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DIRECT DECOMPOSITIONS OF GROUPS WITH FINITELY GENERATED COMMUTATOR QUOTIENT GROUP

RONALD HIRSHON

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Abstract

Let G/G' be finitely generated and let $G = B_1 \times A_1 = B_2 \times A_2 = \ldots = B_i \times A_i = \ldots$ with each B_i isomorphic to a fixed group B which obeys the maximal condition for normal subgroups. Then the A_i represent only finitely many isomorphism classes. We give an example with B infinite cyclic, G/G' free abelian of infinite (countable) rank and such that G is decomposed as above with no two A_i isomorphic.

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1. Introduction

If A and B satisfy $A \times J \approx B \times J$, J infinite cyclic, then A and B have the same finite homomorphic images (Hirshon (1977)). It is easy to choose $A \not\approx B$, A finitely generated and $A \times J \approx B \times J$ (see Hirshon (1969), Baumslag (1977)). These observations lead to a straightforward way of constructing two finitely generated non-isomorphic groups with the same finite homomorphic images. In certain cases A and B can be made to have other prescribed conditions (for example, A and B can be finitely generated torsion-free nilpotent (Hirshon (1977)).

P. Pickel (1974) has constructed an infinitude of isomorphism classes of finitely generated metabelian groups all of which have the same finite homomorphic images. It seems natural to wonder if the ideas above can lead to results like Pickel's. In particular, we ask: When can a group G have infinitely many decompositions

(1) $$G = \langle w_1 \rangle \times A_1 = \langle w_2 \rangle \times A_2 = \ldots = \langle w_i \rangle \times A_i = \ldots$$

with each $\langle w_i\rangle \approx J$ and no two A_i isomorphic? The main purpose of this paper is to show that if G/G' is finitely generated then in a sequence of decompositions (1) of G, the A_i must represent finitely many isomorphism classes. In particular, this holds if G is finitely generated or if G obeys the maximal condition for normal groups.

The above question (1) has, in many cases, a strong connection to the more general question: When can a group G have infinitely many decompositions

$$(1)^* \qquad G = B_1 \times A_1 = B_2 \times A_2 = \ldots = B_i \times A_i = \ldots$$

with each B_i isomorphic to a fixed group B and no two A_i isomorphic? To see the connection, we define a group B to be J replaceable if the isomorphism $C\times B \approx D\times B$ always implies $C\times J \approx D\times J$. For example, a group which obeys the maximal condition for normal subgroups is J replaceable (Hirshon (1977)). Hence if B satisfies the maximal condition for normal subgroups and G/G' is finitely generated in (1)*, the A_i represent finitely many isomorphism classes. Some more general J replaceable groups are given in Hirshon (1978), (1979).

Finally, we give an example of a sequence (1) in which G/G' is free abelian of infinite (countable) rank and the A_i represent infinitely many isomorphism classes.

2. The main result

THEOREM. *Let* $G = \langle w_i\rangle \times A_i$, $i \geqslant 1$, $\langle w_i\rangle \approx J$. *Let* G/G' *be finitely generated. Then the* A_i *represent finitely many isomorphism classes.*

PROOF. We proceed indirectly and suppose that the A_i represent an infinite number of distinct isomorphism classes. Without loss of generality, we may suppose that if $i<j$, then $A_i \not\approx A_j$. Let A_{i_j} $1 \leqslant j < \infty$ be an infinite sequence of distinct A's and let L be their intersection. One can verify that G/L is a torsion-free abelian group. Hence G/L is a finitely generated free abelian group of rank r where r is at most the rank of the torsion-free part of G/G'. We choose the A_{i_j} so that r is as small as possible. Let $B_j = A_{i_j}$ and $u_j = w_{i_j}$. Note that if K represents the intersection of any infinite number of distinct B's, then $K = L$. For $L \subset K \subset G$ and G/K is a free abelian homomorphic image of G/L. If $L \neq K$, the rank of G/K would be less than r. If $Z(B_1) \subset L$ then all the B_i are isomorphic to B_1. For example, if $u_2 = u_1^s b_1$, $b_1 \in B_1$ then b_1 is central and $b_1 \in B_2$ so

$$\langle u_2\rangle \times B_2 = \langle u_2 b_1^{-1} \times B_2 = \langle u_1^s\rangle \times B_2 = \langle u_1\rangle \times B_1,$$

so that $s = 1$ or $s = -1$ and $B_2 \approx B_1$. Hence we may suppose that we can find $b_1 \notin L$, $b_1 \in Z(B_1)$. For infinitely many i then $b_1 \notin B_i$. Hence we may assume that

$b_1 \notin B_i$ for all i (or else we merely change notation). Hence $[(\langle u_1\rangle \times \langle b_1\rangle) Z(B_i)]/Z(B_i)$ is isomorphic to J. Hence $\langle u_1\rangle \times \langle b_1\rangle = \langle g_i\rangle \times \langle b_i\rangle \approx J \times J$ where

$$\langle b_i\rangle = Z(B_i) \cap (\langle u_1\rangle \times \langle b_1\rangle). \text{ Also if } b_i = u_1^{m_i} b_1^{n_i} \text{ then } m_i \neq 0$$

or else n_i is 1 or -1 and $b_1 \in B_i$. Clearly $(m_i, n_i) = 1$.

Note that $r = 1$ implies that all the B_i are identical. Hence $r \geqslant 2$ and B_i/L is of rank $r-1 = q$. Let $B_1 = \langle h_1\rangle \langle h_2\rangle \dots \langle h_q\rangle L$. Let $b_1 = h_1^{c_1} h_2^{c_2} \dots h_q^{c_q}$, mod L. Hence

$$(2) \qquad b_i = u_1^{m_i} h_1^{n_i c_1} h_2^{n_i c_2} \dots h_q^{n_i c_q} l_i, \quad l_i \in L.$$

Let d_i be the greatest common divisor of the integer $m_i, n_i c_1, n_i c_2, \dots, n_i c_q$. If $d_i = 1$, then a set of free generators for G/L may be chosen which contains $b_i L$. This would imply that $\langle b_i\rangle$ is a direct factor of G and hence a direct factor of B_i. But then by Lemma 1 of Hirshon (1977) $B_i \approx B_1$. Hence $d_i \neq 1$, $i > 1$. Since $(m_i, n_i) = 1$, d_i is a divisor of each of the c_j. It follows that we may choose a largest possible divisor d of the c_j, $d > 1$, such that we may write for infinitely many i, that b_i has a dth root, mod L. That is for some $v_i \in G$

$$(3) \qquad b_i = v_i^d, \quad \text{mod } L.$$

Again for ease of notation we may assume that (3) holds for all i (or simply pass to a subsequence and change notation). Let $B_i = \langle c_{i1}\rangle \langle c_{i2}\rangle \dots \langle c_{iq}\rangle L$. Consider the group G_i generated by $u_i^d, c_{il}^d, \dots, c_{iq}^d$ and L. G_i is of index d^r in G. Since G/L has only finitely many subgroups of a given index, infinitely many of the G_i are identical. The argument that follows will depend on the fact that infinitely many of the G_i are identical but not on the nature of the indices i_k with $G_{i_1} = G_{i_2} = \dots$. Hence to avoid an awkward notation let us suppose that we have

$$G_2 = G_4 = G_6 = G_8 = \dots = G_{2n} = \dots.$$

Since $b_{2j} \in G_{2j} = G_2$, we may write

$$(4) \qquad b_{2j} = u_2^{de_j} c_{21}^{de_{j1}} c_{22}^{de_{j2}} \dots c_{2q}^{de_{jq}}, \quad \text{mod } L.$$

Let f_j be the greatest common divisor of the integers $e_j, e_{j1}, e_{j2}, \dots, e_{jq}$. By comparing (2) and (4) with $i = 2j$ we see df_j is a divisor of the integers $c_1, c_2, \dots, c_q$. If $f_j \neq 1$ for infinitely many j, we can find a prime p diving infinitely many f_j and (4) may then be written as

$$b_{2j} = (\tilde{v}_j)^{dp}, \quad \text{mod } L, \quad \text{for some } \tilde{v}_j \in G,$$

for infinitely many j. Since dp is a divisor of $c_1, c_2, \dots, c_q$ this contradicts the maximality of d. Without loss of generality we assume then that $f_j = 1$ for all j.

Now let $\bar{B}_i$ be the subgroup of B generated by L and $c_{i1}^d, c_{i2}^d, \ldots, c_{iq}^d$ so that $G_i = \langle u_i^d \rangle \times \bar{B}_i$. Note that $f_j = 1$ implies that $\langle b_{2j} \rangle$ is a direct factor of G_2 and hence of $\bar{B}_{2j}$. Hence from Lemma 1 of Hirshon (1977) we may deduce that $\bar{B}_2 \approx \bar{B}_{2j}$ for all j. In fact if $Z = Z(G_2)$ one can verify that we can find an isomorphism θ_{2j} of $\bar{B}_2$ onto $\bar{B}_{2j}$ with $b^{-1}\theta_{2j} b \in Z$, $b \in \bar{B}_2$ and such that θ_{2j} fixes L pointwise. Hence for each j we may express the generators c_{2i}^d, $1 \leqslant i \leqslant q$ in terms of the generators $c_{2i,i}^d$, $1 \leqslant i \leqslant q \pmod{ZL}$. Note that modulo the subgroup ZL the matrix of exponents connecting these generators is unimodular. After regrouping terms, we end up with for each j

$$c_{2i}^d = u_{ji}^d z_{ji} l_{ji},$$

where $u_{ji}, u_{j2}, \ldots, u_{jq}$ are free generators of B_{2j}, mod L, and $z_{ji} \in Z$ and $l_{ji} \in L$. Now consider q-tuplets $\bar{S} = (s_1, s_2, \ldots, s_q)$ where $s_i \in Z$. If $\bar{R} = (r_1, r_2, \ldots, r_q)$ define $\bar{R} \approx \bar{S}$ if $r_i = s_i$, mod $Z^d L$, for all i. Since G/L is finitely generated, the above equivalence relation determines finitely many equivalence classes. Let $\bar{R}_i = (z_{i1} z_{i2} \ldots z_{iq})$, $i \geqslant 1$. Choose fixed integer p, m, $p < m$ and with $\bar{R}_p \approx \bar{R}_m$. Hence for $1 \leqslant f \leqslant q$, we have $z_{pf} = z_{mf} z_f^d l_f$, $z_f \in Z$, $l_f \in L$. Hence $(u_{pf}^{-1} u_{mf} z_f^{-1})^d = l_f l_{pf} l_{mf}^{-1}$. Since G/L is torsion free we have $u_{pf}^{-1} u_{mf} z_f^{-1} = \bar{l}_f \in L$. Now it is easy to check that the map

$$u_{pf} \longrightarrow u_{mf}(\bar{l}_f)^{-1}, \quad 1 \leqslant f \leqslant q, \qquad l \longrightarrow l, \quad l \in L,$$

induces an isomorphism of B_{2p} onto B_{2m} contrary to the assumption that no two B's are isomorphic.

We have actually shown somewhat more than the assertion of the theorem. Note that if we have a decomposition (1) with the A_i representing infinitely many isomorphism classes then our arguments show that G/L is not finitely generated. But if $Q_0 = G$, $Q_n = A_1 \cap A_2 \cap \ldots \cap A_n$, $n \geqslant 1$, then either $Q_i = Q_{i+1}$ or $Q_i/Q_{i+1} \approx J$. Also G/Q_n is a finitely generated free abelian group of rank at most n. Moreover, for infinitely many i, $Q_i/Q_{i+1} \approx J$ or else ultimately $Q_i = Q_{i+1}$. This would imply that $L = Q_m$ for some m contradicting the fact that G/L is not finitely generated. We may state then the

COROLLARY. *If in* (1) *the* A_i *represent infinitely many isomorphism classes,* G *has a descending sequence of normal groups* $L_1, L_2, L_3, \ldots, L_n, \ldots$ *with* G/L_n *free abelian of rank* n.

PROOF. Let $L_1 = Q_1 = A_1$. Suppose L_n has been defined with $L_n = Q_{i_n}$ for some i_n. Let i_{n+1} be the smallest integer $i_{n+1} > i_n$ such that $Q_{i_{n+1}}$ is a proper subgroup of Q_n. Take $L_{n+1} = Q_{i_{n+1}}$.

3. An example

The idea of the following example may be briefly outlined as follows. We choose an infinite sequence of groups A_i, $i = 1, 2, \ldots$, with no two A_j isomorphic and such that $A_i \times J \approx B_i \times J$, $A_i \not\approx B_i$. Let G be the (restricted) Cartesian product $G = A_1 \times A_2 \times A_3 \times \ldots$. Then

$$J \times G = (J \times A_i) \times (\underset{k \neq i}{\times} A_k) \approx (J \times B_i) \times (\underset{k \neq i}{\times} A_k) \approx J \times G_i,$$

where

$$G_i = B_i \times (\underset{k \neq i}{\times} A_k).$$

We will show that we can choose the A_i so that if $s \neq t$ then $G_2 \not\approx G_t$.

To construct the A_i, let $p_1, p_2, p_3, \ldots$ be a sequence of distinct primes with $p_i \equiv 3$, mod 8. Let

$$A_i = \langle c_i, d_i;\ c^{-1} d_i c_i = d_i^2,\ d_i^{p_i} = 1 \rangle$$

and let

$$B_i = \langle e_i, f_i;\ e_i^{-1} f_i e_i = f_i^4,\ f_i^{p_i} = 1 \rangle.$$

Then

$$A_i \times J \approx B_i \times J \quad \text{and} \quad A_i \not\approx B_i$$

(see Baumslag (1974)).

Using the A_i, B_i as above, we assert that no two of the G_i are isomorphic. For say, for example, that $G_1 \approx G_2$ and θ is an isomorphism of G_2 onto G_1. Clearly we must have $d_1 \theta = f_1^u$, $(u, p) = 1$, $p = p_1$. Let us write $c_1 \theta = e_1^h f_1^j a$ where a is in $A_2 \times A_3 \times \ldots$. But then from $c_1^{-1} d_1 c_1 = d_1^2$ we obtain $e_1^{-h} f_1^u e_1^h = f_1^{2u}$. Hence $h \neq 0$ and by a change in generators if necessary we may assume $h > 0$. But the last equality implies $f_1^{4^h u} = f_1^{2u}$. Hence $4^h \equiv 2$, mod p. Hence $2^{2h-1} \equiv 1$, mod p. But then if r is the order of 2, mod p, this says that r is a divisor of $2h-1$ so that r is odd. But since $2^{p-1} \equiv 1$, mod p, r is also a divisor of $p-1$ and hence of $(p-1)/2$. Hence $2^{(p-1)/2} \equiv 1$, mod p. However, $2^{(p-1)/2} \equiv -1$, mod p (Hardy and Wright (1960), Theorems 83 and 95) a contradiction.

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Mathematics Department
Polytechnic Institute of New York
333 Jay Street
Brooklyn, N.Y. 11201
U.S.A.