R})$ is nonsingular (i.e. has invertible images) and uniformly bounded, i.e.

$$M := \max\left\{\sup_{k\in\mathbb{Z}^+}\left\|A(k)\right\|, \sup_{k\in\mathbb{Z}^+}\left\|A^{-1}(k)\right\|\right\} < \infty.$$

We denote $j \mapsto x(j, m, \xi)$ and $j \mapsto y(j, m, \eta)$ the solutions of (2.1) and (2.2) which, respectively, take the value ξ and η when $j = m$.

The function $f : \mathbb{Z}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ is such that there exist sequences $\mu, \gamma : \mathbb{Z}^+ \to \mathbb{R}^+$ verifying, for every $k \in \mathbb{Z}^+$ and every pair $(y, \tilde{y}) \in \mathbb{R}^d \times \mathbb{R}^d$, the following: $|f(k, y) - f(k, \tilde{y})| \leqslant \gamma(k)|y - \tilde{y}|, |f(k, y)| \leqslant \mu(k)$.

$$|f(k, y) - f(k, \tilde{y})| \leqslant \gamma(k)|y - \tilde{y}|, \quad |f(k, y)| \leqslant \mu(k).$$

2.1. Nonuniform dichotomies and Green's operator

In [11], the authors assume that the linear part has nonuniform strong exponential dichotomy on $\mathbb{Z}$. For our purpose, we endow (2.1) to a more general nonautonomous hyperbolicity, which it satisfies on $\mathbb{Z}^+$. Namely,

(d1) System (2.1) has a nonuniform dichotomy, i.e. there are two invariant complementary projectors $P(\cdot)$ and $Q(\cdot)$ such that $P(n) + Q(n) = I$ for every $n \in \mathbb{Z}^+$, a nonnegative sequence D and a strictly decreasing sequence convergent to zero h with $h(0) = 1$ such that

$$\begin{cases} \left\|\Phi(k, n)P(n)\right\| \leqslant D(n)\left(\dfrac{h(k)}{h(n)}\right), & \forall\ k \geqslant n \geqslant 0 \\ \left\|\Phi(k, n)Q(n)\right\| \leqslant D(n)\left(\dfrac{h(n)}{h(k)}\right), & \forall\ 0 \leqslant k \leqslant n, \end{cases}$$

where $\Phi(k, n)$ is the transition matrix for (2.1).

REMARK 2.1. The transition matrix for (2.1) is the matrix function $\Phi : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathcal{M}_{d\times d}(\mathbb{R})$ given by

$$\Phi(k,n) = \begin{cases} A(k-1)A(k-2)\cdots A(n) & \text{if } k > n, \\ I & \text{if } k = n, \\ A^{-1}(k)A^{-1}(k+1)\cdots A^{-1}(n-1) & \text{if } k < n. \end{cases}$$

REMARK 2.2. Projectors $P(\cdot)$ and $Q(\cdot)$ have been called invariant, which means they are under the assumption that for every $k, n \in \mathbb{Z}^+$ they satisfy

$$P(k)\Phi(k,n) = \Phi(k,n)P(n) \quad \text{and} \quad Q(k)\Phi(k,n) = \Phi(k,n)Q(n).$$

REMARK 2.3. Green's operator for system (2.1) associated with the dichotomy (d1) is the matrix function $\mathcal{G} : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathcal{M}_{d\times d}(\mathbb{R})$ given by

$$\mathcal{G}(k,n) = \begin{cases} \Phi(k,n)P(n) & \forall\ k \geqslant n \geqslant 0, \\ -\Phi(k,n)Q(n) & \forall\ 0 \leqslant k < n. \end{cases}$$

Additionally it is easily deduced that for every $n \in \mathbb{Z}^+$ and every $k \neq n-1$

$$\mathcal{G}(k+1,n) = A(k)\mathcal{G}(k,n) \quad \text{and} \quad \mathcal{G}(n,n) = I + A(n-1)\mathcal{G}(n-1,n).$$

The following properties establish a relation between Green's operator $\mathcal{G}$ associated with the dichotomy established in (d1) and the sequences μ and γ. These conditions are used in [5] in order to obtain that (2.1) and (2.2) are topologically equivalent on $\mathbb{Z}^+$.

(d2)

$$\sum_{j=0}^{\infty} \big\|\mathcal{G}(k,j+1)\big\|\, \mu(j) < \infty \quad \text{for every } k \in \mathbb{Z}^+.$$

(d3)

$$\sum_{j=0}^{\infty} \big\|\mathcal{G}(k,j+1)\big\|\, \gamma(j) \leqslant q < 1 \quad \text{for every } k \in \mathbb{Z}^+.$$

The property that we establish below has been used [5] in order to ensure that solutions for system (2.2) can be uniquely backwards continued.

(d4) The sequence γ and the matrix function A verify

$$\Big\|A^{-1}(\ell)\gamma(\ell)\Big\| < 1 \quad \text{for every } \ell \in \mathbb{Z}^+.$$

2.2. C^r-topological equivalence on $\mathbb{Z}^+$

We recall the concept of topological equivalence for discrete systems.

DEFINITION 2.4. *Systems (2.1) and (2.2) are $\mathbb{Z}^+$-topologically equivalent if there is a function $H : \mathbb{Z}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ that satisfies:*

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

(i) *If* $x(k)$ *is a solution of* (2.1), *then* $H[k, x(k)]$ *is a solution of* (2.2).

(ii) $H(k, u) - u$ *is bounded over* $\mathbb{Z}^+ \times \mathbb{R}^d$.

(iii) *For each fixed* $k \in \mathbb{Z}^+$, *the map* $u \mapsto H(k, u)$ *is a homeomorphism of* $\mathbb{R}^d$.

Moreover, the function $u \mapsto G(k, u) = H^{-1}(k, u)$ *satisfies (ii) and (iii) and maps solutions of* (2.2) *into solutions of* (2.1).

As explained before, it can be interesting to study which properties can be deduced when studying the differentiability of such homeomorphisms. This idea motivates the following definition.

DEFINITION 2.5. *Systems* (2.1) *and* (2.2) *are* C^r*-topologically equivalent on* $\mathbb{Z}^+$ *if they are topologically equivalent on* $\mathbb{Z}^+$ *and* $u \mapsto H(k, u)$ *is a diffeomorphism of class* C^r, *with* $r \geqslant 1$ *for each fixed* $k \in \mathbb{Z}^+$.

The following result establishes a topological equivalence between systems (2.1) and (2.2). We highlight it, together with its proof, because it gives us the tools that will facilitate the presentation of our main results.

PROPOSITION 2.6 [**5**, theorem 2.1]. *If conditions (d1)–(d4) hold, then systems* (2.1) *and* (2.2) *are* $\mathbb{Z}^+$*-topologically equivalent.*

To facilitate the reading of this article, we distinguish some relevant facts in the proof of theorem [**5**, theorem 2.1]. Let us define

$$
\begin{aligned}
w^*(k; (m, \eta)) &= -\sum_{j=0}^{\infty} \mathcal{G}(k, j+1) f(j, y(j, m, \eta)) \\
&= -\sum_{j=0}^{k-1} \Phi(k, j+1) P(j+1) f(j, y(j, m, \eta)) \\
&\quad + \sum_{j=k}^{\infty} \Phi(k, j+1) Q(j+1) f(j, y(j, m, \eta)).
\end{aligned}
$$

On the contrary, for each $(m, \xi) \in \mathbb{Z}^+ \times \mathbb{R}^d$ we define the map $\Theta : \ell^\infty(\mathbb{Z}^+, \mathbb{R}^d) \to \ell^\infty(\mathbb{Z}^+, \mathbb{R}^d)$ given by

$$(\Theta\phi)(k; (m, \xi)) = \sum_{j=0}^{\infty} \mathcal{G}(k, j+1) f(j, x(j, m, \xi) + \phi(j; (m, \xi))).$$

Using Banach's fixed point theorem we conclude the existence of a unique fixed point

$$z^*(k; (m, \xi))) = \sum_{j=0}^{\infty} \mathcal{G}(k, j+1) f(j, x(j, m, \xi) + z^*(j; (m, \xi))).$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

It is easily verified that this map is a solution for the initial value problem

$$\begin{cases} z(k+1) = A(k)z(k) + f(k, x(k,m,\eta) + z(k)) \\ \qquad z(0) = -\displaystyle\sum_{j=0}^{\infty} \Phi(0, j+1)Q(j+1)f(j, x(j,m,\eta) + z^*(j;(m,\xi))). \end{cases}$$

By unicity of solutions we have

$$x(k,m,\xi) = x(k,p,x(p,m,\xi)), \quad \forall\ k,p,m \in \mathbb{Z}^+,$$

and analogously, we can verify

$$z^*(k;(m,\xi)) = z^*(k;(p,x(p,m,\xi))), \quad \forall\ k,p,m \in \mathbb{Z}^+. \tag{2.3}$$

For every fixed $k \in \mathbb{Z}^+$, consider the maps $H(k,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ and $G(k,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$\begin{cases} H(k,\xi) = \xi + \displaystyle\sum_{j=0}^{\infty} \mathcal{G}(k, j+1)f(j, x(j,k,\xi) + z^*(j;(k,\xi))) \\ \qquad\quad = \xi + z^*(k;(k,\xi)) \end{cases}$$

and

$$\begin{cases} G(k,\eta) = \eta - \displaystyle\sum_{j=0}^{\infty} \mathcal{G}(k, j+1)f(j, y(j,k,\eta)) \\ \qquad\quad = \eta + w^*(k;(k,\eta)). \end{cases}$$

It has been proved that H and $G = H^{-1}$ define the homeomorphisms that show the topological equivalence between (2.1) and (2.2), i.e.

$$H[j, x(j,m,\xi)] = y(j,m,H(m,\xi)) \tag{2.4}$$

and

$$G[j, y(j,m,\eta)] = x(j,m,G(m,\eta)).$$

To study additional properties of G, note that for $n < k$ we have

$$y(n,k,\eta) = \Phi(n,k)\eta - \sum_{j=n}^{k-1} \Phi(n, j+1)f(j, y(j,k,\eta)),$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

which implies

$$\begin{aligned}\Phi(k,n)y(n,k,\eta) &= \eta - \sum_{j=n}^{k-1}\Phi(k,j+1)f(j,y(j,k,\eta))\\ &= \eta - \sum_{j=n}^{k-1}\Phi(k,j+1)P(j+1)f(j,y(j,k,\eta))\\ &\quad - \sum_{j=n}^{k-1}\Phi(k,j+1)Q(j+1)f(j,y(j,k,\eta)).\end{aligned}$$

In particular, for $n=0$ we have

$$\begin{aligned}\Phi(k,0)y(0,k,\eta) &= \eta - \sum_{j=0}^{k-1}\Phi(k,j+1)P(j+1)f(j,y(j,k,\eta))\\ &\quad - \sum_{j=0}^{k-1}\Phi(k,j+1)Q(j+1)f(j,y(j,k,\eta))\\ &= \eta - \sum_{j=0}^{k-1}\Phi(k,j+1)P(j+1)f(j,y(j,k,\eta))\\ &\quad + \sum_{j=k}^{\infty}\Phi(k,j+1)Q(j+1)f(j,y(j,k,\eta))\\ &\quad - \sum_{j=0}^{\infty}\Phi(k,j+1)Q(j+1)f(j,y(j,k,\eta))\\ &= \eta - \sum_{j=0}^{\infty}\mathcal{G}(k,j+1)f(j,y(j,k,\eta))\\ &\quad - \sum_{j=0}^{\infty}\Phi(k,j+1)Q(j+1)f(j,y(j,k,\eta))\\ &= G(k,\eta) - \Phi(k,0)\sum_{j=0}^{\infty}\Phi(0,j+1)Q(j+1)f(j,y(j,k,\eta)).\end{aligned}$$

Hence, using the definition of $k \mapsto w^*(0;(k,\eta))$ we conclude

$$\begin{aligned}G(k,\eta) &= \Phi(k,0)\left(y(0,k,\eta) + \sum_{j=0}^{\infty}\Phi(0,j+1)Q(j+1)f(j,y(j,k,\eta))\right)\\ &= \Phi(k,0)\left(y(0,k,\eta) + w^*(0;(k,\eta))\right). \end{aligned} \tag{2.5}$$

To prove the C^r-topologically equivalence, in [**5**] it is supposed that system (2.1) admits a nonuniform contraction, that is, in condition (d1) it is supposed that $P(\cdot) = I_n$. Namely, we have

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

(D1) System (2.1) has a nonuniform dichotomy with projector $P(\cdot) = I_n$, i.e. there exist a nonnegative sequence D and a monotone decreasing sequence convergent to zero h, with $h(0) = 1$, such that

$$\|\Phi(k,n)\| \leqslant D(n)\frac{h(k)}{h(n)}, \quad \forall\, k \geqslant n \geqslant 0.$$

Furthermore, if we consider the property

(d5) The map $u \mapsto f(k,u)$ and its derivatives with respect to u up to the order r $(r \geqslant 1)$ are continuous functions of $(k,u) \in \mathbb{Z}^+ \times \mathbb{R}^d$,

we have the following result.

Proposition 2.7 [**5**, lemma 2.3]. *If conditions (D1), and (d2)–(d5) hold, then systems* (2.1) *and* (2.2) *are C^r-topologically equivalent on $\mathbb{Z}^+$.*

Remark 2.8. Note that when replacing (d1) with (D1), the form of Green's operator is simpler, hence conditions (d2) and (d3) are also considerably easier to handle. Also, in this case, the map $k \mapsto w^*(0;(k,\eta))$ is identically zero, which allowed the authors to prove the differentiability of $u \mapsto G(k,u)$.

Remark 2.9. Note that (d4) and (d5) imply that the map $y \mapsto A(k)y + f(k,y)$ is invertible with C^r inverse for each $k \in \mathbb{Z}^+$. Thus, we obtain that the map $\eta \mapsto y(0,k,\eta)$ is C^r for every $k \in \mathbb{Z}^+$.

3. Diffeomorphism of discrete topological equivalence under a dichotomy

This section is devoted to study the differentiability properties of the topological equivalence. To obtain some class of differentiability we give sufficient conditions for this purpose: the first one will allow us to obtain that G is of class C^1 and H is differentiable except for a set of zero Lebesgue measure. The second condition helps to achieve C^1 topological equivalence.

3.1. Almost C^1 topological equivalence

The main goal of this subsection is to prove that $\eta \mapsto w^*(0;(m,\eta))$ is of class C^1. Later, we show that G is of class C^1 while H is almost everywhere differentiable. For this purpose, we consider the following property:

(d6) For each fixed $m \in \mathbb{Z}^+$, the sequences D, h and γ satisfy

$$\sum_{j=m+1}^{\infty}\left(D(j+1)h(j+1)\gamma(j)\left[\prod_{p=m}^{j-1}\|A(p)\| + \gamma(p)\right]\right) < +\infty.$$

Lemma 3.1. *If conditions (d1)–(d6) hold, with $r = 1$ on (d5), then $\eta \mapsto w^*(0;(m,\eta))$ is a differentiable map for every fixed $m \in \mathbb{Z}^+$.*

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

Proof. Fix $\eta \in \mathbb{R}^d$ and let $(\delta_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a convergent to zero sequence such that $\delta_n \neq 0$ for every $n \in \mathbb{N}$. Choose and fix $m \in \mathbb{Z}^+$, we define

$$\psi_n(j) = \mathcal{G}(0, j+1)\frac{\begin{array}{c} f(j, y(j,m,\eta+\delta_n)) - f(j, y(j,m,\eta)) \\ -(\partial f/\partial u)(j, y(j,m,\eta))(\partial y/\partial \eta)(j,m,\eta)\delta_n \end{array}}{|\delta_n|},$$

note that as $\eta \mapsto y(j,m,\eta)$ is continuous, then $\lim_{n\to\infty} y(j,m,\eta+\delta_n) = y(j,m,\eta)$. Applying (d5), we have

$$\lim_{n\to\infty} \psi_n(j) = 0.$$

In the proof of proposition 2.6, in [5, p. 11], the authors define for $j < m$

$$\mathcal{C}_m(j) := \prod_{i=j}^{m-1} \frac{\|A^{-1}(i)\|}{1 - \|A^{-1}(i)\gamma(i)\|}, \tag{3.1}$$

and it is proved that

$$|y(j,m,\eta) - y(j,m,\tilde{\eta})| \leqslant \mathcal{C}_m(j)|\eta - \tilde{\eta}|.$$

However, we know

$$y(m+1, m, \eta) = A(m)\eta + f(m, \eta),$$

hence

$$y(m+1,m,\eta) - y(m+1,m,\tilde{\eta}) = A(m)(\eta - \tilde{\eta}) + f(m,\eta) - f(m,\tilde{\eta}),$$

thus

$$\begin{aligned} |y(m+1,m,\eta) - y(m+1,m,\tilde{\eta})| &\leqslant \|A(m)\|\,|\eta - \tilde{\eta}| + |f(m,\eta) - f(m,\tilde{\eta})| \\ &\leqslant (\|A(m)\| + \gamma(m))|\eta - \tilde{\eta}|. \end{aligned} \tag{3.2}$$

For $j \geqslant m$, we define $\mathcal{B}_m(m) = 1$ and

$$\mathcal{B}_m(j) := \prod_{i=m}^{j-1} \|A(i)\| + \gamma(i), \quad \forall\, j > m, \tag{3.3}$$

which, by (3.2), allows us to write for $j \geqslant m$

$$|y(j,m,\eta) - y(j,m,\tilde{\eta})| \leqslant \mathcal{B}_m(j)|\eta - \tilde{\eta}|. \tag{3.4}$$

Now, we define

$$\mathcal{A}_m(j) := \begin{cases} \mathcal{C}_m(j) & j < m \\ \mathcal{B}_m(j) & j \geqslant m, \end{cases} \tag{3.5}$$

hence, for every $j \in \mathbb{Z}^+$

$$|y(j,m,\eta) - y(j,m,\tilde{\eta})| \leqslant \mathcal{A}_m(j)|\eta - \tilde{\eta}|.$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

Thus,

$$\left\|\frac{\partial y}{\partial \eta}(j,m,\eta)\right\| = \lim_{\delta\to 0}\frac{|y(j,m,\eta+\delta)-y(j,m,\eta)|}{|\delta|} \leqslant \mathcal{A}_m(j). \tag{3.6}$$

Similarly,

$$\left\|\frac{\partial f}{\partial u}(j,u)\right\| = \lim_{\delta\to 0}\frac{|f(j,u+\delta)-f(j,u)|}{|\delta|} \leqslant \gamma(j).$$

Hence,

$$\begin{aligned}
&|\psi_n(j)|\\
&\leqslant \big\|\mathcal{G}(0,j+1)\big\| \frac{\begin{array}{c}|f(j,y(j,m,\eta+\delta_n))-f(j,y(j,m,\eta))|\\ +\big|(\partial f/\partial u)(j,y(j,m,\eta))(\partial y/\partial\eta)(j,m,\eta)\delta_n\big|\end{array}}{|\delta_n|}\\
&\leqslant \big\|\mathcal{G}(0,j+1)\big\| \frac{\gamma(j)|y(j,m,\eta+\delta_n)-y(j,m,\eta)|+\gamma(j)\big\|(\partial y/\partial\eta)(j,m,\eta)\big\|\,|\delta_n|}{|\delta_n|}\\
&\leqslant \big\|\mathcal{G}(0,j+1)\big\|\,\gamma(j)\left(\frac{|y(j,m,\eta+\delta_n)-y(j,m,\eta)|}{|\delta_n|}+\left\|\frac{\partial y}{\partial\eta}(j,m,\eta)\right\|\right)\\
&\leqslant 2\big\|\mathcal{G}(0,j+1)\big\|\,\gamma(j)\mathcal{A}_m(j).
\end{aligned}$$

Besides, by using (d3) and (d6) we have

$$\begin{aligned}
&\sum_{j=0}^{\infty}\big\|\mathcal{G}(0,j+1)\big\|\,\gamma(j)\mathcal{A}_m(j)\\
&\quad\leqslant \sum_{j=0}^{m-1}\big\|\mathcal{G}(0,j+1)\big\|\,\gamma(j)\mathcal{C}_m(j)+\sum_{j=m}^{\infty}\big\|\mathcal{G}(0,j+1)\big\|\,\gamma(j)\mathcal{B}_m(j)\\
&\quad\leqslant \sum_{j=0}^{m-1}\big\|\mathcal{G}(0,j+1)\big\|\,\gamma(j)\max_{i<m}\mathcal{C}_m(i)+\sum_{j=m}^{\infty}D(j+1)h(j+1)\gamma(j)\mathcal{B}_m(j)\\
&\quad\leqslant q\max_{i<m}\mathcal{C}_m(i)+\sum_{j=m}^{\infty}D(j+1)h(j+1)\gamma(j)\mathcal{B}_m(j)<+\infty.
\end{aligned}$$

Finally, by using Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\begin{array}{l}w^*(0;(m,\eta+\delta_n))-w^*(0;(m,\eta))\\ \qquad+\left[\sum_{j=0}^{\infty}\mathcal{G}(0,j+1)(\partial f/\partial u)(j,y(j,m,\eta))(\partial y/\partial\eta)(j,m,\eta)\right]\delta_n\end{array}}{|\delta_n|}\\
&\quad=-\lim_{n\to\infty}\sum_{j=0}^{\infty}\psi_n(j)=-\sum_{j=0}^{\infty}\left(\lim_{n\to\infty}\psi_n(j)\right)=0,
\end{aligned}$$

which implies $\eta\mapsto w^*(0;(m,\eta))$ is differentiable. □

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

Corollary 3.2. *If conditions (d1)–(d6) hold, with $r = 1$ on (d5), then for every $m \in \mathbb{Z}^+$*

$$\frac{\partial w^*(0;(m,\eta))}{\partial \eta} = -\sum_{j=0}^{\infty} \mathcal{G}(0,j+1)\frac{\partial f}{\partial u}(j,y(j,m,\eta))\frac{\partial y}{\partial \eta}(j,m,\eta).$$

Moreover, $\eta \mapsto w^*(0;(m,\eta))$ is a C^1 map for every fixed $m \in \mathbb{Z}^+$.

Proof. The identity is an immediate consequence of the previous lemma. From this identity and condition (d6), it follows easily, using Lebesgue's dominated convergence theorem, that $\eta \mapsto \partial w^*(0;(m,\eta))/\partial\eta$ is continuous. Thus, $\eta \mapsto w^*(0;(m,\eta))$ is a C^1 map. □

Corollary 3.3. *If conditions (d1)–(d6) hold, with $r = 1$ (d5), then for every $k \in \mathbb{Z}^+$ $\eta \mapsto G(k,\eta)$ is a C^1 map. Moreover, for every fixed $k \in \mathbb{Z}^+$, $\xi \mapsto H(k,\xi)$ is almost everywhere differentiable, i.e. it is a differentiable map except for a set of zero Lebesgue measure.*

Proof. By proposition 2.6 we know that systems (2.1) and (2.2) are topologically equivalent. By lemma 3.1 and corollary 3.2, $\eta \mapsto w^*(0;(k,\eta))$ is C^1 differentiable for every $k \in \mathbb{Z}^+$, and as stated in remark 2.9, $\eta \mapsto y(0,k,\eta)$ is as well. Then (2.5) allows us to conclude $\eta \mapsto G(k,\eta)$ is a C^1 map.

For every $k \in \mathbb{Z}^+$, let us call $\mathcal{N}_k := \{\xi \in \mathbb{R}^d : \det(\partial G/\partial\eta)(k,H(k,\xi)) = 0\}$. We have $\mathcal{N}_k$ has zero Lebesgue measure by Sard's theorem and for every $\xi \notin \mathcal{N}_k$, as $G(k,H(k,\xi)) = \xi$, we have

$$\frac{\partial G}{\partial \eta}(k,H(k,\xi))\frac{\partial H}{\partial \xi}(k,\xi) = I,$$

and so $(\partial H/\partial\xi)(k,\xi) = [(\partial G/\partial\eta)(k,H(k,\xi))]^{-1}$, which completes the proof. □

Corollary 3.4. *Suppose system (2.1) admits a nonuniform exponential dichotomy, which means there are two complementary invariant projectors $P(\cdot)$ and $Q(\cdot)$ and constants $C, \lambda, \varepsilon > 0$ such that*

$$\begin{cases} \|\Phi(k,n)P(n)\| \leqslant C\,\mathrm{e}^{-\lambda(k-n)+\varepsilon n}, & \forall\ k \geqslant n \geqslant 0 \\ \|\Phi(k,n)Q(n)\| \leqslant C\,\mathrm{e}^{\lambda(k-n)+\varepsilon n}, & \forall\ 0 \leqslant k \leqslant n. \end{cases}$$

Furthermore, suppose that for every $k \in \mathbb{Z}^+$ $u \mapsto f(k,u)$ is a C^1 function such that $u \mapsto (\partial f/\partial u)(k,u)$ is a bounded map that satisfies

$$|f(k,u)| \leqslant \kappa\,\mathrm{e}^{-\varepsilon(k+1)} \tag{3.7}$$

and

$$\left\|\frac{\partial f}{\partial u}(k,u)\right\| \leqslant \nu\,\mathrm{e}^{-\varepsilon(k+1)}a^k, \tag{3.8}$$

for given $\nu, \kappa > 0$ and $a \in (0,1)$. Suppose also that $M\nu\,\mathrm{e}^{-\varepsilon} < 1$. Then, if $aM\,\mathrm{e}^{-\lambda} < 1$, for sufficiently small $\nu > 0$, $\eta \mapsto G(m,\eta)$ is C^1 map and $\xi \mapsto H(m,\xi)$ is almost everywhere differentiable.

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

Proof. Condition (d1) is easily verified with $D(n) = C\,\mathrm{e}^{\varepsilon n}$ and $h(n) = \mathrm{e}^{-\lambda n}$. It is easy to see that condition (3.8) implies that

$$|f(k,y) - f(k,\tilde{y})| \leqslant \nu\,\mathrm{e}^{-\varepsilon(k+1)} a^k |y - \tilde{y}|,$$

so $\gamma(k) = \nu\,\mathrm{e}^{-\varepsilon(k+1)} a^k$ and $\mu(k) = \kappa\,\mathrm{e}^{-\varepsilon(k+1)}$. Hence condition (d2) is immediately satisfied and, provided ν is sufficiently small, so is (d3). Note that

$$\left\| A^{-1}(k)\gamma(k) \right\| \leqslant M\nu\,\mathrm{e}^{-\varepsilon} < 1,$$

thus condition (d4) is verified. Condition (d5) is satisfied by hypothesis, with $r = 1$. Now consider the map $j \mapsto \mathcal{B}_m(j)$ defined in (3.3). It is easy to see that for $j \geqslant m$ we have

$$\mathcal{B}_m(j) \leqslant \left(M + \nu\,\mathrm{e}^{-\varepsilon}\right)^{j-m} \leqslant \left(M + \nu\,\mathrm{e}^{-\varepsilon}\right)^{j},$$

thus

$$\begin{aligned}\sum_{j=m+1}^{\infty} D(j+1)h(j+1)\gamma(j)\mathcal{B}_m(j) &\leqslant \sum_{j=0}^{\infty} D(j+1)h(j+1)\gamma(j)\left(M + \nu\,\mathrm{e}^{-\varepsilon}\right)^{j} \\ &\leqslant C\nu\,\mathrm{e}^{-\lambda} \sum_{j=0}^{\infty} \left[a\,\mathrm{e}^{-\lambda}\left(M + \nu\,\mathrm{e}^{-\varepsilon}\right)\right]^{j},\end{aligned}$$

which, as $aM\,\mathrm{e}^{-\lambda} < 1$, is finite provided ν is sufficiently small, so condition (d6) is satisfied. Applying corollary 3.3, the statement is verified. □

Remark 3.5. In the previous corollary, if we allow $\varepsilon = 0$, we recover the classic exponential dichotomy. Furthermore, we would like to impose weaker conditions, in particular $M\,\mathrm{e}^{-\lambda} < 1$, and hence $a = 1$, but it is a well-known fact in literature that this is impossible for both the exponential and nonuniform exponential dichotomies, since for both cases in general we have $M\,\mathrm{e}^{-\lambda} \geqslant 1$ [**19**, p. 823]. Nevertheless, note that given any linear system (2.1), a wide family of perturbations f can be found such that $M\nu\,\mathrm{e}^{-\varepsilon} < 1$ and $aM\,\mathrm{e}^{-\lambda} < 1$, since both inequalities can be achieved by setting f small enough and with a fast enough decreasing rate.

Remark 3.6. Corollary 3.4 has been inspired by [**11**, theorem 2]. Our approach and that result consider (3.8), the same dichotomy and both results work for ν sufficiently small. Nevertheless, we imposed the extra hypothesis $aM\,\mathrm{e}^{-\lambda} < 1$ and conditions (3.7) and (d4).

Moreover, our result gives topologically equivalence on $\mathbb{Z}^+$ with maps such that $\eta \mapsto G(m,\eta)$ is a C^1 map and $\xi \mapsto H(m,\xi)$ is almost everywhere differentiable, in contrast to the mentioned theorem, which gives C^1-topologically equivalence on $\mathbb{Z}$. However, our approach does not impose a Lipschitz condition for the derivative of the perturbation, neither relies on spectral properties of the dichotomy, which is the core of that result.

3.2. Examples

This subsection is devoted to give examples that allow us to show different ways to obtain the conditions in corollary 3.3.

Example 3.7. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}$ three sequences such that

$$0 < \alpha \leqslant a_n, b_n \leqslant c_n^{-1} \leqslant 1 \leqslant c_n \leqslant M, \tag{3.9}$$

for given $M, \alpha > 0$, and let $(c_n)_{n\in\mathbb{N}}$ be monotone nondecreasing. Consider $A(n) \in \mathcal{M}_{3\times 3}(\mathbb{R})$ the diagonal matrix given by

$$A(n) = \begin{pmatrix} a_n & 0 & 0 \\ 0 & b_n & 0 \\ 0 & 0 & c_n \end{pmatrix}, \tag{3.10}$$

and consider system (2.1) associated with this sequence of matrices. Let $(r_n)_{n\in\mathbb{N}}$ be a summable nonnegative sequence and define $(\gamma_n)_{n\in\mathbb{N}}$ the sequence given by

$$\gamma_n = \frac{r_n c_n}{r_{n-1} + \cdots + r_1 + 1}.$$

Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be a bounded, differentiable Lipschitz (with constant $\leqslant 1$) map, define $f : \mathbb{Z}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(k, x) = \gamma_k g(x)$ and consider system (2.2) associated with this perturbation and the previous linear system. Then, if $Mr_n < \alpha$ for every $n \in \mathbb{N}$ and

$$\sup_{k\in\mathbb{Z}^+} \frac{c_{k-1} r_{k-1}}{r_{k-2} + r_{k-3} + \cdots + 1} + \sum_{j=1,\ j\neq k-1}^{\infty} r_j < 1,$$

systems (2.1) and (2.2) are topologically equivalent on $\mathbb{Z}^+$. Moreover, for every $k \in \mathbb{Z}^+$, $\eta \mapsto G(k, \eta)$ is a C^1 map and $\xi \mapsto H(k, \xi)$ is almost everywhere differentiable.

In fact, since $0 \leqslant a_n, b_n \leqslant c_n$, then $\|A(n)\| = c_n \leqslant M$ and $\|A^{-1}(n)\| \leqslant \alpha^{-1}$. Furthermore, let $P(\cdot)$ and $Q(\cdot)$ be

$$P(n) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q(n) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.11}$$

Then, if we consider $\Phi(m, n)$ be the transition matrix for (2.1), it is clear that

$$\begin{cases} \|\Phi(k, n)P(n)\| \leqslant \displaystyle\prod_{j=n}^{k-1} \max\{a_j, b_j\}, & \forall\ k \geqslant n \geqslant 0 \\ \|\Phi(k, n)Q(n)\| \leqslant \displaystyle\prod_{j=k}^{n-1} c_j^{-1}, & \forall\ 0 \leqslant k < n. \end{cases}$$

Hence, setting $h(n) = \prod_{p=1}^{n-1} c_p^{-1}$ and $D(n) = 1$, condition (d1) follows. Note also

$$\left\|A(n)^{-1}\gamma_n\right\| \leqslant \alpha^{-1} \frac{r_n c_n}{r_{n-1} + \cdots + r_1 + 1} \leqslant M\alpha^{-1} r_n,$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

thus (d4) is verified. It is easy to inductively prove

$$\gamma_j = r_j c_j \left(\prod_{p=1}^{j-1} 1 + \frac{\gamma_p}{c_p}\right)^{-1} = r_j c_j \prod_{p=1}^{j-1} \frac{c_p}{\gamma_p + c_p}.$$

Note that for every $k \in \mathbb{Z}^+$ we have

$$\begin{aligned}
\sum_{j=0}^{\infty} \|\mathcal{G}(k, j+1)\| \gamma_j &\leqslant \sum_{j=0}^{k-1} \frac{h(k)}{h(j+1)} \gamma_j + \sum_{j=k}^{\infty} \frac{h(j+1)}{h(k)} \gamma_j \\
&\leqslant \sum_{j=1}^{k-1} \left(\prod_{p=j+1}^{k-1} c_p^{-1}\right) \frac{c_j r_j}{r_{j-1} + \cdots + r_1 + 1} \\
&\quad + \sum_{j=k}^{\infty} r_j c_j \left(\prod_{p=k}^{j} c_p^{-1}\right) \left(\prod_{p=1}^{j-1} 1 + \frac{\gamma_p}{c_p}\right)^{-1} \\
&\leqslant \sum_{j=1}^{k-1} \left(\prod_{p=j+1}^{k-1} c_p^{-1}\right) \frac{c_{k-1} r_j}{r_{j-1} + \cdots + r_1 + 1} \\
&\quad + \sum_{j=k}^{\infty} r_j \left(\prod_{p=0}^{k-1} 1 + \frac{\gamma_p}{c_p}\right)^{-1} \left(\prod_{p=k}^{j-1} \frac{1}{c_p + \gamma_p}\right) \\
&< \frac{c_{k-1} r_{k-1}}{r_{k-2} + r_{k-3} + \cdots + 1} + \sum_{j=1,\ j \neq k-1}^{\infty} r_j,
\end{aligned}$$

hence conditions (d3) and (d2) are satisfied with $\mu_j = \gamma_j \|g\|_\infty$. Thus, applying proposition 2.6 it follows that the systems are topologically equivalent on $\mathbb{Z}^+$.

Condition (d5) is verified by hypothesis. Now, recall the map $j \mapsto \mathcal{B}_m(j)$ defined in (3.3). It is easy to see $m \geqslant n$ implies $\mathcal{B}_m(j) \leqslant \mathcal{B}_n(j)$, hence for a fixed $m \in \mathbb{Z}^+$

$$\mathcal{B}_m(j) \leqslant \mathcal{B}_1(j) \leqslant \prod_{p=1}^{j-1} c_p + \gamma_p.$$

So, we have

$$\begin{aligned}
\sum_{j=m+1}^{\infty} D(j+1) h(j+1) \gamma_j \mathcal{B}_m(j) &\leqslant \sum_{j=1}^{\infty} h(j+1) \gamma_j \mathcal{B}_1(j) \\
&\leqslant \sum_{j=1}^{\infty} \gamma_j \prod_{p=1}^{j} c_p^{-1} \prod_{p=1}^{j-1} (c_p + \gamma_p)
\end{aligned}$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

$$\leqslant \sum_{j=1}^{\infty} \gamma_j c_j^{-1} \left(\prod_{p=1}^{j-1} 1 + \frac{\gamma_p}{c_p} \right)$$
$$= \|r\|_1 < \infty.$$

Hence, (d6) is fulfilled. Finally, applying corollary 3.3, we show our claim.

REMARK 3.8. Condition $\sup_{k\in\mathbb{Z}^+}(c_{k-1}r_{k-1}/(r_{k-2}+r_{k-3}+\cdots+1))+\sum_{j=1,j\neq k-1}^{\infty} r_j < 1$ is obtained, for example, if the increasing rate of (c_n) is smaller than the growing rate of the partial sums of (r_n) and $\|r\|_1 < 1$. Another simpler case is if $M\|r\|_1 < 1$.

EXAMPLE 3.9. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}$ three sequences as in (3.9), $A(n)$ as in (3.10) and system (2.1) associated with this sequence of matrices. Let $(r_n)_{n\in\mathbb{N}}$ be a summable nonnegative sequence and define $(\gamma_n)_{n\in\mathbb{N}}$ the sequence given by

$$\gamma_n = \frac{r_n}{r_{n-1}+\cdots+r_1+1}.$$

Let $g:\mathbb{R}^3\to\mathbb{R}^3$ be a bounded, differentiable Lipschitz (with constant $\leqslant 1$) map, define $f:\mathbb{Z}^+\times\mathbb{R}^3\to\mathbb{R}^3$ given by $f(k,x)=\gamma_k g(x)$ and consider system (2.2) associated with this perturbation and the previous linear system. Then, if $r_n<\alpha$ for every $n\in\mathbb{N}$ and $\|r\|_1<1$, systems (2.1) and (2.2) are topologically equivalent on $\mathbb{Z}^+$. Moreover, for every $k\in\mathbb{Z}^+$, $\eta\mapsto G(k,\eta)$ is a C^1 map and $\xi\mapsto H(k,\xi)$ is almost everywhere differentiable.

Indeed, conditions (d1), (d4) and (d5) follow easily in the same fashion as in the previous example, with $P(\cdot)$ and $Q(\cdot)$ as in (3.11). It is easy to inductively prove

$$\gamma_j = r_j \left(\prod_{p=1}^{j-1} 1+\gamma_p \right)^{-1}.$$

Note that for a fixed $k\in\mathbb{Z}^+$

$$\begin{aligned}\sum_{j=0}^{\infty}\left\|\mathcal{G}(k,j+1)\right\|\gamma_j &\leqslant \sum_{j=1}^{k-1}\frac{h(k)}{h(j+1)}\gamma_j+\sum_{j=k}^{\infty}\frac{h(j+1)}{h(k)}\gamma_j\\ &\leqslant \sum_{j=1}^{k-1}\left(\prod_{p=j+1}^{k-1}c_p^{-1}\right)\frac{r_j}{r_{j-1}+\cdots r_1+1}\\ &\quad+\sum_{j=k}^{\infty}\left(\prod_{p=k}^{j}c_p^{-1}\right)\frac{r_j}{r_{j-1}+\cdots r_1+1}\\ &<\|r\|_1<1,\end{aligned}$$

hence proving (d3) and (d2) with $\mu_j = \gamma_j \|g\|_\infty$. Thus, by proposition 2.6 the systems are topologically equivalent on $\mathbb{Z}^+$. Now, for a fixed $m \in \mathbb{Z}^+$ we have

$$\begin{aligned}\sum_{j=m}^{\infty} D(j+1)h(j+1)\gamma(j)\mathcal{B}_m(j) &\leqslant \sum_{j=1}^{\infty} h(j+1)\gamma(j)\mathcal{B}_1(j)\\ &\leqslant \sum_{j=1}^{\infty} r_j \left(\prod_{p=1}^{j-1} 1+\gamma_p\right)^{-1} \left(\prod_{p=1}^{j} c_p^{-1}\right) \left(\prod_{p=1}^{j-1} c_p+\gamma_p\right)\\ &\leqslant \sum_{j=1}^{\infty} r_j c_j^{-1} \left(\prod_{p=1}^{j-1} \frac{1+\gamma_p c_p^{-1}}{1+\gamma_p}\right)\\ &\leqslant \|r\|_1 < \infty.\end{aligned}$$

Hence, (d6) is fulfilled. Thus, applying corollary 3.3 our claim is proved.

3.3. C^1 topological equivalence

We show, in this subsection, that the topological equivalence between systems (2.1) and (2.2) is of class C^1. Moreover, we prove that the derivatives of $G(k,\xi)$ and $H(k,\xi)$ are bounded as a function of ξ variable for each fixed k. To obtain our main result, we introduce the following condition:

(d7) For each fixed $m \in \mathbb{Z}^+$, the map $j \mapsto \mathcal{A}_m(j)$ defined in (3.5) verifies

$$\sum_{j=0}^{\infty} \left\|\mathcal{G}(m,j+1)\right\| \gamma(j)\mathcal{A}_m(j) < 1.$$

Remark 3.10. To the best of our knowledge, condition (d7) first appeared in [**1**], when Backes and Dragičević were also studying the smooth linearization of nonautonomous systems without spectral conditions to analyse coupled systems.

Lemma 3.11. *If conditions (d1)–(d7) hold, with $r=1$ on (d5), then $\eta \mapsto w^*(m;(m,\eta))$ is a C^1 map for every $m \in \mathbb{Z}^+$. Moreover, in that case we have*

$$\frac{\partial w^*(m;(m,\eta))}{\partial \eta} = -\sum_{j=0}^{\infty} \mathcal{G}(m,j+1)\frac{\partial f}{\partial u}(j,y(j,m,\eta))\frac{\partial y}{\partial \eta}(j,m,\eta)$$

and

$$\left\|\frac{\partial w^*}{\partial \eta}(m;(m,\eta))\right\| < 1. \tag{3.12}$$

Note that the proof of this lemma follows in the same fashion as lemma 3.1 and corollary 3.2. Moreover, statement (3.12) is a direct consequence of condition (d7).

Now we can establish the main result of this section.

Theorem 3.12. *If conditions (d1)–(d7) hold, with $r = 1$ on (d5), then systems (2.1) and (2.2) are C^1-topologically equivalent on $\mathbb{Z}^+$. Moreover, in this case both $\xi \mapsto H_\xi(m, \xi)$ and $\eta \mapsto G_\eta(m, \eta)$ are bounded maps for every fixed $m \in \mathbb{Z}^+$.*

Proof. It is enough to note that $\eta \mapsto G(m, \eta) = \eta + w^*(m; (m, \eta))$ is differentiable, and as the norm of the derivative of $w^*(m; (m, \eta))$, by lemma 3.11, is smaller than 1, then the derivative of $G(m, \eta)$ is invertible everywhere. Moreover, as $G(m, \eta) - \eta$ is bounded, thus $G(m, \eta) \to \infty$ when $|\eta| \to \infty$. Hence, by [**16**, corollary 2.1], $\xi \mapsto G(m, \xi)$ is a C^1 diffeomorphism. In particular, $\xi \mapsto H(m, \xi)$ is also differentiable.

Now, observe that

$$G(m, \eta) - G(m, \tilde{\eta}) = \eta - \tilde{\eta} + w^*(m; (m, \eta)) - w^*(m; (m, \tilde{\eta})),$$

thus, under condition (d7) it is easy to see

$$|G(m, \eta) - G(m, \tilde{\eta})| \leqslant 2|\eta - \tilde{\eta}|.$$

On the other hand, observe that as

$$H(m, \xi) = \xi + \sum_{j=0}^{\infty} \mathcal{G}(m, j+1) f(j, x(j, m, \xi) + z^*(j; (m, \xi))) = \xi + z^*(m; (m, \xi)),$$

then, using (2.3) we obtain

$$H[j, x(j, m, \xi))] = x(j, m, \xi) + z^*(j; (j, x(j, m, \xi))) = x(j, m, \xi) + z^*(j; (m, \xi)).$$

Thus,

$$H(m, \xi) = \xi + \sum_{j=0}^{\infty} \mathcal{G}(m, j+1) f(j, H[j, x(j, m, \xi)]),$$

and, using (2.4), we conclude

$$H(m, \xi) = \xi + \sum_{j=0}^{\infty} \mathcal{G}(m, j+1) f(j, y(j, m, H(m, \xi))).$$

Now, observe

$$\begin{aligned}
&|H(m, \xi) - H(m, \tilde{\xi})| \\
&\quad \leqslant |\xi - \tilde{\xi}| + \sum_{j=0}^{\infty} \left\| \mathcal{G}(m, j+1) \right\| \left| f(j, y(j, m, H(m, \xi))) - f(j, y(j, m, H(m, \tilde{\xi}))) \right| \\
&\quad \leqslant |\xi - \tilde{\xi}| + \sum_{j=0}^{\infty} \left\| \mathcal{G}(m, j+1) \right\| \gamma(j) \left| y(j, m, H(m, \xi)) - y(j, m, H(m, \tilde{\xi})) \right| \\
&\quad \leqslant |\xi - \tilde{\xi}| + \sum_{j=0}^{\infty} \left\| \mathcal{G}(m, j+1) \right\| \gamma(j) \mathcal{A}_m(j) \left| H(m, \xi)) - H(m, \tilde{\xi}) \right|.
\end{aligned}$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

Hence, by using (d7)

$$\frac{|H(m,\xi)-H(m,\tilde{\xi})|}{|\xi-\tilde{\xi}|} \leqslant \frac{1}{1-\sum_{j=0}^{\infty}\big\|\mathcal{G}(m,j+1)\big\|\,\gamma(j)\mathcal{A}_m(j)}.$$

□

REMARK 3.13. Suppose all conditions from corollary 3.4 are verified. By theorem 3.12, it is enough to verify condition (d7) in order to have that (2.1) and (2.2) are C^1-topologically equivalent on $\mathbb{Z}^+$. Now, assume we can find a linear system (2.1) such that $M\,\mathrm{e}^{-\lambda+\varepsilon} < \mathrm{e}^{\varepsilon}$. It is easy to see that we can obtain a family of perturbations f such that:

- $M\,\mathrm{e}^{-\lambda+\varepsilon} < \mathrm{e}^{\varepsilon} - M\nu$,
- denoting $b = M\,\mathrm{e}^{-\lambda}/a(1-M\nu\,\mathrm{e}^{-\varepsilon})$, it is also verified

$$C\nu\left(\frac{M}{|M\mathrm{e}^{-\lambda}-a(1-M\nu\,\mathrm{e}^{-\varepsilon})|}+\mathrm{e}^{-\lambda}+\frac{\mathrm{e}^{-2\lambda}a(M+\nu\,\mathrm{e}^{-\varepsilon})}{1-a\,\mathrm{e}^{-\lambda}(M+\nu\,\mathrm{e}^{-\varepsilon})}\right)<1,\ \text{ if } b\neq 1,$$

or

$$C\nu\left(\mathrm{e}^{\lambda}\sup_{m\in\mathbb{Z}^+} ma^m+\mathrm{e}^{-\lambda}+\frac{\mathrm{e}^{-2\lambda}a(M+\nu\,\mathrm{e}^{-\varepsilon})}{1-a\,\mathrm{e}^{-\lambda}(M+\nu\,\mathrm{e}^{-\varepsilon})}\right)<1, \quad \text{if } b=1,$$

since both these conditions can easily be achieved for small enough $\nu > 0$.

Under these conditions, it is easy to prove $b > 0$ and $ab < 1$. Then, considering $j \mapsto \mathcal{C}_m(j)$ as defined in (3.1), for $j < m$ we have

$$\mathcal{C}_m(j) \leqslant \left(\frac{M}{1-M\nu\,\mathrm{e}^{-\varepsilon}}\right)^{m-j},$$

hence,

$$\begin{aligned}
&\sum_{j=0}^{\infty}\big\|\mathcal{G}(m,j+1)\big\|\,\gamma(j)\mathcal{A}_m(j)\\
&\quad\leqslant \sum_{j=0}^{m-1}\big\|\Phi(m,j+1)P(j+1)\big\|\,\nu\,\mathrm{e}^{-\varepsilon(j+1)}a^j\mathcal{C}_m(j)\\
&\qquad+\big\|\mathcal{G}(m,m+1)\big\|\,\nu\,\mathrm{e}^{-\varepsilon(m+1)}a^m\\
&\qquad+\sum_{j=m+1}^{\infty}\big\|-\Phi(m,j+1)Q(j+1)\big\|\,\nu\,\mathrm{e}^{-\varepsilon(j+1)}a^j\mathcal{B}_m(j)\\
&\quad\leqslant \sum_{j=0}^{m-1} C\nu\,\mathrm{e}^{\lambda}a^m b^{m-j}+C\nu\,\mathrm{e}^{-\lambda}a^m+\sum_{j=m+1}^{\infty}C\nu\,\mathrm{e}^{-\lambda}a^m\left[a\,\mathrm{e}^{-\lambda}(M+\nu\,\mathrm{e}^{-\varepsilon})\right]^{j-m}.
\end{aligned}$$

Thus, if $b \neq 1$, we have

$$\begin{aligned}\sum_{j=0}^{\infty}\left\|\mathcal{G}(m,j+1)\right\|\gamma(j)\mathcal{A}_m(j) &\leqslant C\nu\,\mathrm{e}^{\lambda}b\left(\frac{(ab)^m-a^m}{b-1}\right)+C\nu\,\mathrm{e}^{-\lambda}a^m+\frac{C\nu\,\mathrm{e}^{-2\lambda}a^{m+1}(M+\nu\,\mathrm{e}^{-\varepsilon})}{1-a\,\mathrm{e}^{-\lambda}(M+\nu\,\mathrm{e}^{-\varepsilon})}\\ &\leqslant C\nu\left(\frac{M}{|M\,\mathrm{e}^{-\lambda}-a(1-M\nu\,\mathrm{e}^{-\varepsilon})|}+\mathrm{e}^{-\lambda}+\frac{\mathrm{e}^{-2\lambda}a(M+\nu\,\mathrm{e}^{-\varepsilon})}{1-a\,\mathrm{e}^{-\lambda}(M+\nu\,\mathrm{e}^{-\varepsilon})}\right),\end{aligned}$$

and if $b = 1$, we have

$$\begin{aligned}\sum_{j=0}^{\infty}\left\|\mathcal{G}(m,j+1)\right\|\gamma(j)\mathcal{A}_m(j) &\leqslant C\nu\,\mathrm{e}^{\lambda}ma^m+C\nu\,\mathrm{e}^{-\lambda}a^m+\frac{C\nu\,\mathrm{e}^{-2\lambda}a^{m+1}(M+\nu\,\mathrm{e}^{-\varepsilon})}{1-a\,\mathrm{e}^{-\lambda}(M+\nu\,\mathrm{e}^{-\varepsilon})}\\ &\leqslant C\nu\left(\mathrm{e}^{\lambda}\sup_{m\in\mathbb{Z}^+}ma^m+\mathrm{e}^{-\lambda}+\frac{\mathrm{e}^{-2\lambda}a(M+\nu\,\mathrm{e}^{-\varepsilon})}{1-a\,\mathrm{e}^{-\lambda}(M+\nu\,\mathrm{e}^{-\varepsilon})}\right).\end{aligned}$$

Thus, under these hypotheses, condition (d7) is verified. However, as mentioned in remark 3.5, it is impossible to find a linear system (2.1) such that $M\,\mathrm{e}^{-\lambda+\varepsilon} < \mathrm{e}^{\varepsilon}$. In other words, using the tools developed in this study, theorem 3.12 fails to be applicable to the exponential or nonuniform exponential case; the latter dichotomy is the one considered in corollary 3.4.

4. Second derivative

We once again consider expression (2.5) in order to study the second derivative of the homeomorphism of topological equivalence. Taking into account remark 2.9, the map $\eta \mapsto y(0, m, \eta)$ is of class C^r $(r \geqslant 1)$ for every fixed $m \in \mathbb{Z}^+$ if conditions (d1)–(d5) are satisfied; hence the class of differentiability of the homeomorphism relies on the differentiability of the maps $\eta \mapsto w^*(0; (m, \eta))$ and $\eta \mapsto w^*(m; (m, \eta))$.

LEMMA 4.1. *Suppose conditions (d1)–(d6) hold, and (d5) is fulfilled with $r = 2$. Also, there are functions $\Gamma : \mathbb{Z}^+ \to \mathbb{R}^+$ and $\pi_m : \mathbb{Z}^+ \to \mathbb{R}^+$ such that*

$$\left\|\frac{\partial^2 f}{\partial u^2}(j,u)\right\| \leqslant \Gamma(j), \quad \textit{for every } j \in \mathbb{Z}^+ \tag{4.1}$$

and

$$\left\|\frac{\partial^2 y}{\partial \eta^2}(j,m,\eta)\right\| \leqslant \pi_m(j), \quad \textit{for every } j \geqslant m. \tag{4.2}$$

Finally, suppose that for each fixed $m \in \mathbb{Z}^+$, the sequences satisfy

$$\sum_{j=m}^{\infty}\left(D(j+1)h(j+1)\left\{\pi_m(j)\gamma(j)+\Gamma(j)\left[\prod_{p=m}^{j-1}\|A(p)\|+\gamma(p)\right]^2\right\}\right)<+\infty. \tag{4.3}$$

Then $\eta \mapsto w^*(0;(m,\eta))$ is a C^2 map.

Proof. By corollary 3.2 we have

$$\frac{\partial w^*(0;(m,\eta))}{\partial \eta} = -\sum_{j=0}^{\infty}\mathcal{G}(0,j+1)\frac{\partial f}{\partial u}(j,y(j,m,\eta))\frac{\partial y}{\partial \eta}(j,m,\eta)$$

for every fixed $m \in \mathbb{Z}^+$. Hence, in order to prove the differentiability of this map, it is enough to find a summable function that uniformly (with respect to η) dominates

$$\mathcal{G}(0,j+1)\left[\frac{\partial f}{\partial u}(j,y(j,m,\eta))\frac{\partial^2 y}{\partial \eta^2}(j,m,\eta)+\frac{\partial^2 f}{\partial u^2}(j,y(j,m,\eta))\left(\frac{\partial y}{\partial \eta}(j,m,\eta)\right)^2\right]$$

for every $j \geqslant 0$, which is verified by hypothesis. Therefore, we apply the same strategy as in the proof of lemma 3.1 and corollary 3.2. □

COROLLARY 4.2. *If conditions (d1)–(d6) hold, and (d5) is satisfied with $r=2$, and conditions of lemma 4.1 are satisfied, then $\eta \mapsto G(k,\eta)$ is a C^2 map for every $k \in \mathbb{Z}^+$. Moreover, for every fixed $k \in \mathbb{Z}^+$, $\xi \mapsto H(k,\xi)$ is almost everywhere two times differentiable, i.e. it is a map two times differentiable except for a set of zero Lebesgue measure.*

Note that corollary 4.2 follows easily in the same fashion as the proof of corollary 3.3. Actually, for a fixed $k \in \mathbb{Z}^+$, it can be deduced that outside of the set of zero Lebesgue measure $\mathcal{N}_k$ (defined in the proof of corollary 3.3) where $\xi \mapsto H(k,\xi)$ is not differentiable, $\xi \mapsto H(k,\xi)$ is two times differentiable.

Now we can establish the main result of this section.

THEOREM 4.3. *If conditions (d1)–(d7), with $r=2$ on (d5), and conditions from lemma 4.1 hold, then systems (2.1) and (2.2) are C^2-topologically equivalent on $\mathbb{R}^+$. Moreover, in this case both $\xi \mapsto H_\xi(m,\xi)$ and $\eta \mapsto G_\eta(m,\eta)$ are bounded maps for every fixed $m \in \mathbb{Z}^+$.*

Proof. By lemma 4.1 $\eta \mapsto w^*(0;(k,\eta))$ is a C^2 map, thus using (2.5) we can deduce $\eta \mapsto G(k,\eta)$ is a C^2 map. Now, observe that as $G(k,H(k,\xi))=\xi$, and by theorem 3.12 both $G(k,\cdot)$ and $H(k,\cdot)$ are C^1 maps, then

$$\frac{\partial G}{\partial \xi}\left(k,H(k,\xi)\right)\cdot\frac{\partial H}{\partial \xi}(k,\xi)=I,$$

thus

$$\frac{\partial^2 G}{\partial \xi^2}\left(k,H(k,\xi)\right)\cdot\left(\frac{\partial H}{\partial \xi}(k,\xi)\right)^2+\frac{\partial G}{\partial \xi}\left(k,H(k,\xi)\right)\cdot\frac{\partial^2 H}{\partial \xi^2}(k,\xi)=0,$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

hence,

$$\frac{\partial^2 H}{\partial \xi^2}(k,\xi) = -\left(\frac{\partial G}{\partial \xi}\left(k, H(k,\xi)\right)\right)^{-1} \cdot \frac{\partial^2 G}{\partial \xi^2}\left(k, H(k,\xi)\right) \cdot \left(\frac{\partial H}{\partial \xi}(k,\xi)\right)^2,$$

which is valid, since, as stated before, under condition (d7) the first derivative of $G(k,\cdot)$ is invertible everywhere. Furthermore, both $\xi \mapsto H_\xi(k,\xi)$ and $\eta \mapsto G_\eta(k,\eta)$ are bounded maps for every $k \in \mathbb{Z}^+$, such as in theorem 3.12. □

Now we proceed to study some technical results in order to establish a concrete example of corollary 4.2.

Lemma 4.4. *Suppose conditions (d4)–(d5) hold, with $r = 2$ on (d5). Furthermore, suppose there exist a function $\Gamma : \mathbb{Z}^+ \to \mathbb{R}^+$ that verifies (4.1). Then, for every $j \geqslant m$*

$$\left\|\frac{\partial^2 y}{\partial \eta^2}(j,m,\eta)\right\| \leqslant \sum_{i=0}^{j-m-1}\left[\Gamma(m+i)\prod_{p=m}^{m+i-1}\Big(\|A(p)\| + \gamma(p)\Big)^2 \prod_{p=m+i+1}^{j-1}\|A(p)\| + \gamma(p)\right].$$

Proof. Fix $m \in \mathbb{Z}^+$. We know $\eta \mapsto (\partial y/\partial \eta)(j,m,\eta)$ is well defined for every $j \geqslant m$. Consider the matrix initial value problem:

$$\begin{cases} z(j+1) &= \left[A(j) + \dfrac{\partial f}{\partial u}(j, y(j,m,\eta)\right] z(j) \\ z(m) &= I. \end{cases} \tag{4.4}$$

From the proof of theorem 2.7 we can deduce $j \mapsto z(j,m,\eta) = (\partial y/\partial \eta)(j,m,\eta)$ is solution of (4.4), hence,

$$z(m+1,m,\eta) = \left[A(m) + \frac{\partial f}{\partial u}(m, y(m,m,\eta))\right] z(m,m,\eta) = A(m) + \frac{\partial f}{\partial u}(m,\eta).$$

Then, we obtain

$$\big\|z(m+1,m,\eta) - z(m+1,m,\tilde{\eta})\big\| = \left\|\frac{\partial f}{\partial u}(m,\eta) - \frac{\partial f}{\partial u}(m,\tilde{\eta})\right\| \leqslant \Gamma(m)|\eta - \tilde{\eta}|,$$

or equivalently

$$\frac{\big\|z(m+1,m,\eta) - z(m+1,m,\tilde{\eta})\big\|}{|\eta - \tilde{\eta}|} \leqslant \Gamma(m).$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

Now, set $\hat{m} \geqslant m+1$. Note that

$$\begin{aligned}
&z(\hat{m}+1, m, \eta) - z(\hat{m}+1, m, \tilde{\eta}) \\
&\quad = \left[A(\hat{m}) + \frac{\partial f}{\partial u}(\hat{m}, y(\hat{m}, m, \eta))\right] z(\hat{m}, m, \eta) \\
&\qquad - \left[A(\hat{m}) + \frac{\partial f}{\partial u}(\hat{m}, y(\hat{m}, m, \tilde{\eta}))\right] z(\hat{m}, m, \tilde{\eta}) \\
&\quad = A(\hat{m}) \left[z(\hat{m}, m, \eta) - z(\hat{m}, m, \tilde{\eta})\right] \\
&\qquad + \frac{\partial f}{\partial u}(\hat{m}, y(\hat{m}, m, \eta)) \left[z(\hat{m}, m, \eta) - z(\hat{m}, m, \tilde{\eta})\right] \\
&\qquad + \left[\frac{\partial f}{\partial u}(\hat{m}, y(\hat{m}, m, \eta)) - \frac{\partial f}{\partial u}(\hat{m}, y(\hat{m}, m, \tilde{\eta}))\right] z(\hat{m}, m, \tilde{\eta}),
\end{aligned}$$

from where it follows that

$$\begin{aligned}
\left\|z(\hat{m}+1, m, \eta) - z(\hat{m}+1, m, \tilde{\eta})\right\| \leqslant{}& \left(\left\|A(\hat{m})\right\| + \gamma(\hat{m})\right) \left\|z(\hat{m}, m, \eta) - z(\hat{m}, m, \tilde{\eta})\right\| \\
&+ \Gamma(\hat{m}) |y(\hat{m}, m, \eta) - y(\hat{m}, m, \tilde{\eta})| \left\|z(\hat{m}, m, \tilde{\eta})\right\|,
\end{aligned}$$

hence, using (3.4) and (3.6), we obtain

$$\begin{aligned}
\left\|z(\hat{m}+1, m, \eta) - z(\hat{m}+1, m, \tilde{\eta})\right\| \leqslant{}& \left(\left\|A(\hat{m})\right\| + \gamma(\hat{m})\right) \left\|z(\hat{m}, m, \eta) - z(\hat{m}, m, \tilde{\eta})\right\| \\
&+ \Gamma(\hat{m}) \left(\prod_{i=m}^{\hat{m}-1} \left\|A(i)\right\| + \gamma(i)\right)^2 |\eta - \tilde{\eta}|.
\end{aligned}$$

Thus, defining for $j \geqslant 1$

$$\begin{aligned}
\phi_{j,m} &= \frac{\left\|z(m+j, m, \eta) - z(m+j, m, \tilde{\eta})\right\|}{|\eta - \tilde{\eta}|}, \\
\alpha_{j,m} &= \left\|A(m+j)\right\| + \gamma(m+j)
\end{aligned}$$

and

$$\beta_{j,m} = \Gamma(m+j) \left(\prod_{p=m}^{m+j-1} \left\|A(p)\right\| + \gamma(p)\right)^2,$$

we have $\phi_{1,m} \leqslant \Gamma(m)$ and

$$\phi_{j+1,m} \leqslant \alpha_{j,m}\phi_{j,m} + \beta_{j,m}.$$

Thus, defining $\beta_{0,m} = \Gamma(m)$ it is easy to inductively prove

$$\frac{\left\|z(m+j, m, \eta) - z(m+j, m, \tilde{\eta})\right\|}{|\eta - \tilde{\eta}|} = \phi_{j,m} \leqslant \sum_{i=0}^{j-1} \left(\beta_{i,m} \prod_{p=i+1}^{j-1} \alpha_{p,m}\right),$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

and as $z(j,m,\eta) = (\partial y/\partial\eta)(j,m,\eta)$, this implies that for every $j \geqslant m$

$$\begin{aligned}\left\|\frac{\partial^2 y}{\partial\eta^2}(j,m,\eta)\right\| &\leqslant \sum_{i=0}^{j-m-1}\left(\beta_{i,m}\prod_{p=i+1}^{j-m-1}\alpha_{p,m}\right)\\ &= \sum_{i=0}^{j-m-1}\left[\Gamma(m+i)\prod_{p=m}^{m+i-1}\Big(\big\|A(p)\big\|+\gamma(p)\Big)^2\prod_{p=m+i+1}^{j-1}\big\|A(p)\big\|+\gamma(p)\right].\end{aligned}$$

□

Corollary 4.5. *Suppose conditions (d4)–(d5) hold, with $r = 2$ on (d5). If $\gamma(j) = \nu\,\mathrm{e}^{-\varepsilon(j+1)}a^j$ and $\Gamma(j) = \zeta\,\mathrm{e}^{-\varepsilon(j+1)}a^j$ satisfies (4.1), then if $a(M+\nu) < \mathrm{e}^{\varepsilon}$, for every $j \geqslant m$*

$$\left\|\frac{\partial^2 y}{\partial\eta^2}(j,m,\eta)\right\| \leqslant \frac{\zeta\left(a\,\mathrm{e}^{-\varepsilon}\right)^m}{\mathrm{e}^{\varepsilon}(M+\nu)-a(M+\nu)^2}\left[(M+\nu)^{j-m}-\Big(a\,\mathrm{e}^{-\varepsilon}(M+\nu)^2\Big)^{(j-m)}\right].$$

Proof. Applying lemma 4.4 and noticing $\big\|A(p)\big\| + \gamma(p) \leqslant M+\nu$ for every $p \in \mathbb{Z}^+$, we have

$$\begin{aligned}\left\|\frac{\partial^2 y}{\partial\eta^2}(j,m,\eta)\right\| &\leqslant \sum_{i=0}^{j-m-1}\left[\zeta\,\mathrm{e}^{-\varepsilon(m+i+1)}a^{m+i}\prod_{p=m}^{m+i-1}\Big(\big\|A(p)\big\|+\nu\,\mathrm{e}^{-\varepsilon(p+1)}\Big)^2\right.\\ &\qquad\left.\times\prod_{p=m+i+1}^{j-1}\big\|A(p)\big\|+\nu\,\mathrm{e}^{-\varepsilon(p+1)}\right]\\ &\leqslant \zeta\,\mathrm{e}^{-\varepsilon(m+1)}a^m\sum_{i=0}^{j-m-1}\left[\mathrm{e}^{-\varepsilon i}a^i\prod_{p=m}^{m+i-1}(M+\nu)^2\prod_{p=m+i+1}^{j-1}M+\nu\right]\\ &= \zeta\,\mathrm{e}^{-\varepsilon(m+1)}a^m(M+\nu)^{j-m-1}\sum_{i=0}^{j-m-1}\left[a\,\mathrm{e}^{-\varepsilon}(M+\nu)\right]^i\\ &= \zeta\,\mathrm{e}^{-\varepsilon(m+1)}a^m(M+\nu)^{j-m-1}\frac{1-\left[a\,\mathrm{e}^{-\varepsilon}(M+\nu)\right]^{j-m}}{1-a\,\mathrm{e}^{-\varepsilon}(M+\nu)}\\ &= \frac{\zeta(\mathrm{e}^{-\varepsilon}a)^m}{\mathrm{e}^{\varepsilon}(M+\nu)-a(M+\nu)^2}\left[(M+\nu)^{j-m}-(\mathrm{e}^{-\varepsilon}a(M+\nu)^2)^{(j-m)}\right].\end{aligned}$$

□

Theorem 4.6. *Suppose system (2.1) admits a nonuniform exponential dichotomy, which means there are two complementary invariant projectors $P(\cdot)$ and $Q(\cdot)$ and*

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

constants $C, \lambda, \varepsilon > 0$ *such that*

$$\begin{cases} \left\|\Phi(k,n)P(n)\right\| \leqslant C\,\mathrm{e}^{-\lambda(k-n)+\varepsilon n}, & \forall\ k \geqslant n \geqslant 0 \\ \left\|\Phi(k,n)Q(n)\right\| \leqslant C\,\mathrm{e}^{\lambda(k-n)+\varepsilon n}, & \forall\ 0 \leqslant k \leqslant n. \end{cases}$$

Furthermore, suppose that for every $k \in \mathbb{Z}^+$ $u \mapsto f(k,u)$ *is a* C^2 *function such that:*

$$|f(k,u)| \leqslant \kappa\,\mathrm{e}^{-\varepsilon(k+1)},$$
$$\left\|\frac{\partial f}{\partial u}(k,u)\right\| \leqslant \nu\,\mathrm{e}^{-\varepsilon(k+1)}a^k$$

and

$$\left\|\frac{\partial f}{\partial u}(k,u) - \frac{\partial f}{\partial u}(k,\tilde{u})\right\| \leqslant \zeta\,\mathrm{e}^{-\varepsilon(k+1)}a^k|u-\tilde{u}|, \tag{4.5}$$

for given $\kappa, \nu, \zeta > 0$ *and* $a \in (0,1)$. *Suppose also that* $M\nu\,\mathrm{e}^{-\varepsilon} < 1$ *and* $a(M+\nu) < \mathrm{e}^{\varepsilon}$. *Then, if* $aM^2\,\mathrm{e}^{-\lambda} < 1$, *for sufficiently small* $\nu > 0$, $\eta \mapsto G(k,\eta)$ *is a* C^2 *map and* $\xi \mapsto H(k,\xi)$ *is almost everywhere two times differentiable for every fixed* $k \in \mathbb{Z}^+$.

Proof. By corollary 3.4 we know conditions (d1)–(d6) are satisfied, since $aM^2\,\mathrm{e}^{-\lambda} > aM\,\mathrm{e}^{-\lambda}$. In particular, (d3) is verified with $\gamma(j) = \nu\,\mathrm{e}^{-\varepsilon(j+1)}a^j$ and (d5) is satisfied with $r = 2$.

It is easy to see that (4.5) implies (4.1) from lemma 4.1 with $\Gamma(j) = \zeta\,\mathrm{e}^{-\varepsilon(j+1)}a^j$. Once again consider the map $j \mapsto \mathcal{B}_m(j)$ defined in (3.3). As in the proof of corollary 3.4, it is easy to see for $j \geqslant m$:

$$\mathcal{B}_m(j)^2 \leqslant (M+\nu)^{2(j-m)} \leqslant (M+\nu)^{2j}, \quad \forall\ m \in \mathbb{Z}^+.$$

Hence,

$$\begin{aligned} &\sum_{j=m}^{\infty} D(j+1)h(j+1)\Gamma(j)\left[\prod_{p=m}^{j-1}\left\|A(p)\right\| + \gamma(p)\right]^2 \\ &\leqslant \sum_{j=0}^{\infty} D(j+1)h(j+1)\Gamma(j)\,(M+\nu)^{2j} \\ &\leqslant C\zeta\,\mathrm{e}^{-\lambda}\sum_{j=0}^{\infty}\left[a\,\mathrm{e}^{-\lambda}\,(M+\nu)^2\right]^j. \end{aligned}$$

As $aM^2\mathrm{e}^{-\lambda} < 1$, then for small enough ν we have

$$\sum_{j=m}^{\infty}\left(D(j+1)h(j+1)\Gamma(j)\left[\prod_{p=m}^{j-1}\left\|A(p)\right\| + \gamma(p)\right]^2\right) < +\infty. \tag{4.6}$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

On the other hand, using corollary 4.5 we have

$$\left\| \frac{\partial^2 y}{\partial \eta^2}(j,m,\eta) \right\| \leqslant \frac{\zeta(\mathrm{e}^{-\varepsilon}a)^m}{\mathrm{e}^{\varepsilon}(M+\nu)-a(M+\nu)^2}\left[(M+\nu)^{j-m}-(\mathrm{e}^{-\varepsilon}a(M+\nu)^2)^{(j-m)}\right] =: \pi_m(j),$$

for every $j \geqslant m$, thus

$$\sum_{j=m}^{\infty} D(j+1)h(j+1)\pi_m(j)\gamma(j) \leqslant \sum_{j=m}^{\infty} K_m \left\{ \frac{\left[\mathrm{e}^{-\lambda}a(M+\nu)\right]^j}{(M+\nu)^m} - \frac{\left[\mathrm{e}^{-\lambda-\varepsilon}a^2(M+\nu)^2\right]^j}{\left[\mathrm{e}^{-\varepsilon}a(M+\nu)^2\right]^m} \right\} < +\infty,$$

for a small enough ν, where $K_m = C\nu\zeta(\mathrm{e}^{-\varepsilon}a)^m\,\mathrm{e}^{-\lambda}\mathrm{e}^{\varepsilon}(M+\nu)-a(M+\nu)^2$. Thus, the previous expression along with (4.6) implies condition (4.3) from lemma 4.1. Hence, applying corollary 4.2 the theorem follows. □

Remark 4.7. This theorem extends corollary 3.4, with some extra hypothesis. Note that we impose the existence of a function Γ that satisfies (4.1). This implies a Lipschitz condition for $\partial f/\partial u$, which, as mentioned in remark 3.6, was used by the authors in [**11**] in order to prove the first class of differentiability for the topological equivalence.

Remark 4.8. Similarly as we commented on remark 3.5, given any linear system (2.1) with nonuniform exponential dichotomy, a wide family of perturbations f can be found such that $M\nu\,\mathrm{e}^{-\varepsilon} < 1$, $a(M+\nu) < \mathrm{e}^{\varepsilon}$ and $aM^2\,\mathrm{e}^{-\lambda} < 1$, since all of these inequalities can be achieved by setting f small enough and with a fast enough decreasing rate.

On the other hand, by theorem 4.3, it would be enough to verify (d7) in order to achieve C^2-topological equivalence. Nevertheless, as explained in remark 3.13, this cannot be proved with the tools developed in this study.

5. Conjectures for high-order derivatives

In this section we seek conditions that generalize our previous hypothesis in order to obtain higher-order derivatives for the homeomorphism. To begin we introduce a new condition to replace (d5):

(d5′) For every fixed $k \in \mathbb{Z}^+$, $u \mapsto f(k,u)$ is a C^r map and there are functions $\Gamma_s : \mathbb{Z}^+ \to \mathbb{R}^+$, $0 \leqslant s \leqslant r$, $m \in \mathbb{Z}^+$ such that:

$$\left\| \frac{\partial^s f}{\partial u^s}(j,u) \right\| \leqslant \Gamma_s(j), \quad \text{for every } j \in \mathbb{Z}^+.$$

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

CONJECTURE 5.1. *Suppose that conditions (d1), (d4) and (d5′) are verified. Then, there are functions* $\pi_{s,m} : \mathbb{Z}^+ \to \mathbb{R}^+$, *for every* $1 \leqslant s \leqslant r$ *and* $m \in \mathbb{Z}^+$ *such that*

$$\left\| \frac{\partial^s y}{\partial \eta^s}(j, m, \eta) \right\| \leqslant \pi_{s,m}(j), \quad \textit{for every } j \geqslant m. \tag{5.1}$$

Moreover, the family of functions $\{\pi_{s,m} : 1 \leqslant s \leqslant r,\ m \in \mathbb{Z}^+\}$ *can be recursively found, i.e. each* $\pi_{s,m}$ *can be written as a function of those* Γ_t *and* $\pi_{t,m}$ *with* $t \leqslant s$.

Note that conjecture 5.1 is true for $r = 1$ and $r = 2$, since in the proof of lemma 3.1 we showed that $\pi_{1,m} = \mathcal{A}_m$ verifies (5.1) for $s = 1$ and in lemma 4.4 we proved that

$$\pi_{2,m}(j) := \sum_{i=0}^{j-m-1} \left[\Gamma_2(m+i) \prod_{p=m}^{m+i-1} \Big(\|A(p)\| + \Gamma_1(p) \Big)^2 \prod_{p=m+i+1}^{j-1} \|A(p)\| + \Gamma_1(p) \right]$$

satisfies (5.1) for $s = 2$. Moreover, the identity

$$\pi_{2,m}(j) = \pi_{1,m}(j) \sum_{i=0}^{j-m-1} \frac{\Gamma_2(m+i)}{\|A(m+i)\| + \Gamma_1(m+i)} \pi_{1,m}(m+i)$$

shows the recursive relation we established. With these new notations, let us consider the following conditions:

(d2′) For each fixed $m \in \mathbb{Z}^+$, we have

$$\sum_{j=m}^{\infty} \|\mathcal{G}(0, j+1)\| \, \Gamma_0(j) < +\infty.$$

(d6′) For each fixed $m \in \mathbb{Z}^+$, we have

$$\sum_{j=m}^{\infty} \|\mathcal{G}(0, j+1)\| \, \Gamma_1(j) \pi_{1,m}(j) < +\infty.$$

(d8) For each fixed $m \in \mathbb{Z}^+$, we have

$$\sum_{j=m}^{\infty} \|\mathcal{G}(0, j+1)\| \, \{\pi_{2,m}(j)\Gamma_1(j) + \Gamma_2(j)\pi_{1,m}(j)\} < +\infty.$$

A short calculus shows that (d2′) is equivalent to (d2). It is immediate that (d6) is a more exigent condition than (d6′), while (d6′) does not imply (d6) in general, but it is easy to see that if we replace (d6) with (d6′) in all of our previous results, all conclusions follow in the same fashion. Similarly, (d8) replaces condition (4.3) from lemma 4.1.

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

The following construction shows the utility of this conjecture. Suppose conjecture 5.1 is true. Choose a fixed $m \in \mathbb{Z}^+$ and consider the set of functions:

$$\mathfrak{S}_m = \left\{ \Gamma_s \prod_{k=1}^{r} \pi_{k,m}^{e_k} : 0 \leqslant s \leqslant r, e_k \in \mathbb{Z}^+ \right\}.$$

Consider the module $\mathbb{Z}[\mathfrak{S}_m]$, that is, all $\mathbb{Z}$-linear combinations of the functions in $\mathfrak{S}_m$, and define the $\mathbb{Z}$-linear map $\mathbb{D}_m : \mathbb{Z}[\mathfrak{S}_m] \to \mathbb{Z}[\mathfrak{S}_m]$ given by

- $\mathbb{D}_m(\Gamma_s) = \Gamma_{s+1}\pi_{1,m}$, for every $0 \leqslant s < r - 1$ and $\mathbb{D}_m(\Gamma_r) = 0$.
- $\mathbb{D}_m(\pi_{s,m}) = \pi_{s+1,m}$, for every $1 \leqslant s < r - 1$ and $\mathbb{D}_m(\pi_{r,m}) = 0$.
- $\mathbb{D}_m(fg) = \mathbb{D}_m(f)g + f\mathbb{D}_m(g)$, for every $f, g \in \mathbb{Z}[\mathfrak{S}_m]$ such that $fg \in \mathbb{Z}[\mathfrak{S}_m]$.

Now we can introduce our final condition:

(d9) For each fixed $m \in \mathbb{Z}^+$ and every $0 \leqslant s \leqslant r$, we have

$$\sum_{j=m}^{\infty} \left\| \mathcal{G}(0, j+1) \right\| \mathbb{D}_m^s(\Gamma_0)(j) < +\infty.$$

It is easy to see (d9) corresponds to (d2′) if $r = 0$, to (d2′) and (d6′) if $r = 1$ and to (d2′), (d6′) and (d8) if $r = 2$. Thus, we can rewrite some of our previous results as follows:

- **Corollary 3.3.** If conditions (d1), (d3), (d4), (d5′) and (d9) with $r = 1$ are verified, then $\eta \mapsto G(k, \eta)$ is a C^1 map and $\xi \mapsto H(k, \xi)$ is almost everywhere differentiable for every fixed $k \in \mathbb{Z}^+$.
- **Theorem 3.12.** If conditions (d1), (d3), (d4), (d5′) and (d9) with $r = 1$, and (d7) are verified, then systems (2.1) and (2.2) are C^1-topologically equivalent on $\mathbb{Z}^+$.
- **Corollary 4.2.** If conditions (d1), (d3), (d4), (d5′) and (d9) with $r = 2$ are verified, then $\eta \mapsto G(k, \eta)$ is a C^2 map and $\xi \mapsto H(k, \xi)$ is almost everywhere two times differentiable for every fixed $k \in \mathbb{Z}^+$.
- **Theorem 4.3.** If conditions (d1), (d3), (d4), (d5′) and (d9) with $r = 2$, and (d7) are verified, then systems (2.1) and (2.2) are C^2-topologically equivalent on $\mathbb{Z}^+$.

Naturally, this raises the following conjectures:

CONJECTURE 5.2. *Suppose conjecture 5.1 is true for some $r \in \mathbb{N}$, $r \geqslant 3$. If conditions (d1), (d3), (d4), (d5′) and (d9) are verified, then systems (2.1) and (2.2) are topologically equivalent on $\mathbb{Z}^+$ with maps H and G such that for every fixed $k \in \mathbb{Z}^+$ $\eta \mapsto G(k, \eta)$ is a C^r map and $\xi \mapsto H(k, \xi)$ is almost everywhere r times differentiable.*

https://doi.org/10.1017/prm.2023.25 Published online by Cambridge University Press

Conjecture 5.3. *Suppose conjecture 5.1 is true for some $r \in \mathbb{N}$, $r \geqslant 3$. If conditions (d1), (d3), (d4), (d5′), (d7) and (d9) are verified, then systems (2.1) and (2.2) are C^r-topologically equivalent on $\mathbb{Z}^+$.*

Remark 5.4. Notice that in (5.1) the bounds for each partial derivative of order $1 \leqslant s \leqslant r$ are uniform in η for each $j \geqslant m$ where m is the initial time of the solution of (2.2). Thus, these conjectures are interesting in the sense that is not enough that the partial derivatives exist in order to conclude that the homeomorphisms to have some class of differentiability (actually they always exists if (d5′) is granted), rather the uniform boundedness of these derivatives is what plays an important role concerning the regularity of such homeomorphisms, and this is what was showed in the proof of the cases $r = 1, 2$.

Remark 5.5. It is easy to see that a positive response to conjecture 5.1 immediately implies a positive response to conjecture 5.2, that in turn implies a positive response to conjecture 5.3.

Acknowledgments
We are very grateful to an anonymous referee. We deeply appreciate the constructive criticism, kind suggestions and careful insight which assisted us to obtain a better understanding and writing of these results.

Financial support
This research has been partially supported by ANID, Beca Nacional de Magister 22200774 and FONDECYT Regular 1200653.

References

1 L. Backes and D. Dragičević, Smooth linearization of nonautonomous coupled systems. arXiv:2202.12367.
2 L. Barreira and C. Valls. A Grobman–Hartman theorem for nonuniformly hyperbolic dynamics. *J. Differ. Equations* **228** (2006), 285–310.
3 G. R. Belickiĭ. Functional equations and the conjugacy of local diffeomorphisms of a finite smoothness class. *Dokl. Akad. Nauk SSSR* **202** (1972), 255–258.
4 G. R. Belickiĭ. Equivalence and normal forms of germs of smooth mappings. *Uspekhi Mat. Nauk* **33** (1978), 95–155, 263.
5 Á. Castañeda, P. González and G. Robledo. Topological equivalence of nonautonomous difference equations with a family of dichotomies on the half line. *Commun. Pure Appl. Anal.* **20** (2021), 511–532.
6 Á. Castañeda, P. Monzón and G. Robledo. Smoothness of topological equivalence on the half-line for nonautonomous systems. *Proc. R. Soc. Edinburgh Sect. A* **150** (2020), 2484–2502.
7 Á. Castañeda and G. Robledo. Differentiability of Palmer's linearization theorem and converse result for density functions. *J. Differ. Equations* **259** (2015), 4634–4650.
8 C. V. Coffman and J. J. Schäffer. Dichotomies for linear difference equations. *Math. Ann.* **172** (1967), 139–166.
9 W. A. Coppel. *Dichotomies in stability theory.* Lecture Notes in Mathematics (Berlin: Springer-Verlag, 1978).
10 L. V. Cuong, T. S. Doan and S. Siegmund. A Sternberg theorem for nonautonomous differential equations. *J. Dyn. Differ. Equations* **31** (2019), 1279–1299.
11 D. Dragičević, W. Zhang and W. Zhang. Smooth linearization of nonautonomous difference equations with a nonuniform dichotomy. *Math. Z.* **292** (2019), 1175–1193.

12 D. Dragičević, W. Zhang and W. Zhang. Smooth linearization of nonautonomous differential equations with a nonuniform dichotomy. *Proc. Lond. Math. Soc.* **121** (2020), 32–50.

13 P. Hartman. A lemma in the theory of structural stability of differential equations. *Proc. Am. Math. Soc.* **11** (1960), 610–620.

14 P. E. Kloeden and M. Rasmussen *Nonautonomous dynamical systems*. Mathematical Surveys and Monographs, vol. 176 (Providence, RI: AMS, 2011).

15 K. J. Palmer. A generalization of Hartman's linearization theorem. *J. Math. Anal. Appl.* **41** (1973), 753–758.

16 R. Plastock. Homeomorphisms between Banach spaces. *Trans. Am. Math. Soc.* **200** (1974), 169–183.

17 C. C. Pugh. On a theorem of P. Hartman. *Am. J. Math.* **91** (1969), 363–367.

18 R. J. Sacker and G. R. Sell. A spectral theory for linear differential systems. *J. Differ. Equations* **27** (1978), 320–358.

19 J. L. Shi and K. Q. Xiong. On Hartman's linearization theorem and Palmer's linearization theorem. *J. Math. Anal. Appl.* **192** (1995), 813–832.

20 S. Sternberg. Local contractions and a theorem of Poincaré. *Am. J. Math.* **79** (1957), 809–824.

21 S. Sternberg. On the structure of local homeomorphisms of Euclidian n-space, I. *Am. J. Math.* **80** (1958), 623–631.