*J. Austral. Math. Soc.* (*Series A*) 31 (1981), 163–174

# CONVERGENCE IN $k$TH VARIATION AND $RS_k$ INTEGRALS

U. DAS and A. G. DAS

(Received 10 November 1978; revised 6 December 1979)

Communicated by E. Strzelecki

**Abstract**

In recent papers, Russell introduced the notions of functions of bounded $k$th variation ($BV_k$ functions) and the $RS_k$ integral. Das and Lahiri enriched Russell's works along with a convergence formula of $RS_k$ integrals depending on the convergence of integrands. In this paper a convergence theorem analogous to Arzela's dominated convergence theorem has been presented. An investigation to the convergence in $k$th variation has been made leading to some convergence theorems of $RS_k$ integrals depending on the convergence of integrators.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 26 A 42; secondary 26 A 45.

## 1. Preliminaries and definitions

A. M. Russell (1973) obtained the definition of functions of bounded $k$th variation ($BV_k$ functions) along with certain properties of $BV_k$ functions. A. G. Das and B. K. Lahiri (1980a) introduce the notion of $AC_k$ functions and produce certain relations between $AC_k$- and $BV_k$-functions. Russell (1975) obtained later the definition of an integral (the $RS_k$ integral) together with some important properties of the integral. Das and Lahiri (1980b) obtained some other properties of the integral and certain modifications of some results of Russell (1975). A convergence theorem of $RS_k$ integrals appears in Das and Lahiri (1980b) depending on the convergence of integrals. In the present paper the authors present a convergence theorem analogous to Arzela's dominated convergence theorem. The authors also feel that there is an interest in obtaining convergence theorems of $RS_k$ integrals depending on the convergence of integrators. To this end it is desirable to investigate the convergence in $k$th variation. In





https://doi.org/10.1017/S1446788700033449 Published online by Cambridge University Press

the sequel, we shall need the following definitions and results from Das and Lahiri (1980b), Russell (1973) and Russell (1975).

Let $a'$, $a$, $b$, $b'$ be fixed real numbers such that $a' < a < b < b'$ and let $k$ be a positive integer greater than 1. The real-valued functions that occur are defined at least in $[a, b]$.

DEFINITION 1. We denote by $\Pi(x_0, \dots, x_n)$ a subdivision of the closed interval $[a, b]$ of the form

$$a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b.$$

DEFINITION 2. We denote by $\Gamma(x_{-k+1}, \dots, x_{n+k-1})$ a subdivision of the closed interval $[a, b]$ of the form

$$a' \leqslant x_{-k+1} < \cdots < x_0 = a < x_1 < \cdots < x_n$$
$$= b < x_{n+1} < \cdots < x_{n+k-1} \leqslant b'.$$

The norm of the subdivision $\Gamma$, denoted by $\|\Gamma\|$, is the number $\max_{-k+2 \leqslant i \leqslant n+k-1}(x_i - x_{i-1})$. The norm of the subdivision $\Pi$, $\|\Pi\|$, is the number $\max_{1 \leqslant i \leqslant n}(x_i - x_{i-1})$.

DEFINITION 3. Let $x_0, x_1, \dots, x_k$ be $k + 1$ distinct points, not necessarily in the linear order, belonging to $[a, b]$. Define the $k$th divided difference of $f$ as

$$Q_k(f; x_0, x_1, \dots, x_k) = \sum_{i=0}^{k} \left[ f(x_i) \Big/ \prod_{\substack{j=0 \\ j \neq i}}^{k} (x_i - x_j) \right].$$

DEFINITION 4. A function $f$ defined on $[a, b]$ is said to be $k$-convex on $[a, b]$ if and only if $Q_k(f; x_0, x_1, \dots, x_k) \geqslant 0$ for all choices of the distinct points $x_0, x_1, \dots, x_k$ in $[a, b]$.

DEFINITION 5. Let $x, x_1, \dots, x_k$ be $k + 1$ distinct points in $[a, b]$. Suppose that $h_i = x_i - x$ when $i = 1, 2, \dots, k$ and that $0 < |h_1| < |h_2| < \cdots < |h_k|$. Then define the $k$th Riemann $*$ derivative by

$$D^k f(x) = k! \lim_{h_k \to 0} \lim_{h_{k-1} \to 0} \cdots \lim_{h_1 \to 0} Q_k(f; x, x_1, \dots, x_k),$$

if the iterated limit exists. The right and left Riemann $*$ derivatives are defined in the obvious way.

When the $k$th Riemann derivative, in the sense of Bullen (1971), exists for $h_0 = 0$, it coincides with the $k$th Riemann $*$ derivative.

DEFINITION 6. The total $k$th variation of $f$ in $[a, b]$ is defined by

$$V_k[f; a, b] = \sup_{\Pi} \sum_{i=0}^{n-k} (x_{i+k} - x_i) |Q_k(f; x_i, x_{i+1}, \dots, x_{i+k})|.$$

If $V_k[f; a, b] < \infty$, we say that $f$ is of bounded $k$th variation on $[a, b]$ and write $f \in BV_k[a, b]$. The summations over which the supremum is taken are called approximating sums for $V_k[f; a, b]$.

DEFINITION 7. The total outer $k$th variation of $f$ on $[a, b]$ is defined by

$$W_k[f; a, b] = \sup_{\Gamma} \sum_{i=-k+1}^{n-1} (x_{i+k} - x_i)|Q_k(f; x_i, \ldots, x_{i+k})|.$$

If $W_k[f; a, b] < \infty$ we say that $f$ is of bounded outer $k$th variation on $[a, b]$ and write $f \in BW_k[a, b]$.

DEFINITION 8. The integral $\int_a^b f(x)(d^k g(x)/dx^{k-1})$ is the real number $I$, if it exists uniquely and if for each $\varepsilon > 0$ there is a real number $\delta(\varepsilon) > 0$ such that when $x_i \leqslant \xi_i \leqslant x_{i+k}$, $i = -k+1, \ldots, n-1$,

$$\left| I - \sum_{i=-k+1}^{n-1} f(\xi_i)\left[Q_{k-1}(g; x_{i+1}, \ldots, x_{i+k})\right] - \left[Q_{k-1}(g; x_i, \ldots, x_{i+k-1})\right] \right| < \varepsilon$$

whenever $\|\Gamma\| < \delta(\varepsilon)$.

If the integral exists we will write $(f, g) \in RS_k[a, b]$, and we will refer to the integral as an $RS_k$ integral.

DEFINITION 9. If in Definition 8 we consider only $\Pi$ subdivision of $[a, b]$, so that we necessarily consider only functions $f$ and $g$ defined on $[a, b]$, then we obtain an $RS_k^*$ integral, $*\int_a^b f(x)(d^k g(x)/dx^{k-1})$.

The notations and further definitions which are not noted here may be seen in Russell (1973) and Russell (1975). We simply note the following results from Das and Lahiri (1980b) for ready references.

THEOREM 1. *Suppose that the $(k-1)$th Riemann $*$ derivatives of $g$ exist at $a$ and $b$. A necessary and sufficient condition that $(f, g) \in RS_k[a, b]$ is that $(f, g) \in RS_k^*[a, b]$. In either case*

$$\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = *\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

THEOREM 2. *Let $D^{k-1}g(c)$ exist where $a < c < b$. If $g$ is $k$ convex in $[a', b']$ and $(f, g) \in RS_k[a, b]$, then $(f, g) \in RS_k[a, c]$ and $(f, g) \in RS_k[c, b]$, and*

$$\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = \int_a^c f(x)\frac{d^k g(x)}{dx^{k-1}} + \int_c^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

https://doi.org/10.1017/S1446788700033449 Published online by Cambridge University Press

THEOREM 3. *Let* $\{f_p(x)\}$ *be a sequence of functions which converges uniformly to* $f(x)$ *on* $[a', b']$. *If* $g$ *is* $k$-*convex in* $[a', b']$ *and for all* $p$, $(f_p, g) \in RS_k[a, b]$, *then* $(f, g) \in RS_k[a, b]$ *and*

$$\lim_{p\to\infty} \int_a^b f_p(x)\frac{d^k g(x)}{dx^{k-1}} = \int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

REMARK 1. We remark that Theorems 2 and 3 can also be obtained for $RS_k^*$ integrals.

## 2. Convergence in $k$th variation

Let $\{F_p(x)\}$ be a sequence of real functions in $[a, b]$ which is assumed, throughout the section, to be convergent and to converge to $F(x)$, say.

It is easily observed that $V_k[F; a, b] \leqslant \liminf_{p\to\infty} V_k[F_p; a, b]$.

PROPERTY $A_k$. A sequence $\{F_p(x)\}$ is said to satisfy Property $A_k$ on $[a, b]$ if a subdivision $\Pi_0(\xi_0, \xi_1, \ldots, \xi_\mu)$, $\mu \geqslant 2k$, of $[a, b]$ and a positive integer $q$ exist such that

$$|Q_k(F_p; x_0, x_1, \ldots, x_k)| \geqslant |Q_k(F_q; x_0, x_1, \ldots, x_k)|$$

when $p > q$ and for each set of $k+1$ distinct points $x_r$, $r = 0, 1, \ldots, k$, belonging to $[\xi_i, \xi_{i+2k}]$, $i = 0, 1, \ldots, \mu - 2k$.

REMARK 2.1. The case $k = 1$ demands a simpler definition:

A sequence $\{F_p(x)\}$ is said to satisfy Property $A_1$ on $[a, b]$ if a subdivision $\Pi_0(a = \xi_0, \xi_1, \ldots, \xi_\mu = b)$ of $[a, b]$ and a positive integer $q$ exist such that

$$|F_p(x_1) - F_p(x_0)| \geqslant |F_q(x_1) - F_q(x_0)|$$

when $p > q$ and for every pair $x_0$, $x_1$ belonging to $[\xi_i, \xi_{i+1}]$, $i = 0, 1, \ldots, \mu - 1$. We observe that for distinct elements $x_0$, $x_1$ the above inequality is the same as that in Property $A_k$ $(k = 1)$, but the fundamental difference is that the containing subintervals are disjoint save the end points in this case contrary to the case of $k \geqslant 2$.

Consider the sequence $\{F_p(x)\}$ in $[a, b]$ defined by

$$F_p(x) = a_p x^n, \qquad n \geqslant k, |a_p| \geqslant |a_q| \quad \text{for } p > q.$$

By Milne-Thomson (1965), §1.31 p. 7, we obtain

$$Q_k(F_p; x_0, x_1, \ldots, x_k) = a_p \sum x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_k^{\alpha_k}$$

where the summation is extended to all positive integers including zero which satisfy the relation $\sum_{r=0}^k \alpha_r = n - k$. Obviously then $\{F_p(x)\}$ possesses property $A_k$.

Let $\mathcal{C}$ denote the collection of all $\Pi$ subdivisions of $[a, b]$ and let

$$V_k[\varphi; \Pi] = \sum_{i=0}^{n-k} (x_{i+k} - x_i)|Q_k(\varphi; x_i, \ldots, x_{i+k})|.$$

Lemma 2.1. $\lim_{p\to\infty} V_k[F_p; \Pi] = V_k[F; \Pi]$ *for every* $\Pi \in \mathcal{C}$.

Proof. Let $\varepsilon > 0$ be arbitrary. We consider a subdivision $\Pi(x_0, x_1, \ldots, x_\mu)$ of $[a, b]$, and let $\delta = \min_{0 \leqslant i \leqslant \mu-1}(x_{i+1} - x_i)$. There exists a positive integer $p_i$ $(i = 0, 1, \ldots, \mu)$ such that $|F_p(x_i) - F(x_i)| < \varepsilon\delta^k/\mu(k+1)(b-a)$ whenever $p \geqslant p_i$. Then for $p \geqslant P = \max_i p_i$ and for each $i$, $0 \leqslant i \leqslant \mu - k$

$$\begin{aligned} &\big|\,|Q_k(F_p; x_i, \ldots, x_{i+k})| - |Q_k(F; x_i, \ldots, x_{i+k})|\,\big| \\ &\quad \leqslant \sum_{r=i}^{i+k} \left| \{F_p(x_r) - F(x_p)\} \Big/ \prod_{\substack{s=i\\ s\neq r}}^{i+k} (x_r - x_s) \right|, \quad \text{by Definition 3} \\ &\quad < \varepsilon/\mu(b-a). \end{aligned}$$

It then follows that for $p \geqslant P$

$$\begin{aligned} &\Bigg| \sum_{i=0}^{\mu-k} (x_{i+k} - x_i)|Q_k(F_p; x_i, \ldots, x_{i+k})| \\ &\qquad - \sum_{i=0}^{\mu-k} (x_{i+k} - x_i)|Q_k(F; x_i, \ldots, x_{i+k})| \Bigg| \\ &\quad \leqslant \sum_{i=0}^{\mu-k} (x_{i+k} - x_i)\varepsilon/\mu(b-a) < \varepsilon \end{aligned}$$

and the lemma is proved.

Lemma 2.2. *If $K$ is a finite positive number and if for all $p$, $V_k[F_p; a, b] \leqslant K$, then $V_k[F; a, b] \leqslant K$.*

Proof. The proof follows directly from Definition 6 or else easily using Lemma 2.1.

Lemma 2.3. *If the sequence $\{F_p(x)\}$ possesses Property $A_k$ on $[a, b]$ and if $V_k[F_p; a, b] > K$ for all $p$, $K$ being a finite positive number, then a subdivision $\Pi \in \mathcal{C}$ exists such that $V_k[F_p; \Pi] > K$ for all $p$.*

Proof. A subdivision $\Pi_0(\xi_0, \xi_1, \ldots, \xi_\mu)$ of $[a, b]$ and a positive integer $q$ exist such that

$$|Q_k(F_p; x_i, x_{i+1}, \ldots, x_{i+k})| \geqslant |Q_k(F_q; x_i, x_{i+1}, \ldots, x_{i+k})|$$

https://doi.org/10.1017/S1446788700033449 Published online by Cambridge University Press

when $p > q$ and for each set of $k+1$ distinct points $x_r$, $r = i, \ldots, i+k$, belonging to $[\xi_i, \xi_{i+2k}]$, $i = 0, 1, \ldots, \mu - k$.

Let $\Pi_1 \in \mathcal{C}$ which contains all the points of subdivision of $\Pi_0$. Using Property $A_k$ it is easily seen that

$$V_k[F_p; \Pi_1] \geqslant V_k[F_q; \Pi_1] \quad \text{for all } p > q. \tag{2.1}$$

Since $V_k[F_i; a, b] > K$, $1 \leqslant i \leqslant q$ an element $\Pi_2 \in \mathcal{C}$ exists such that

$$V_k[F_i; \Pi_2] > K \quad \text{for each } i,\ 1 \leqslant i \leqslant q. \tag{2.2}$$

Let $\Pi$ be a subdivision in $\mathcal{C}$ containing all the points of subdivisions of $\Pi_1$ and $\Pi_2$. By Russell (1973), Theorem 3, and the inequalities (2.1) and (2.2) above, it follows that

$$V_k[F_p; \Pi] > K \quad \text{for all } p.$$

This proves the lemma.

THEOREM 2.1. *If $\{F_p(x)\}$ and all its subsequences possess Property $A_k$ on $[a, b]$ and if $V_k[F; a, b] < K$, then $V_k[F_p; a, b] \leqslant K$ for all $p$ except possibly a finite number.*

PROOF. If possible, we suppose that the theorem is false. There exists a sequence of positive integers $\{p_i\}$ with $p_i \to \infty$ such that $V_k[F_{p_i}; a, b] > K$. Applying Lemma 2.3 and then Lemma 2.1, it follows that $V_k[F; a, b] \geqslant K$. The contradiction proves the theorem.

THEOREM 2.2. *If $\{F_p(x)\}$ and all its subsequences possess the Property $A_k$ on $[a, b]$ and $V_k[F_p; a, b]$ is finite for each $p$, then*

$$\lim_{p\to\infty} V_k[F_p; a, b] = V_k[F; a, b].$$

PROOF. We are to dispose of the following two cases:

$$\text{(I) } V_k[F; a, b] < +\infty \quad \text{and} \quad \text{(II) } V_k[F; a, b] = +\infty.$$

*Case* I. Let $K$ denote a positive number such that $V_k[F; a, b] < K$. Then, by Theorem 2.1, there exists an integer $p_0$ such that

$$V_k[F_p; a, b] \leqslant K \quad \text{for } p \geqslant p_0.$$

Let $\Lambda = \overline{\lim}\, V_k[F_p; a, b]$ and $\lambda = \underline{\lim}\, V_k[F_p; a, b]$. There exists a sequence $\{p_i\}$ of positive integers such that $\lim_{i\to\infty} V_k[F_{p_i}; a, b] = \Lambda$.

If $\varepsilon > 0$ is arbitrary, an integer $i_0$ exists such that

$$\Lambda - \varepsilon < V_k[F_{p_i}; a, b] < \Lambda + \varepsilon \quad \text{when } i \geqslant i_0.$$

So, by Lemma 2.2,

$$V_k[F; a, b] \leqslant \Lambda + \varepsilon. \tag{2.3}$$

https://doi.org/10.1017/S1446788700033449 Published online by Cambridge University Press

Again, by Lemma 2.3, an element $\Pi \in \mathcal{C}$ exists such that $V_k[F_{p_i}; \Pi] > \Lambda - \varepsilon$ for $i \geqslant i_0$. Letting $i \to \infty$, we obtain, by Lemma 2.1, $V_k[F; \Pi] \geqslant \Lambda - \varepsilon$ and so

(2.4) $$V_k[F; a, b] \geqslant \Lambda - \varepsilon.$$

Combining (2.3) and (2.4) we get $\Lambda - \varepsilon \leqslant V_k[F; a, b] \leqslant \Lambda + \varepsilon$. As $\varepsilon > 0$ is arbitrary, it follows that $V_k[F; a, b] = \Lambda$. It can similarly be shown that $V_k[F; a, b] = \lambda$ and hence

$$\lim_{p \to \infty} V_k[F_p; a, b] = V_k[F; a, b].$$

*Case* II. In this case the sequence $\{V_k[F_p; a, b]\}$ cannot be bounded. If possible, let $\underline{\lim}\ V_k[F_p; a, b] = \lambda$. Then as in Case I, it follows that $V_k[F; a, b] = \lambda$ which contradicts the hypothesis. Hence $\lim_{p\to\infty}[F_p; a, b] = +\infty$. This completes the proof.

NOTE 2.1. If $g$ is $k$-convex in $[a, c]$ and $k$-concave in $[c, b]$ where $a < c < b$ and if $D^{k-1}g(x)$ exists everywhere in $[a, b]$, then

$$(k-1)!\, V_k[g; a, b] = |D^{k-1}g(a) - D^{k-1}g(c)| + |D^{k-1}g(c) - D^{k-1}g(b)|.$$

This result enables us sometimes to evaluate $V_k[g; a, b]$ independently.

REMARK 2.2. For the validity of Theorem 2.2, the convergence of the sequence $\{F_p(x)\}$ or even the uniform convergence is not sufficient. This is shown by the following example.

Let $F_p(x) = (1 - \cos px)/p^2$, $0 \leqslant x \leqslant \pi$. Clearly $\{F_p(x)\}$ converges uniformly to $F(x) \equiv 0$ in $[0, \pi]$. We observe that $F_p'(x)$ exists in $[0, \pi]$ and $F_p'(x) = (\sin px)/p$, $0 \leqslant x \leqslant \pi$. Also, in view of Russell (1973), Theorems 7 and 13, and 2-convex property of $F_p(x)$ in a subinterval in which $F_p'(x)$ is increasing, we have

$$V_2(F_p; 0, \pi) = V(F_p'; 0, \pi) = 2 \quad \text{for all } p.$$

But $V_2(F; 0, \pi) = 0$ and so

$$\lim_{p\to\infty} V_2(F_p; 0, \pi) \neq V_2(F; 0, \pi).$$

## 3. Sequence of $RS_k$ integrals

We consider a $\Gamma(x_{-k+1}, \ldots, x_{n+k-1})$ subdivision of $[a, b]$ and make the definitions $M_i$, $m_i$, $S$, $s$ as in Russell (1975), Lemma 4. We note here that Lemma 4 of Russell (1975) is still true if $f$ is simply bounded in $[a', b']$. It can further be observed that no lower approximating sum can exceed any upper approximating

https://doi.org/10.1017/S1446788700033449 Published online by Cambridge University Press

sum for $RS_k$ integral with $g$ $k$-convex in $[a', b']$. We define

$$\underline{\int_a^b} f(x)\frac{d^k g(x)}{dx^{k-1}} = \sup_{\Gamma} s \quad \text{and} \quad \int_a^{\bar{b}} f(x)\frac{d^k g(x)}{dx^{k-1}} = \inf_{\Gamma} S.$$

It readily follows that if $g$ is $k$-convex in $[a', b']$, then

$$\underline{\int_a^b} f(x)\frac{d^k g(x)}{dx^{k-1}} \leqslant \int_a^{\bar{b}} f(x)\frac{d^k g(x)}{dx^{k-1}};$$

and $(f, g) \in RS_k[a, b]$ if and only if the equality sign holds.

Following Luxemburg (1971) it is not difficult to obtain an Arzela's dominated convergence theorem for $RS_k$ integral:

THEOREM 3.1. *Let $g(x)$ be $k$-convex in $[a', b']$ and let $\{f_p(x)\}$ be a sequence of functions which converges to $f(x)$ in $[a', b']$. If for all $p$, $(f_p, g) \in RS_k[a, b]$ and $(f, g) \in RS_k[a, b]$ and if there exists a constant $M > 0$ satisfying $|f_p(x)| \leqslant M$ for all $x \in [a, b]$ and for all $p$, then*

$$\lim_{p\to\infty} \int_a^b f_p(x)\frac{d^k g(x)}{dx^{k-1}} = \int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

To establish the proof of Theorem 3.1, we simply require Theorem 1 of Russell (1975), Theorem 3 of §1, and the obvious inequality $\int_a^b \varphi(x)(d^k h(x)/dx^{k-1}) \geqslant 0$ for $\varphi \geqslant 0$, $h$ being $k$-convex in $[a', b']$ and $(\varphi, h) \in RS_k[a, b]$.

For the sake of simplicity we prove the remaining results for $RS_k^*$ integral. These can also be proved for $RS_k$ integral by proving the results of Section 2 for outer $k$th variation.

LEMMA 3.1. *If $(f, g) \in RS_k^*[a, b]$ and $f$ bounded in $[a, b]$, then*

$$\left| {}_*\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} \right| \leqslant M(f)V_k[g; a, b],$$

*where* $M(f) = \max_{a\leqslant x\leqslant b}|f(x)|$.

PROOF. Consider any $\Pi(x_0, x_1, \ldots, x_n)$ subdivision of $[a, b]$ and choose $\xi_i$, $x_i \leqslant \xi_i \leqslant x_{i+k}$, $0 \leqslant i \leqslant n-k$, arbitrarily. The lemma follows from the inequalities

$$\begin{aligned}\left|\sum_{i=0}^{n-k} f(\xi_i)\left[Q_{k-1}(g; x_{i+1}, \ldots, x_{i+k}) - Q_{k-1}(g; x_i, \ldots, x_{i+k-1})\right]\right| \\ \leqslant M(f)\sum_{i=0}^{n-k}|Q_{k-1}(g; x_{i+1}, \ldots, x_{i+k}) - Q_{k-1}(g; x_i, \ldots, x_{i+k-1})| \\ \leqslant M(f)V_k[g; a, b].\end{aligned}$$

https://doi.org/10.1017/S1446788700033449 Published online by Cambridge University Press

THEOREM 3.2. *Let $f$ be bounded in $[a, b]$ and let $\{g_p(x)\}$ be a sequence of functions which converges to $g(x)$ in $[a, b]$ with $\{V_k[g_p; a, b]\}$ converging to $V_k[g; a, b]$. If for all $p$, $(f, g_p) \in RS_k^*[a, b]$ and also $(f, g) \in RS_k^*[a, b]$, then*

$$\lim_{p\to\infty} *\int_a^b f(x)\frac{d^k g_p(x)}{dx^{k-1}} = *\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

PROOF. Let $\varepsilon > 0$ be arbitrary, then there exists a positive integer $p_0$ such that for $p \geqslant p_0$

$$|V_k[g_p; a, b] - V_k[g; a, b]| < \varepsilon/M(f),$$

where $M(f) = \max_{a \leqslant x \leqslant b}|f(x)|$.

Using Russell (1975), Theorem 2, and then Lemma 3.1, we obtain

$$\begin{aligned}\left| *\int_a^b f(x)\frac{d^k g_p(x)}{dx^{k-1}} - *\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}\right| &= \left| *\int_a^b f(x)\frac{d^k\{g_p(x) - g(x)\}}{dx^{k-1}}\right| \\ &\leqslant M(f)V_k[g_p - g; a, b] \\ &< \varepsilon\end{aligned}$$

whenever $p \geqslant p_0$. This proves the theorem.

Convergence of $\{V_k[g_p; a, b]\}$ to $V_k[g; a, b]$ in Theorem 3.2 may be obtained by Property $A_k$. In that case Theorem 3.2 takes the form:

THEOREM 3.3. *Let $f$ be bounded in $[a, b]$ and let $\{g_p(x)\}$ be a sequence of functions which converges to $g(x)$ in $[a, b]$. Let $\{g_p(x)\}$ and all its subsequences possess Property $A_k$ and $V_k[g_p; a, b]$ is finite for all $p$. If for all $p$, $(f, g_p) \in RS_k^*[a, b]$ and also $(f, g) \in RS_k^*[a, b]$, then*

$$\lim_{p\to\infty} *\int_a^b f(x)\frac{d^k g_p(x)}{dx^{k-1}} = *\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

We now present a convergence formula similar to that for Stieltjes-integral in Natanson (1961), Theorem 3, p. 233. For this purpose we prove the following two lemmas.

LEMMA 3.2. *If $g \in BV_k[a, b]$, then*
(a) *$D^r g(x)$ are continuous in $[a, b]$, $1 \leqslant r \leqslant k-2$, for $k \geqslant 3$,*
(b) *$D^{k-1}g(x)$ exists in $[a, b]$ except possibly a countable set of points.*

PROOF. (a) Utilising Russell (1973), Theorem 19, the proof is obtained from that of Bullen (1971), Theorem 7(a), simply omitting the last sentence.

https://doi.org/10.1017/S1446788700033449 Published online by Cambridge University Press

The proof may also follow from Russell (1973), Theorem 12 and Milne–Thomson (1965), §1.2(2), p. 6.

(b) In view of Russell (1973), Theorem 19, and Bullen (1971), Theorem 6, $D_+^{k-1}g(x)$ exists in $[a, b)$ and $D_-^{k-1}g(x)$ exists in $(a, b]$. Also if $a < x_0 < x_1 < \cdots < x_{k-1} \leqslant x \leqslant y_0 < y_1 < \cdots < y_{k-1} \leqslant y < z_0 < \cdots < z_{k-1} \leqslant b$, then

$$D_+^{k-1}g(a) \leqslant D_-^{k-1}g(x) \leqslant D_+^{k-1}g(x) \leqslant D_-^{k-1}g(y) \leqslant D_+^{k-1}g(y) \leqslant D_-^{k-1}g(b).$$

Thus $D_-^{k-1}g(x)$, $D_+^{k-1}g(x)$ are monotonic increasing respectively in $(a, b]$, $[a, b)$ and so are continuous in $[a, b]$ except possibly a countable set of points. It, then, follows that $D_-^{k-1}g(x) = D_+^{k-1}g(x)$ in $[a, b]$ except possibly a countable set of points. The lemma is then immediate if $k = 2$. If $k \geqslant 3$, the lemma follows in view of Part(a) above and Bullen (1973), Corollary 3(b).

LEMMA 3.3. *Let $\{g_p(x)\}$ be a sequence of functions, converging uniformly to the function $g(x)$ in $[a, b]$. If $g(x)$ and each $g_p(x)$ belong to $BV_k[a, b]$, then*

$$\lim_{p\to\infty} *\int_a^b \frac{d^k g_p(x)}{dx^{k-1}} = *\int_a^b \frac{d^k g(x)}{dx^{k-1}}.$$

PROOF. The existence of the above integrals follows from Russell (1975), Theorem 11. If $\Pi(\alpha_0, \alpha_1, \ldots, \alpha_n)$ is any subdivision of $[a, b]$ and $S^*(\Pi, 1, g_p)$, $S^*(\Pi, 1, g)$ denote respectively the approximating sums for the above integrals, then

$$S^*(\Pi, 1, g_p) = Q_{k-1}(g_p; \alpha_{n-k}, \ldots, \alpha_n) - Q_{k-1}(g_p; \alpha_0, \ldots, \alpha_{k-1}),$$
$$S^*(\Pi, 1, g) = Q_{k-1}(g; \alpha_{n-k}, \ldots, \alpha_n) - Q_{k-1}(g; \alpha_0, \ldots, \alpha_{k-1}).$$

By Russell (1973), Theorem 4, the approximating sums are bounded independent of $\Pi$.

Let $\varepsilon > 0$ be arbitrary. There exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$\left| S^*(\Pi, 1, g) - *\int_a^b \frac{d^k g(x)}{dx^{k-1}} \right| < \frac{1}{4}\varepsilon \quad \text{whenever } \|\Pi\| < \delta_1.$$

Since $\{g_p(x)\}$ converges uniformly to $g(x)$, there exists a positive integer $p_0$ such that for any $\Pi$-subdivision of $[a, b]$

$$|S^*(\Pi, 1, g_p) - S^*(\Pi, 1, g)| < \frac{1}{4}\varepsilon \quad \text{whenever } p \geqslant p_0.$$

It then follows that

$$\text{(3.1)} \qquad \left| S^*(\Pi, 1, g_p) - *\int_a^b \frac{d^k g(x)}{dx^{k-1}} \right| < \frac{1}{2}\varepsilon \quad \text{whenever } \|\Pi\| < \delta_1 \text{ and } p \geqslant p_0.$$

Also for each $p$ we can choose $\delta_2 = \delta_2(\varepsilon, p) > 0$ such that

$$\text{(3.2)} \qquad \left| S^*(\Pi, 1, g_p) - *\int_a^b \frac{d^k g_p(x)}{dx^{k-1}} \right| < \frac{1}{2}\varepsilon \quad \text{whenever } \|\Pi\| < \delta_2.$$

For each $p \geqslant p_0$ choose $\delta_2$ and then choose a fixed $\Pi$-subdivision of $[a, b]$ with $\|\Pi\| < \delta = \min(\delta_1, \delta_2)$. Then from (3.1) and (3.2) we obtain

$$\left| *\int_a^b \frac{d^k g_p(x)}{dx^{k-1}} - *\int_a^b \frac{d^k g(x)}{dx^{k-1}} \right| < \varepsilon \quad \text{whenever } p \geqslant p_0.$$

This proves the lemma.

THEOREM 3.4. *Let $f(x)$ be continuous in $[a, b]$ and let $\{g_p(x)\}$ be a sequence of functions which converges uniformly to a finite function $g(x)$ in $[a, b]$. If $K$ is a fixed positive number and $V_k[g_p; a, b] \leqslant K$ for all $p$, then*

$$\lim_{p\to\infty} *\int_a^b f(x)\frac{d^k g_p(x)}{dx^{k-1}} = *\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

PROOF. Clearly each $g_p \in BV_k[a, b]$. By Lemma 2.2, $V_k[g; a, b] \leqslant K$ and so $g \in BV_k[a, b]$. The existence of the integrals are, then, ensured by Russell (1975), Theorem 11.

We now establish the equality.

By Lemma 3.2, there exists a subset $E$ of $[a, b]$, where $[a, b] - E$ is countable, such that $g$ and each $g_p$ possess $(k-1)$th Riemann $*$ derivatives at each point of $E$. Let $\varepsilon > 0$ be arbitrary. There exist finite subintervals $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, m-1$, $x_0 = a$, $x_m = b$, $x_i \in E$, $1 \leqslant i \leqslant m-1$, of $[a, b]$ such that oscillation of $f(x)$ in each subinterval is less than $\varepsilon/3K$. In view of Russell (1973), Theorem 19, and Russell (1975), Theorem 1 and Theorem 2 of §1,

$$*\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = \sum_{i=0}^{m-1} *\int_{x_i}^{x_{i+1}} [f(x) - f(x_i)]\frac{d^k g(x)}{dx^{k-1}} + \sum_{i=0}^{m-1} f(x_i) *\int_{x_i}^{x_{i+1}} \frac{d^k g(x)}{dx^{k-1}}.$$

By Lemma 3.1 and Russell (1973), Theorem 7,

$$\left| \sum_{i=0}^{m-1} *\int_{x_i}^{x_{i+1}} [f(x) - f(x_i)]\frac{d^k g(x)}{dx^{k-1}} \right| < \sum_{i=0}^{m-1} \frac{\varepsilon}{3K} V_k[g; x_i, x_{i+1}] = \frac{\varepsilon}{3K} V_k[g; a, b] < \varepsilon/3.$$

https://doi.org/10.1017/S1446788700033449 Published online by Cambridge University Press

We find, therefore, that

$$(3.3)\qquad *\int_a^b f(x)\frac{d^kg(x)}{dx^{k-1}} = \sum_{i=0}^{m-1} f(x_i) * \int_{x_i}^{x_{i+1}} \frac{d^kg(x)}{dx^{k-1}} + \theta\varepsilon/3 \qquad (|\theta| < 1).$$

In the same way, we can show that for all $p$

(3.4)

$$*\int_a^b f(x)\frac{d^kg_p(x)}{dx^{k-1}} = \sum_{i=0}^{m-1} f(x_i) * \int_{x_i}^{x_{i+1}} \frac{d^kg_p(x)}{dx^{k-1}} + \theta_p\varepsilon/3 \qquad (|\theta_p| < 1).$$

By Lemma 3.3, there exist $p_i$, $i = 0, 1, \ldots, m-1$, such that

$$(3.5)\qquad \left| * \int_{x_i}^{x_{i+1}} \frac{d^kg_p(x)}{dx^{k-1}} - * \int_{x_i}^{x_{i+1}} \frac{d^kg(x)}{dx^{k-1}} \right| < \varepsilon/3mM \quad \text{whenever } p > p_i,$$

where $M = \sup_{a<x<b}|f(x)|$. Choosing $p_0 = \max_{0<i<m-1} p_i$ we obtain, from (3.3), (3.4) and (3.5)

$$\left| * \int_a^b f(x)\frac{d^kg_p(x)}{dx^{k-1}} - * \int_a^b f(x)\frac{d^kg(x)}{dx^{k-1}} \right| < \varepsilon \quad \text{whenever } p > p_0.$$

This proves the theorem.

## References


P. S. Bullen (1971), 'A criterion for *n* convexity', *Pacific J. Math.* **36**, 81–98.

A. G. Das and B. K. Lahiri (1980a), 'On absolutely *k*th continuous functions', *Fund. Math.* **105**, 159–169.

A. G. Das and B. K. Lahiri (1980b), 'On $RS_k$ integrals', *Comment. Math. Univ. Carolinae*, Sec. 1, **22**, to appear.

W. A. J. Luxemburg (1971), 'Arzela's dominated convergence theorem for the Riemann integral', *Amer. Math. Monthly* **78**, 970–979.

L. M. Milne-Thomson (1965), *The calculus of finite differences* (Macmillan).

I. P. Natanson (1961), *Theory of functions of a real variable*, Volume I (Ungar, revised addition).

A. M. Russell (1973), 'Functions of bounded *k*th variation', *Proc. London Math. Soc.* (3), **26**, 547–563.

A. M. Russell (1975), 'Stieltjes-type integrals', *J. Austral. Math. Soc. Ser. A* **20**, 431–448.



Department of Mathematics
University of Kalyani
West Bengal, India