COMPOSITIO MATHEMATICA

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Compositio Math. **146** (2010), 129–144.

doi:10.1112/S0010437X09004461

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

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Abstract

We introduce in this paper a hypercohomology version of the resonance varieties and obtain some relations to the characteristic varieties of rank one local systems on a smooth quasi-projective complex variety M. A logarithmic resonance variety is also considered and, as an application, we determine the first characteristic variety of the configuration space of n distinct labeled points on an elliptic curve. Finally, for a logarithmic 1-form α on M we investigate the relation between the resonance degree of α and the codimension of the zero set of α on a good compactification of M. This question was inspired by the recent work by Cohen, Denham, Falk and Varchenko.

1. Introduction

Let M be a connected CW-complex with finitely many cells in each dimension and let $\mathbb{T}(M) = \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ be the character variety of M. This is an algebraic group whose identity irreducible component is an algebraic torus $\mathbb{T}^0(M) = (\mathbb{C}^*)^{b_1(M)}$.

The *characteristic varieties* of M are the jumping loci for the cohomology of M, with coefficients in rank one local systems:

$$\mathcal{V}_k^j(M) = \{\mathcal{L} \in \mathbb{T}(M) \mid \dim H^j(M, \mathcal{L}) \geqslant k\}. \tag{1}$$

When $j = 1$, we use the simpler notation $\mathcal{V}_k(M) = \mathcal{V}_k^1(M)$. The characteristic varieties of M are Zariski closed subvarieties in $\mathbb{T}(M)$.

It is usual to consider the following 'linear algebra' approximation of the characteristic varieties. The *resonance varieties* of M are the jumping loci for the cohomology of the complex $H^*(H^*(M, \mathbb{C}), \alpha\wedge)$, namely

$$\mathcal{R}_k^j(M) = \{\alpha \in H^1(M, \mathbb{C}) \mid \dim H^j(H^*(M, \mathbb{C}), \alpha\wedge) \geqslant k\}. \tag{2}$$

When $j = 1$, we use the simpler notation $\mathcal{R}_k(M) = \mathcal{R}_k^1(M)$.

If M is 1-formal, then the tangent cone theorem (see [DPS09, Theorem A]) says that the exponential mapping

$$\exp : H^1(M, \mathbb{C}) \to H^1(M, \mathbb{C}^*) = \mathbb{T}(M)$$

induces a germ isomorphism $(\mathcal{R}_k(M), 0) = (\mathcal{V}_k(M), 1)$. On the other hand, when M is not 1-formal, strange things may happen, for example the irreducible components of the resonance varieties $\mathcal{R}_k(M)$ may fail to be linear, see § 5.

In this paper, we assume that M is a connected smooth quasi-projective variety and investigate to what extent (a version of) the above statement is true without any formality assumption.

Received 13 November 2008, accepted in final form 22 April 2009.
2000 Mathematics Subject Classification 14C30, 14F40 (primary), 14H52, 32S22 (secondary).
Keywords: twisted cohomology, logarithmic de Rham complex, zeroes of 1-forms, Hodge theory.

Our idea is to regard $\mathcal{R}_k^j(M)$ as an *upper bound* for the tangent cone $TC_1(\mathcal{V}_k^j(M))$ of the corresponding characteristic variety at the trivial representation $1 \in \mathbb{T}(M)$ and to determine a *lower bound* $ETC_1(\mathcal{V}_k^j(M))$ of this tangent cone $TC_1(\mathcal{V}_k^j(M))$ by using a hypercohomology version of the resonance varieties.

More precisely, the inclusion

$$TC_1(\mathcal{V}_k^j(M)) \subset \mathcal{R}_k^j(M) \tag{3}$$

is known to hold in general, see [Lib02]. On the other hand, for any subvariety $W \subset \mathbb{T}(M)$ with $1 \in W$ we define the *exponential tangent cone* $ETC_1(W)$ such that $ETC_1(W) \subset TC_1(W)$. Our first main result says that one can determine to a certain extent the exponential tangent cone $ETC_1(\mathcal{V}_k^j(M))$ using the hypercohomology group $\mathbb{H}^j(\Omega_X^*(\log D), \alpha\wedge)$ (see Theorem 3.1 and Corollary 3.2). Here X is a good compactification of M and $(\Omega_X^*(\log D), \alpha)$ is the corresponding logarithmic de Rham complex with the differential given by the cup-product by the 1-form $\alpha \in H^0(X, \Omega_X^1(\log D))$.

The relation between the usual resonance varieties and the new hypercohomology ones is explained in Corollary 4.2, in terms of the E_2-degeneration of a twisted Hodge–Deligne spectral sequence. We introduce next the *first logarithmic resonance variety* $L\mathcal{R}_1(M)$ and restate the *logarithmic Castelnuovo–de Franchis theorem* due to Bauer (see [Bau97, Theorem 1.1]) using this notion in Proposition 4.5. (For the classical version of the Castelnuovo–de Franchis theorem see [Cat91].) The relation of this new logarithmic resonance variety to the tangent cone $TC_1(\mathcal{V}_k(M))$ is described in Corollary 4.6.

The similarity in structure of $L\mathcal{R}_1(M)$, for M an arbitrary variety, to the structure of $\mathcal{R}_1(M)$, for M an 1-formal variety, is surprising: both of them are unions of linear subspaces V_i with $V_i \cap V_j = 0$ for $i \neq j$.

As a first application, we determine in Proposition 5.1 the positive dimensional irreducible components of the characteristic variety $\mathcal{V}_1(M_{1,n})$, where $M_{1,n}$ is the *configuration space of n distinct labeled points on an elliptic curve C*. This example exhibits the special role played by the two-dimensional isotropic subspaces coming from fibrations $f : M \to S$, where S is a punctured elliptic curve. The fact that these subspaces are special was noticed by Catanese in [Cat00, Theorem 2.11].

In the final section we apply our results to the following problem of current interest. Let $\mathcal{A} = \{H_1, \ldots, H_d\}$ be an essential central arrangement of hyperplanes in $\mathbb{C}^{n+1}$. Let $f_j = 0$ be a linear form defining H_j and consider the logarithmic 1-form $\alpha_j = (df_j/f_j)$ on $M_0 = \mathbb{C}^{n+1} \setminus \bigcup_{j=1,d} H_j$. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ consider the logarithmic 1-form

$$\alpha_\lambda = \lambda_1\alpha_1 + \cdots + \lambda_d\alpha_d.$$

If $\sum_{j=1,d} \lambda_j = 0$, then α_λ can be regarded as a 1-form on the corresponding projective hyperplane arrangement complement $M = M_0/\mathbb{C}^*$. The study of the zero set $Z(\alpha_\lambda)$ of this 1-form α_λ on M is obviously related to the study of the critical locus of the associated multi-valued *master function*

$$\Phi_\lambda = \prod_{j=1,d} f_j^{\lambda_j}.$$

This in turn is related to the solutions of the $s\ell_n$ Knizhnik–Zamolodchikov equation via the Bethe ansatz, see [SV91, SV03].

The results in the final section have been inspired by the joint work of Cohen, Denham, Falk and Varchenko, see [Den07, Fal07]. They investigate the relation between the dimension of the zero set $Z(\alpha_\lambda)$ and the resonance properties of the logarithmic 1-form α_λ. Our setting is more

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general and the new idea is to consider the zeroes of 1-forms not only on M, but also on a good compactification X of M; see Theorem 6.1 and the following corollaries.

We say that $\alpha \in H^{1,0}(M) \cup H^{1,1}(M)$ is *resonant in degree* p if $H^j(H^*(M,\mathbb{C}),\alpha\wedge)=0$ for $j<p$ and $H^p(H^*(M,\mathbb{C}),\alpha\wedge)\neq 0$. Theorem 3.1, Corollary 4.2, Remark 4.3 and Theorem 6.1 yield the following result, where this time $Z(\alpha)$ denotes the zero set of α on X.

Corollary 1.1. *Assume that the spectral sequence ${}_{\alpha}E_1^{p,q}$ from Corollary 4.2 degenerates at E_2 for a logarithmic 1-form $\alpha \in (H^{1,0}(M) \cup H^{1,1}(M))$ (for instance, this holds when M is a hyperplane arrangement complement). If α is resonant in degree p, then* $\operatorname{codim} Z(\alpha) \leqslant p$. *In particular, if α is resonant in degree one, then* $\operatorname{codim} Z(\alpha)=1$.

This corollary should be compared to [Fal07, Theorems 1 and 2] and [Den07, Theorem 1]. The example discussed in Remark 6.4 shows that the inequality $\operatorname{codim} Z(\alpha) \leqslant p$ may be strict.

Moreover, Theorem 6.1(i) is similar in spirit to the generic vanishing theorem by Green and Lazarsfeld, see [GL87, Theorem 3.1].

2. Preliminary facts

By Deligne's work [Del72], the cohomology group $H^1(M,\mathbb{Q})$ of a connected smooth quasi-projective variety M has a mixed Hodge structure (for short MHS). Forgetting the rationality of the weight filtration, this MHS consists of two vector subspaces

$$W_1(M)=W_1(H^1(M,\mathbb{C}))\subset H^1(M,\mathbb{C}) \quad \text{and} \quad F^1(M)=F^1H^1(M,\mathbb{C})\subset H^1(M,\mathbb{C}).$$

If we set

$$H^{1,0}(M)=W_1(M)\cap F^1(M), \quad H^{0,1}(M)=W_1(M)\cap \overline{F^1(M)}$$

and

$$H^{1,1}(M)=F^1(M)\cap \overline{F^1(M)},$$

then we have $H^{0,1}(M)=\overline{H^{1,0}(M)}$ and the following direct sum decomposition:

$$H^1(M,\mathbb{C})=H^{1,0}(M)\oplus H^{0,1}(M)\oplus H^{1,1}(M). \tag{4}$$

This direct sum decomposition is a special case of the Deligne splitting, see [PS08, Lemma-Definition 3.4]. Suppose that W is an irreducible component of some characteristic variety $\mathcal{V}_k^j(M)$ such that $1\in W$ and let $E=T_1W$ be the corresponding tangent space. The first key result is due to Arapura, see [Ara97, Theorem 1.1].

Theorem 2.1. *Let M be a smooth quasi-projective irreducible complex variety and let $E=T_1W$ be as above. Assume that either:*

(i) $j=1$*; or*

(ii) $H^1(M,\mathbb{Q})$ *has a pure Hodge structure (of weight one if $H^1(M,\mathbb{C})=H^{1,0}(M)\oplus H^{0,1}(M)$ or two if $H^1(M,\mathbb{C})=H^{1,1}(M)$).*

Then there is a (mixed) Hodge substructure $E_{\mathbb{Q}}$ in $H^1(M,\mathbb{Q})$ such that

$$E=E_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C}$$

and the corresponding component W is just the algebraic torus $\exp(E)$. *In particular, the irreducible components of the tangent cone $TC_1(\mathcal{V}_k^j(M))$ are linear subspaces in $H^1(M,\mathbb{C})$ coming from (mixed) Hodge substructures in $H^1(M,\mathbb{Q})$.*

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It follows that the tangent space $E = T_1(W)$ satisfies the following direct sum decomposition, similar to (4):

$$E = (H^{1,0}(M) \cap E) \oplus (H^{0,1}(M) \cap E) \oplus (H^{1,1}(M) \cap E). \tag{5}$$

With respect to the direct sum decomposition (4), each class $\alpha \in H^1(M, \mathbb{C})$ may be written as

$$\alpha = \alpha^{1,0} + \alpha^{0,1} + \alpha^{1,1}. \tag{6}$$

This yields the following.

COROLLARY 2.2. *Let M be a smooth quasi-projective irreducible complex variety and j be an integer such that the assumptions (i) or (ii) in Theorem 2.1 are satisfied. Then $\alpha \in H^1(M, \mathbb{C})$ is in the tangent cone $TC_1(\mathcal{V}_k^j(M))$ if and only if $\alpha^{1,0}$, $\alpha^{0,1}$ and $\alpha^{1,1}$ are all in the same irreducible component of $TC_1(\mathcal{V}_k^j(M))$.*

The interest in this result comes from the fact that the condition $\alpha^{p,q} \in TC_1(\mathcal{V}_k^j(M))$ above can in turn be checked using our Theorem 3.1, see Corollary 3.2.

We do not know whether these results hold without the assumptions (i) or (ii) in Theorem 2.1 above. It was shown by Simpson in [Sim97, pp. 229–230] that, for a finite CW-complex M, the characteristic variety $\mathcal{V}_k^2(M)$ can be any subvariety defined over $\mathbb{Q}$ in an even-dimensional torus $\mathbb{T}(M) = (\mathbb{C}^*)^{2a}$. In particular, the irreducible components of the characteristic varieties are not necessarily translated subtori in $\mathbb{T}(M)$.

As explained in [Sim97, pp. 229–230], we see that all the characteristic varieties $\mathcal{V}_k^j(M)$ and their tangent cones $TC_1(\mathcal{V}_k^j(M))$ at the origin are defined over $\mathbb{Q}$. Note, however, that this does not imply that the irreducible components of $TC_1(\mathcal{V}_k^j(M))$ (even assumed to be linear) are defined over $\mathbb{Q}$.

DEFINITION 2.3. For a subvariety $W \subseteq \mathbb{T}(M)$, define the *exponential tangent cone* of W at 1 by

$$ETC_1(W) = \{\alpha \in H^1(M, \mathbb{C}) \mid \exp(t\alpha) \in W, \forall t \in \mathbb{C}\}.$$

Note that it is enough to require $\exp(t\alpha) \in W$ for $t \in T$ with T a subset of $\mathbb{C}$ with at least one accumulation point. One has the following general result.

LEMMA 2.4. *For any subvariety $W \subseteq \mathbb{T}(M)$, the following hold.*

(i) *$ETC_1(W) \subset TC_1(W)$.*

(ii) *$ETC_1(W)$ is a finite union of rationally defined linear subspaces of $H^1(M, \mathbb{C})$.*

The first claim is left to the reader (just use the description of the tangent cone as the set of secant limits). For the second claim above, the idea of the proof is the following. First reduce the claim to the case where W is a hypersurface defined by a Laurent polynomial. Then use the well-known fact that the exponential functions $e^{y_1 t}, \ldots, e^{y_r t}$ are linearly independent, provided $y_1, \ldots, y_r$ are all distinct. For details, see [DPS09, Lemma 4.3].

Theorem 2.1 yields the following.

COROLLARY 2.5. *Let M be a smooth quasi-projective irreducible complex variety. Then the equality*

$$ETC_1(\mathcal{V}_k^j(M)) = TC_1(\mathcal{V}_k^j(M))$$

holds if either $j = 1$ or $H^1(M, \mathbb{Q})$ is a pure MHS.

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Recall also that if $H^1(M,\mathbb{Q})$ is pure of weight two, then M is 1-formal. For 1-formal spaces M one has

$$ETC_1(\mathcal{V}_k^1(M)) = TC_1(\mathcal{V}_k^1(M)) = \mathcal{R}_k^1(M);$$

see [DPS09]. When $H^1(M,\mathbb{Q})$ is pure of weight one and M is not compact, the inclusion $TC_1(\mathcal{V}_k^1(M)) \subset \mathcal{R}_k^1(M)$ may be strict as shown in Proposition 5.1.

3. The main result

Let X be a good compactification of the smooth quasi-projective irreducible complex variety M. Then X is smooth, projective and there is a divisor with simple normal crossings $D \subset X$ such that $M = X \backslash D$. Let $(\Omega_X^*(\log D), d)$ denote the logarithmic de Rham complex corresponding to the pair (X, D). It is a locally free sheaf complex on X whose hypercohomology is $H^*(M,\mathbb{C})$. One may replace the differential d by the wedge product by some logarithmic 1-form $\alpha \in H^0(X, \Omega_X^1(\log D)) = F^1(M)$ to get a new sheaf complex $K^* = (\Omega_X^*(\log D), \alpha\wedge)$.

THEOREM 3.1. *Let M be a smooth quasi-projective irreducible complex variety and $\alpha \in H^0(X, \Omega_X^1(\log D)) = F^1(M)$ be a cohomology class in $H^{1,0}(M)$ or in $H^{1,1}(M)$. Then*

$$\alpha \in ETC_1(\mathcal{V}_k^j(M)) \text{ if and only if } \dim \mathbb{H}^j(\Omega_X^*(\log D), \alpha\wedge) \geqslant k.$$

More precisely, denote by $\mathcal{L}_t = \exp(t\alpha) \in \mathbb{T}(M)$ the one-parameter subgroup associated to $\alpha \in F^1(M)$.

(i) *If $\alpha \in H^{1,0}(M)$, then $\dim H^j(M, \mathcal{L}_t) = \dim \mathbb{H}^j(\Omega_X^*(\log D), \alpha\wedge)$ for any $t \in \mathbb{C}^*$.*

(ii) *If $\alpha \in H^{1,1}(M)$, then $\dim H^j(M, \mathcal{L}_t) \geqslant \dim \mathbb{H}^j(\Omega_X^*(\log D), \alpha\wedge)$ for any $t \in \mathbb{C}$ and the equality holds for t in a punctured neighborhood of 0 in $\mathbb{C}$.*

Proof. Consider first the case $\alpha \in H^{1,0}(M)$. Then we apply [Ara97, Theorem 2.1 in § IV] to the trivial unitary line bundle $\mathcal{O}_M$ on M with the trivial connection $d_M : \mathcal{O}_M \to \Omega_M^1$. The Deligne extension in this case is of course $(\mathcal{O}_X, d_X)$. In this first case, one has $\alpha \in H^0(X, \Omega_X^1)$ and we regard α as the regular Higgs field denoted by θ in [Ara97, Theorem 2.1]. It follows that

$$\mathbb{H}^j(\Omega_X^*(\log D), \alpha\wedge) = \mathbb{H}^j(\Omega_X^*(\log D), d - \alpha\wedge) = \mathbb{H}^j(\Omega_X^*(\log D), d - t\alpha\wedge)$$

for all $t \in \mathbb{C}^*$ (see [Ara97, Corollary 2.2 in § IV]). Since the connection $\nabla = d - t\alpha\wedge$ has trivial residues along the D_m, it follows from Deligne [Del70] that

$$H^j(M, \mathcal{L}_t) = \mathbb{H}^j(\Omega_X^*(\log D), d - t\alpha\wedge)$$

for any $t \in \mathbb{C}^*$. This proves the result in this case.

Consider now the case $\alpha \in H^{1,1}(M)$. Then we apply [Ara97, Theorem 2.4 in § IV], again to the trivial unitary line bundle $\mathcal{O}_M$ on M with the trivial connection $d_M : \mathcal{O}_M \to \Omega_M^1$. Here α is identified with a representative in $H^0(X, \Omega_X^1(\log D)) = F^1(M)$, which is denoted by ϕ in *loc.cit.* It follows as above that

$$\mathbb{H}^j(\Omega_X^*(\log D), \alpha\wedge) = \mathbb{H}^j(\Omega_X^*(\log D), d - \alpha\wedge) = \mathbb{H}^j(\Omega_X^*(\log D), d - t\alpha\wedge)$$

for all $t \in \mathbb{C}^*$ (see [Ara97, Corollary 2.5 in § IV]). For t in a punctured neighborhood of 0 in $\mathbb{C}$, the residues of $\nabla = d - t\alpha$ along the D_j are not strictly positive integers. Using again Deligne's results in [Del70] yields the claim in this case, since one has $\mathcal{L}_t \in \mathcal{V}_k^j(M)$ for all t if $k = \dim \mathbb{H}^j(\Omega_X^*(\log D), \alpha\wedge)$. □

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The above theorem yields the following hypercohomology description of the tangent cones $TC_1(\mathcal{V}_k^j(M))$.

Corollary 3.2. *Let M be a smooth quasi-projective irreducible complex variety. Assume that either $j=1$ or $H^1(M,\mathbb{Q})$ is a pure MHS. Let $\alpha=\alpha^{1,0}+\alpha^{0,1}+\alpha^{1,1}$ be the type decomposition of $\alpha \in H^1(M,\mathbb{C})$.*

If $\alpha \in TC_1(\mathcal{V}_k^j(M))$ then $\dim \mathbb{H}^j(\Omega_X^*(\log D), \beta\wedge) \geqslant k$ for any $\beta \in \{\alpha^{1,0}, \alpha^{1,1}, \overline{\alpha^{0,1}}\}$.

Remark 3.3.

(i) For $j=1$, if E and E' are two distinct irreducible components of $TC_1(\mathcal{V}_1(M))$, then $E\cap E'=0$ (see [DPS09, Theorem C, (2)]). It follows that any non-trivial one-parameter subgroup $\mathcal{L}_t=\exp(t\alpha)$ with $\alpha\in TC_1(\mathcal{V}_1(M))$ is contained in exactly one irreducible component W of $\mathcal{V}_1(M)$. This property fails for $j>1$. We have been informed by Suciu that for the central hyperplane arrangement in $\mathbb{C}^4$ given by

$$xyzw(x+y+z)(y-z+w)=0 \tag{7}$$

the resonance variety $\mathcal{R}_1^2(M)=TC_1(\mathcal{V}_1^2(M))$ consists of two three-dimensional components

$$E_1: x_1+x_2+x_3+x_6=x_4=x_5=0 \quad \text{and} \quad E_2: x_2+x_3+x_4+x_5=x_1=x_6=0$$

(the hyperplanes are numbered according to the position of the corresponding factor in the product (7) and x_j is associated with the hyperplane H_j). It follows that the intersection $E_1\cap E_2$ is one-dimensional.

(ii) Again, for $j=1$ and any irreducible component W of $\mathcal{V}_1(M)$, $\dim H^1(M,\mathcal{L})$ is constant for $\mathcal{L}\in W$ *except for finitely many* $\mathcal{L}$ (see [DPS09, Dim07]). We do not know whether this result holds for $j>1$.

Example 3.4. If M is a hyperplane arrangement complement (or, more generally, a *pure variety* M, i.e. a smooth quasi-projective irreducible complex variety such that the Hodge structure $H^k(M,\mathbb{Q})$ is pure of type (k,k) for all k), then the Hodge–Deligne spectral sequence, see Theorem 4.1 below, shows that

$$\mathbb{H}^j(\Omega_X^*(\log D), \alpha\wedge) = H^j(H^*(X), \alpha\wedge)$$

for all j and the result is known, see for instance [DM07, ESV92].

More generally, if M is a smooth quasi-projective irreducible complex variety such that the Hodge structure $H^k(M,\mathbb{Q})$ is pure of type (k,k) for all $k\leqslant m$, then we get

$$\mathbb{H}^j(\Omega_X^*(\log D), \alpha\wedge) = H^j(H^*(X), \alpha\wedge)$$

for all $j\leqslant m$ and an inclusion $\mathbb{H}^m(\Omega_X^*(\log D), \alpha\wedge)\subset H^m(H^*(X),\alpha\wedge)$, see for instance [DM07].

Remark 3.5. Let $\mathbb{T}(M)_e$ denote the connected component of the unit element e in the algebraic group $\mathbb{T}(M)$. It is well-known, see for instance [Ara97], that any local system $\mathcal{L}\in\mathbb{T}(M)_e$ can be represented as $\exp(\alpha)$, for some closed smooth differential 1-form α on M. More precisely, if we denote by

$$\nabla_\alpha : \mathcal{E}_M^0 \to \mathcal{E}_M^1, \quad \nabla_\alpha(f)=d(f)-f\cdot\alpha$$

the corresponding connection on the trivial smooth line bundle $\mathcal{E}_M^0$, then $\mathcal{L}$ is just the sheaf of horizontal sections, i.e. $\mathcal{L}=\ker\nabla_\alpha$. Here $\mathcal{E}_M^k$ denotes the sheaf of smooth $\mathbb{C}$-valued differential k-forms on M. Let $d=d'+d''$ and $\alpha=\alpha'+\alpha''$ be the decompositions according to $(1,0)$ and $(0,1)$ types respectively. In order to use the algebraic/analytic geometry, one has to replace the trivial smooth line bundle $\mathcal{E}_M^0$ by a holomorphic line bundle L on M. This is done by saying that

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the holomorphic sections of L are given locally by the smooth functions s such that $\nabla''_\alpha(s) = 0$, where $\nabla''_\alpha(f) = d''(f) - f \cdot \alpha''$. Then $\nabla'_\alpha(f) = d'(f) - f \cdot \alpha'$ becomes a holomorphic connection on L. The problem is that in general L is no longer a trivial line bundle, i.e. $L \neq \mathcal{O}_M$, and hence the corresponding Deligne extension $(\overline{L}, \overline{\nabla}'_\alpha)$ to a logarithmic connection on X is not easy to describe.

Remark 3.6. Simpson has introduced in [Sim08] a Deligne–Hitchin twistor moduli space of logarithmic λ-connections $M_{\mathrm{DH}}(X, \log D)$, which is an analytic stack and a group relative to $\mathbb{P}^1$. Moreover, Simpson has defined a weight filtration on $M_{\mathrm{DH}}(X, \log D)$ such that the exponential morphism sends the usual weight filtration on $H^1(M, \mathbb{C})$ to the induced weight filtration on $M_{\mathrm{DH}}(X, \log D)^0$, the connected component of the identity representation in $M_{\mathrm{DH}}(X, \log D)$ (see [Sim08, Lemma 6.9]). We have seen above that, when $H^1(M, \mathbb{C})$ is pure of weight two, the study of the characteristic varieties is simpler, since M is 1-formal. Under the same purity hypothesis, several results in [Sim08] get a simpler form, since then one has $Gr_2^W M_{\mathrm{DH}}(X, \log D) = M_{\mathrm{DH}}(X, \log D)$.

4. Relation to the resonance varieties

The complex $\Omega_X^*(\log D)$ has decreasing Hodge filtration F^* which is just the trivial filtration $F^p = \sigma_{\geqslant p}$. The following is one of the key results of Deligne, see [Del72, Corollaire 3.2.13].

Theorem 4.1. *Let M be a smooth quasi-projective irreducible complex variety. The spectral sequence*

$$_F E_1^{p,q} = H^q(X, \Omega_X^p(\log D))$$

associated to the Hodge filtration F on $\Omega_X^*(\log D)$ converges to $H^*(M, \mathbb{C})$ and degenerates at the E_1-level. The filtration induced by this spectral sequence on each cohomology group $H^j(M, \mathbb{C})$ is the Hodge filtration of the canonical MHS on $H^j(M, \mathbb{C})$.

This result yields the following.

Corollary 4.2. *Let M be a smooth quasi-projective irreducible complex variety and $\alpha \in H^0(X, \Omega_X^1(\log D)) = F^1(M)$ be a cohomology class. Then there is a spectral sequence*

$$_\alpha E_1^{p,q} = H^q(X, \Omega_X^p(\log D))$$

associated to the Hodge filtration F on $(\Omega_X^*(\log D), \alpha\wedge)$. This spectral sequence converges to $\mathbb{H}^{p+q}(\Omega_X^*(\log D), \wedge\alpha)$ and the differential d_1 is induced by the cup-product by α. Moreover, one has

$$\dim \mathbb{H}^j(\Omega_X^*(\log D), \alpha\wedge) \leqslant \dim H^j(H^*(X, \mathbb{C}), \alpha\wedge)$$

and equality holds if and only if this spectral sequence degenerates at E_2 (e.g. M is a pure variety).

Proof. First note that by Theorem 4.1 we get $_\alpha E_1^{p,q} = Gr_F^p H^{p+q}(M, \mathbb{C})$.

Since $\alpha \in F^1(M)$ and the cup-product is compatible with the MHS on $H^*(M, \mathbb{C})$ (see [PS08]), it follows that $f_m = \alpha\wedge : H^m(M, \mathbb{C}) \to H^{m+1}(M, \mathbb{C})$ is strictly compatible with the Hodge filtration F, i.e. for any $m, p \in \mathbb{N}$ one has the following:

(i) $f_m(F^p H^m(M, \mathbb{C})) \subset F^{p+1} H^{m+1}(M, \mathbb{C})$; and

(ii) if $\beta \in H^m(M, \mathbb{C})$ satisfies $f_m(\beta) \in F^{p+1} H^{m+1}(M, \mathbb{C})$, then there is $\beta_0 \in F^p H^m(M, \mathbb{C})$ such that $f_m(\beta_0) = f_m(\beta)$.

Set $K_m = \ker f_m$, $I_m = \operatorname{im} f_{m-1}$ and $H_m = K_m/I_m$. Then H_m has an induced F-filtration

$$F^p H_m = \frac{K_m \cap F^p H^m(M,\mathbb{C})}{I_m \cap F^p H^m(M,\mathbb{C})}.$$

Let $g_m^p : Gr_F^p H^m(M,\mathbb{C}) \to Gr_F^{p+1} H^{m+1}(M,\mathbb{C})$ be the mapping induced by f_m. Then $\ker g_m^p$ can be identified with

$$\frac{K_m \cap F^p H^m(M,\mathbb{C}) + F^{p+1} H^m(M,\mathbb{C})}{F^{p+1} H^m(M,\mathbb{C})}$$

and the image $\operatorname{im} g_{m-1}^{p-1}$ can be identified with

$$\frac{I_m \cap F^p H^m(M,\mathbb{C}) + F^{p+1} H^m(M,\mathbb{C})}{F^{p+1} H^m(M,\mathbb{C})}.$$

It follows that one has

$$Gr_F^p H_{p+q} = \ker g_{p+q}^p / \operatorname{im} g_{p+q-1}^{p-1} = {}_\alpha E_2^{p,q}.$$

This proves all the claims in Corollary 4.2. □

Remark 4.3. Assume that the irreducible components of $\mathcal{R}_k^j(M)$ are all linear and come from MHS substructures. (In view of [Voi04, Lemma 2], it is enough to ask that these components are linear and defined over $\mathbb{Q}$ or $\mathbb{R}$.) Then, if the spectral sequence ${}_\alpha E_1^{p,q}$ degenerates at E_2 for all $\alpha \in (H^{1,0}(M) \cup H^{1,1}(M))$, and either $j = 1$ or $H^1(M,\mathbb{Q})$ is pure, we get

$$TC_1(\mathcal{V}_k^j(M)) = \mathcal{R}_k^j(M).$$

To see this, let E be an irreducible component of $\mathcal{R}_k^j(M)$. If $\alpha \in (E^{1,0} \cup E^{1,1})$ is a non-zero element, then by Theorem 3.1 we get $\alpha \in E_1$, where E_1 is an irreducible component of $TC_1(\mathcal{V}_k^j(M))$. Now Theorem 3.1 implies that $E^{1,0} = E_1^{1,0}$ and $E^{1,1} = E_1^{1,1}$. This clearly implies $E = E_1$. This proves our claim in view of the inclusion (3).

Conversely, if we know that $TC_1(\mathcal{V}_k^j(M)) = \mathcal{R}_k^j(M)$ for all $k, j \geqslant 0$, then the spectral sequence ${}_\alpha E_1^{p,q}$ degenerates at E_2 for all $\alpha \in (H^{1,0}(M) \cup H^{1,1}(M))$. This is the case, for instance, for the hyperplane arrangement complements, see [CS99].

The example discussed in the § 5 below shows that this spectral sequence does not necessarily degenerate at E_2.

If M and N are quasi-projective varieties, a *fibration* $f : M \to N$ is a surjective morphism with a connected general fiber (this is called an admissible morphism in [Ara97]). Two fibrations $f : M \to C$ and $f' : M \to C'$ over quasi-projective curves C and C' are said to be *equivalent* if there is an isomorphism $g : C \to C'$ such that $f' = g \circ f$.

Beauville's paper [Bea92], in the case where M is proper, and Arapura's paper [Ara97], in the case where M is non-proper, establish a bijection between the set $\mathcal{E}(M)$ of equivalence classes of fibrations $f : M \to C$ from M to curves C with $\chi(C) < 0$ and the set $\mathcal{I}C_1(M)$ of irreducible components of the first characteristic variety $\mathcal{V}_1(M)$ passing through the unit element 1 of the character group $\mathbb{T}(M)$ of M.

More precisely, the irreducible component associated to an equivalence class $[f] \in \mathcal{E}(M)$ is $W_f = f^*(\mathbb{T}(C))$. The corresponding tangent space is given by $E_f = T_1 W_f = f^*(H^1(C,\mathbb{C}))$. The union of all these tangent spaces is the tangent cone $TC_1(\mathcal{V}_1(M))$, and the tangent cone theorem (see [DPS09, Theorem A]) implies that, when M is 1-formal, one has the equality

$$TC_1(\mathcal{V}_1(M)) = \mathcal{R}_1(M).$$

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This equality imposes very strong conditions on $\mathcal{R}_1(M)$, which may be regarded as special properties enjoyed by the cohomology algebras of 1-formal varieties, in particular of compact Kähler manifolds as in [Voi08]. See also Remark 5.2(ii).

To get a similar result in the general case one may proceed as follows.

Definition 4.4. For any smooth complex quasi-projective variety M, consider the graded subalgebra $F^*(M) \subset H^*(M, \mathbb{C})$ given by $F^k(M) = F^k H^k(M, \mathbb{C}) = H^0(M, \Omega^k_X(\log D))$. We define the *first logarithmic resonance variety* of M by the equality

$$L\mathcal{R}_1(M) = \{\alpha \in F^1(M) \mid H^1(F^*(M), \alpha\wedge) \neq 0\}.$$

Note that $L\mathcal{R}_1(M) \subset \mathcal{R}_1(M) \cap F^1(M)$, but the inclusion may be strict, as in the case $M = M_{1,n}$ described in § 5. On the other hand, $L\mathcal{R}_1(M) = \mathcal{R}_1(M)$ if $H^1(M, \mathbb{Q})$ is pure of weight two, for example when M is a hypersurface complement in $\mathbb{P}^n$. Corollary 4.2 yields

$$\dim H^1(F^*(M), \alpha\wedge) \leqslant \dim \mathbb{H}^1(\Omega^*_X(\log D), \alpha\wedge) \tag{8}$$

for any $\alpha \in L\mathcal{R}_1(M)$.

The first logarithmic resonance variety is not defined topologically, but it enjoys the following very nice property.

Proposition 4.5. *For any smooth connected complex quasi-projective variety M, the following hold.*

(i) *The (strictly positive dimensional) irreducible components of $L\mathcal{R}_1(M)$ are exactly the maximal isotropic subspaces $I \subset F^1(M)$ satisfying $\dim I \geqslant 2$.*

(ii) *If I and I' are distinct irreducible components of $L\mathcal{R}_1(M)$, then $I \cap I' = 0$.*

(iii) *The mapping*

$$[f] \mapsto I_f = f^*(F^1(C)) = f^*(H^0(\tilde{C}, \Omega^1_{\tilde{C}}(\log B)))$$

induces a bijection between the set $\mathcal{E}_0(M)$ of equivalence classes of fibrations $f : M \to C$ with $g^*(C) \geqslant 2$ and the set of (strictly positive dimensional) irreducible components of the first logarithmic resonance variety $L\mathcal{R}_1(M)$.

Here $\tilde{C}$ is a smooth projective model for C, $B = \tilde{C}\setminus C$ is a finite set and $g^*(C) = b_1(C) - g(\tilde{C}) = \dim H^0(\tilde{C}, \Omega^1_{\tilde{C}}(\log B))$.

Proof. Assume that $\alpha \in L\mathcal{R}_1(M)$ is a non-zero 1-form. Let I be a maximal isotropic subspace in $F^1(M)$ (with respect to the usual cup-product) such that $\alpha \in I$. Then $d = \dim I \geqslant 2$.

We can apply the logarithmic Castelnuovo–de Franchis theorem obtained by Bauer in [Bau97, Theorem 1.1], and get a fibration $f : M \to C$ such that $I = I_f$. In particular, $g^*(C) = d \geqslant 2$. Note that $I_f \cap I_g = 0$ for $[f] \neq [g]$ (see Remark 3.3). It follows that

$$L\mathcal{R}_1(M) = \bigcup_{[f]\in\mathcal{E}_0(M)} I_f. \tag{9}$$

Since $\mathcal{E}_0(M)$ is a finite set, it follows that (9) is precisely the decomposition of $L\mathcal{R}_1(M)$ into irreducible components. □

Note that $\chi(C) < 0$ is equivalent to either $g^*(C) \geqslant 2$ or $g(\tilde{C}) = 1$ and $|B| = 1$. It is precisely this latter case that is not covered by the above bijective correspondence, which occurs in the example treated in § 5. For more on this exceptional case refer to [Dim08].

Corollary 4.6. *Let M be a smooth connected complex quasi-projective variety. If $I \neq 0$ is an irreducible component of $L\mathcal{R}_1(M)$, then $I + \overline{I}$ is an irreducible component of $TC_1(\mathcal{V}_1(M))$. Conversely, any irreducible component $E = E_f \neq 0$ of $TC_1(\mathcal{V}_1(M))$, not coming from a fibration $f: M \to S$ onto a once-punctured elliptic curve S, is of this form, with $I = E \cap F^1(M)$.*

In particular, $\alpha \in L\mathcal{R}_1(M)$ if and only if both Hodge type components $\alpha^{1,0}$ and $\alpha^{1,1}$ of α are in the same irreducible component of $L\mathcal{R}_1(M)$.

5. A first application: configuration spaces of n points on elliptic curves

In this section let C be a smooth compact complex curve of genus $g = 1$. Consider the *configuration space* of n distinct labeled points in C,

$$M_{1,n} = C^n \Big\backslash \bigcup_{i<j} \Delta_{ij},$$

where Δ_{ij} is the diagonal $\{s \in C^n \mid s_i = s_j\}$. It is straightforward to check that:

(i) the inclusion $\iota : M_{1,n} \to C^n$ yields an isomorphism $\iota^* : H^1(C^n, \mathbb{C}) \to H^1(M_{1,n}, \mathbb{C})$, in particular $W_1(H^1(M_{1,n}, \mathbb{C})) = H^1(M_{1,n}, \mathbb{C})$;

(ii) using the above isomorphism, the cup-product map

$$\bigwedge^2 H^1(M_{1,n}, \mathbb{C}) \to H^2(M_{1,n}, \mathbb{C})$$

is equivalent to the composite

$$\mu_{1,n}\colon \bigwedge^2 H^1(C^n, \mathbb{C}) \xrightarrow{\cup_{C^n}} H^2(C^n, \mathbb{C}) \twoheadrightarrow H^2(C^n, \mathbb{C})/\langle\{[\Delta_{ij}]\}_{i<j}\rangle, \qquad (10)$$

where $[\Delta_{ij}] \in H^2(C^n, \mathbb{C})$ denotes the dual class of the diagonal Δ_{ij}, and the second arrow is the canonical projection. See [DPS09, § 10] for more details.

Let $\{a, b\}$ be the standard basis of $H^1(C, \mathbb{C}) = \mathbb{C}^2$. Note that the cohomology algebra $H^*(C^n, \mathbb{C})$ is isomorphic to $\bigwedge^*(a_1, b_1, \ldots, a_n, b_n)$. Denote by $(x_1, y_1, \ldots, x_n, y_n)$ the coordinates of $z \in H^1(M_{1,n}, \mathbb{C})$. Using (10), it was shown in [DPS09, § 10] that

$$\mathcal{R}_1(M_{1,n}) = \left\{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0,\, x_i y_j - x_j y_i = 0, \text{ for } 1 \leqslant i < j < n\right\}.$$

Suppose $n \geqslant 3$. Then $\mathcal{R}_1(M_{1,n})$ is the affine cone over the $(n-1)$-fold scroll $S_{1,\ldots,1}$, with 1 repeated $(n-1)$-times, see [Har92, Exercise 8.27]. In particular, $\mathcal{R}_1(M_{1,n})$ is an *irreducible, nonlinear variety.*

Let $\Omega_C = (1, \lambda)$ be a normalized period matrix for the projective curve C. Then $\lambda \in \mathbb{C}$ and $\mathrm{Im}(\lambda) > 0$. It can be shown easily that

$$F^1(M_{1,n}) = H^{1,0}(M_{1,n}) = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid y = \lambda x\} \qquad (11)$$

and similarly $H^{0,1}(M_{1,n}) = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid y = \overline{\lambda} x\}$.

This implies that

$$F^1(M_{1,n}) \subset \mathcal{R}_1(M_{1,n}). \qquad (12)$$

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Let $\alpha = (x, \lambda x)$ with $x \neq 0$. It is easy to see that $\alpha \wedge (x', \lambda x') = 0$ if and only if $x' \in \mathbb{C}x$. It follows that

$$L\mathcal{R}_1(M_{1,n}) = 0. \tag{13}$$

In other words, one has

$${}_{\alpha}E_2^{1,0} = 0.$$

Similarly, $\alpha \wedge (x', \overline{\lambda} x') = 0$ if and only if $x' \in \mathbb{C}x$. Hence

$${}_{\alpha}E_2^{0,1} = \mathbb{C}.$$

We set as above $\mathcal{L}_t = \exp(t\alpha)$. It follows that $\dim H^1(M_{1,n}, \mathcal{L}_t) \leqslant 1$ for all $t \in \mathbb{C}^*$ and $\alpha \in F^1(M_{1,n})$, with equality exactly when $d_2 : {}_{\alpha}E_2^{0,1} \to {}_{\alpha}E_2^{2,0}$ is zero. If we assume that this is the case for all α, then $\mathcal{V}_1(M_{1,n}) = \mathbb{T}(M_{1,n})$, a contradiction, since $TC(\mathcal{V}_1(M)) \subset \mathcal{R}_1(M)$ always (see [Lib02]).

In fact, if W is any component of $\mathcal{V}_1(M_{1,n})$ passing though the origin and containing $\mathcal{L}_t$ with $\dim H^1(M_{1,n}, \mathcal{L}_t) = 1$, then it follows from [Ara97] that $\dim W = 2$ and $W = f^*(\mathbb{T}(S))$ where $f : M_{1,n} \to S$ is an admissible map onto an affine curve S with $b_1(S) = 2$. In other words, S is obtained from a $\mathbb{P}^1$ by deleting 3 points, or S is obtained from a projective genus 1 curve C' by deleting a point, say the unit element 1 of the group structure on C'. The former case is discarded easily by Hodge theory, see the subcase (ii$_a$) in the proof below. The next result says that the mappings in the latter case can be completely described.

PROPOSITION 5.1. *With the above notation, let $f : M_{1,n} \to S$ be an admissible map onto a curve S obtained from a projective genus 1 curve C' by deleting a point. Then $C' = C$ and, up to an isomorphism of C, the map f coincides to one of the maps $f_{ij} : M_{1,n} \to C \backslash \{1\}$, $(s_1, \ldots, s_n) \mapsto s_i s_j^{-1}$ for some $1 \leqslant i < j \leqslant n$.*

In particular, $W_{ij} = f_{ij}^*(\mathbb{T}(S))$ are all the irreducible components of $\mathcal{V}_1(M_{1,n})$ passing through the origin. More precisely, for $1 \leqslant i < j \leqslant n$ consider the two projections $\pi_i, \pi_j : C^n \to C$ onto the ith (respectively jth) factor. Then

$$W_{ij} = \{\pi_i^*(\mathcal{L}) \otimes \pi_j^*(\mathcal{L}^{-1}) \mid \mathcal{L} \in \mathbb{T}(C \backslash \{1\})\}.$$

And there are no translated positive dimensional components in $\mathcal{V}_1(M_{1,n})$.

Proof. For any quasi-projective smooth variety Y such that $H_1(Y, \mathbb{Z})$ is torsion free and $H^1(Y, \mathbb{Q})$ is a pure Hodge structure of weight one, one may define a (generalized) Albanese variety

$$\mathrm{Alb}(Y) = \frac{H^{1,0}(Y)^{\vee}}{H_1(Y, \mathbb{Z})}$$

and a natural mapping $a_Y : Y \to \mathrm{Alb}(Y)$, $y \mapsto \int_{y_0}^{y}$. Here $\vee$ denotes the dual vector space and $y_0 \in Y$ is a fixed point. This Albanese variety is a compact torus and, if Y itself is an abelian variety, the map a_Y is an isomorphism.

If $g : Y \to Z$ is a regular mapping between two varieties as above, there is a functorial induced (regular) homomorphism $g_* : \mathrm{Alb}(Y) \to \mathrm{Alb}(Z)$.

Set for simplicity $M = M_{1,n}$. Then the inclusion $j_M : M \to C^n$ induces an isomorphism $j_{M*} : \mathrm{Alb}(M) \to \mathrm{Alb}(C^n)$. Similarly, the inclusion $j_S : S \to C'$ is an isomorphism $j_{S*} : \mathrm{Alb}(S) \to \mathrm{Alb}(C')$.

The mapping $f : M \to S$ induces, via these isomorphisms, a homomorphism $f_* : \mathrm{Alb}(C^n) \to \mathrm{Alb}(C')$. Since a_{C^n} and $a_{C'}$ are isomorphisms, this yields, up to a translation in C', a

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homomorphism $h(f): C^n \to C'$ such that $f: M \to S$ is just the restriction of this homomorphism. This may happen if and only if the kernel of $h(f)$ is contained in $\bigcup_{i<j} \Delta_{ij}$. Since $\ker(h(f))$ is a codimension one irreducible subgroup in C^n, this is possible if and only if there exists $i < j$ with

$$\ker(h(f)) = \Delta_{ij}.$$

Note that we have $h(f)(s_1, \dots, s_n) = h_1(s_1) \cdot \cdots \cdot h_n(s_n)$, where $h_j: C \to C'$ are homomorphisms for $j = 1, \dots, n$. Let Δ'_{ij} be the subset of Δ_{ij} consisting of all the points $(s_1, \dots, s_n) \in C^n$ such that $s_i = s_j = t$ and $s_m = 1$ for $m \notin \{i, j\}$. Then $\Delta'_{ij} \subset \ker(h(f))$ implies that $h_j(t) = (h_i(t))^{-1}$ for all $t \in C$.

By considering the subset Δ'_{ijk} of Δ_{ij} consisting of all the points $(s_1, \dots, s_n) \in C^n$ such that $s_i = s_j = t$, $s_k = u$ and $s_m = 1$ for $k \notin \{i, j\}$ and $m \notin \{i, j, k\}$, we see that $f_k(u) = 1$ for all $u \in C$.

It follows that the image of the morphism $h(f)_*: H_1(C^n) \to H_1(C')$ is exactly $\operatorname{im} h_{i*} = \operatorname{im} h_{j*}$. Since f is admissible, the fibers of $h(f)$ have to be connected, and this implies that $h(f)_*$ is surjective. Hence h_{i*} is surjective, and this implies that $h_i: C \to C'$ is an isomorphism. Moreover,

$$h(f)(s_1, \dots, s_n) = h_i(s_i)h_j(s_j) = h_i(s_i)h_i(s_j^{-1}) = h_i(s_i s_j^{-1}),$$

which completes the proof of our proposition, except for the last claim.

The translated components W in $\mathcal{V}_1(M)$ may be of one of the following types.

(i) If $\dim W \geqslant 2$, then W should be either a translate of one of the components W_{ij}, or be associated to an admissible mapping $f: M \to C'$, with C' an elliptic curve. Exactly as above one may argue that then f is the restriction of a homomorphism $h(f): C^n \to C'$ with connected fibers. Both cases are impossible, since the corresponding admissible mappings f_{ij} (respectively f) have no multiple fibers. For details, see [Dim07, Theorems 3.6(vi) and 5.3].

(ii) Suppose that $\dim W = 1$. Then using Corollary 5.9 in [Dim07], we see that there are two subcases.

(ii$_\mathrm{a}$) The component W is associated to an admissible mapping $f: M \to \mathbb{C}^*$. This subcase is impossible in the situation at hand, since this would give an injection $f^*: H^1(\mathbb{C}^*, \mathbb{Q}) \to H^1(M, \mathbb{Q})$, in contradiction with the Hodge types of these two cohomology groups.

(ii$_\mathrm{b}$) The component W is associated to an admissible mapping $f: M \to C'$, with C' an elliptic curve. This case was already discarded in (i) above.

Remark 5.2.

(i) Let X be a compactification of the smooth quasi-projective irreducible complex variety M. Assume that the inclusion $j: M \to X$ induces an isomorphism $j^*: H^1(X) \to H^1(M)$ and a monomorphism $j^*: H^2(X) \to H^2(M)$. Then $D = X \setminus M$ has codimension at least two and hence $j_\sharp: \pi_1(M) \to \pi_1(X)$ is an isomorphism. In particular, $\mathcal{V}_1(M) = \mathcal{V}_1(X)$ and $\mathcal{R}_1(M) = \mathcal{R}_1(X)$.
To see this, note that the conditions on j^* are equivalent to $H^2(X, M) = 0$. Let T be a closed tubular neighborhood of D in X. Then, by excision and duality we get

$$\dim H^2(X, M) = \dim H^2(T, \partial T) = \dim H_{2n-2}(T \setminus \partial T) = \dim H_{2n-2}(D) = n(D)$$

where $n(D)$ is the number of $(n-1)$-dimensional irreducible components in D.

(ii) Consider a smooth quasi-projective irreducible complex variety M such that the cohomology group $H^1(M, \mathbb{Q})$ is a pure Hodge structure of weight one. It can be shown that if $\alpha \in \mathcal{R}_1(M)$, then the Hodge components $\alpha^{1,0}$ and $\alpha^{0,1}$ are both in $\mathcal{R}_1(M)$. The converse implication fails, as shown by our discussion above of the case $M = M_{1,n}$, where $F^1(M) \subset \mathcal{R}_1(M)$ and $\overline{F^1(M)} \subset \mathcal{R}_1(M)$, but $\mathcal{R}_1(M)$ is strictly contained in $H^1(M) = F^1(M) + \overline{F^1(M)}$.

6. A second application: twisted cohomology and zeroes of logarithmic 1-forms

As above, let X be a good compactification of the smooth quasi-projective irreducible complex variety M. Let $(\Omega^*_X(\log D), d)$ denote the logarithmic de Rham complex of the pair (X, D) and take a logarithmic 1-form $\alpha \in H^0(X, \Omega^1_X(\log D)) = F^1(M)$. For any point $x \in X$, choose $\alpha_1, \ldots, \alpha_n$ to be a basis of the free module $\Omega^1_X(\log D)_x$ over the corresponding local ring $\mathcal{O}_{X,x}$. Then $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$ for some function germs $a_j \in \mathcal{O}_{X,x}$. The complex

$$K^*_x:\ 0 \to \Omega^0_X(\log D)_x \to \Omega^1_X(\log D)_x \to \cdots \to \Omega^n_X(\log D)_x \to 0$$

where the differential is the wedge product by the germ of α at x can be identified with the Koszul complex of the sequence $(a_1, \ldots, a_n)$ in the ring $\mathcal{O}_{X,x}$. Let I_x be the ideal generated by the germs a_j in the local ring $\mathcal{O}_{X,x}$.

Let $Z(\alpha) \subset X$ be the zero set of α regarded as a section of the locally free sheaf $\Omega^1_X(\log D)$. In other words, for all $x \in X$, the germ of $Z(\alpha)$ at x is exactly the zero set of the ideal I_x.

Let c_x be the codimension of the closed analytic subset $Z(\alpha)$ at the point $x \in X$, i.e. $c_x = \operatorname{codim}(I_x)$. Using the relation between codimension and depth in regular local rings (see [Eis99, Theorem 18.7, p. 455], as well as [Eis99, Theorem 17.4, p. 428 and Theorem 17.6, p. 430]), it follows that

$$H^j(K^*_x) = 0 \quad \text{for all } j < c_x \text{ and } H^{c_x}(K^*_x) \neq 0. \tag{14}$$

Now we use our Theorem 3.1. Let K^* denote the sheaf complex $\Omega^*_X(\log D)$ with differential $\alpha\wedge$. Then there is an E_2-spectral sequence with

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(K^*))$$

converging to the hypercohomology groups $\mathbb{H}^{p+q}(X, K^*)$. Here $\mathcal{H}^q(K^*)$ denotes the qth cohomology sheaf of the complex K^* and one clearly has

$$\mathcal{H}^q(K^*)_x = H^q(K^*_x) \tag{15}$$

for any point $x \in X$ and any integer q. Let $c(\alpha) = \min_{x\in X} c_x$ and $d(\alpha) = n - c(\alpha) = \dim Z(\alpha)$. Equations (14) and (15) imply that $H^p(X, \mathcal{H}^q(K^*)) = 0$ for all $q < c(\alpha)$. Since all the coherent sheaves $\mathcal{H}^q(K^*)$ are supported on $Z(\alpha)$, it follows by the general theory that $H^p(X, \mathcal{H}^q(K^*)) = 0$ for $p > d(\alpha)$. These two vanishing results imply the following result.

Theorem 6.1. *Let M be a smooth quasi-projective irreducible complex variety. Take a 1-form $\alpha \in H^0(X, \Omega^1_X(\log D)) = F^1(M)$ and set $\mathcal{L}_t = \exp(t\alpha) \in \mathbb{T}(M)$. Then the following hold.*

(i) *If $\alpha \in H^{1,0}(M)$, then $H^j(M, \mathcal{L}_t) = 0$ for any $t \in \mathbb{C}^*$ and $j < c(\alpha) = \operatorname{codim} Z(\alpha)$ or $j > 2n - c(\alpha)$. Moreover, one has $H^{c(\alpha)}(M, \mathcal{L}_t) = H^0(X, \mathcal{H}^{c(\alpha)}(K^*))$ and $H^{2n-c(\alpha)}(M, \mathcal{L}_t) = H^{d(\alpha)}(X, \mathcal{H}^n(K^*))$.*

(ii) *If $\alpha \in H^{1,1}(M)$, then the above claims hold for $t \in \mathbb{C}$ generic.*

Note that M is not necessarily affine, and hence it has not necessarily the homotopy type of a CW-complex of dimension at most n, i.e. the above vanishing for $j > 2n - c(\alpha)$ is meaningful.

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The following special cases are easy to handle, using the obvious fact that in these cases the above spectral sequences degenerate at E_2.

Corollary 6.2 (A logarithmic Hopf index theorem). *Let M be a smooth quasi-projective irreducible complex variety, $\alpha \in H^0(X, \Omega^1_X(\log D))$ and $\mathcal{L}_t = \exp(t\alpha) \in \mathbb{T}(M)$. Then, if $\dim Z(\alpha) = 0$, the following hold.*

(i) *If $\alpha \in H^{1,0}(M)$, then $H^j(M, \mathcal{L}_t) = 0$ for any $t \in \mathbb{C}^*$ and $j \neq n$. In addition, one has that*

$$\dim H^n(M, \mathcal{L}_t) = \dim H^0(X, \mathcal{H}^n(K^*)) = |\chi(M)|$$

is the number of zeroes of the 1-form α counted with multiplicities.

(ii) *If $\alpha \in H^{1,1}(M)$, then the above claims hold for $t \in \mathbb{C}$ generic.*

Since the support of the sheaf $\mathcal{H}^n(K^*)$ is finite in the above case, note that $\mathcal{H}^n(K^*) \neq 0$ implies $\dim H^0(X, \mathcal{H}^n(K^*)) > 0$.

Corollary 6.3. *Let M be a smooth quasi-projective irreducible complex variety, and α a 1-form in $H^0(X, \Omega^1_X(\log D)) = F^1(M)$, $\mathcal{L}_t = \exp(t\alpha) \in \mathbb{T}(M)$. Then, if $\dim Z(\alpha) = 1$, the following hold.*

(i) *If $\alpha \in H^{1,0}(M)$, then $H^j(M, \mathcal{L}_t) = 0$ for any $t \in \mathbb{C}^*$ and $j < n-1$ or $j > n+1$. Moreover, one has natural isomorphisms*

$$H^{n-1}(M, \mathcal{L}_t) = H^0(X, \mathcal{H}^{n-1}(K^*)),$$
$$H^n(M, \mathcal{L}_t) = H^0(X, \mathcal{H}^n(K^*)) \oplus H^1(X, \mathcal{H}^{n-1}(K^*)),$$
$$H^{n+1}(M, \mathcal{L}_t) = H^1(X, \mathcal{H}^n(K^*))$$

for any $t \in \mathbb{C}^*$.

(ii) *If $\alpha \in H^{1,1}(M)$, then the above claims hold for $t \in \mathbb{C}$ generic.*

Remark 6.4. At the end of the report [Den07] there is an example of a plane line arrangement complement M (with a one-dimensional translated component in $\mathcal{V}_1(M)$ discovered by Suciu [Suc02]) and of a logarithmic 1-form α such that $c(\alpha) = 1$ but $H^1(M, \mathcal{L}_t) = 0$ for generic t. Since $\chi(M) \neq 0$ in this case, one has $H^1(M, \mathcal{L}_t) \neq 0$ for generic t, and hence α is resonant in degree $p = 2$. Such a possibility is clear by our results above: the corresponding sheaf $\mathcal{H}^1(K^*)$ is definitely non-zero by (14), but the cohomology group $H^0(X, \mathcal{H}^1(K^*))$ may be trivial, i.e. the coherent sheaf $\mathcal{H}^1(K^*)$ may have no non-trivial global sections.

Moreover this situation occurs as soon as M is a hyperplane arrangement complement such that there is a one-dimensional translated component W in $\mathcal{V}_1(M)$. Indeed, by the results in [Dim07], such a component is associated to a surjective morphism $f: M \to \mathbb{C}^*$, with a connected generic fiber and having at least one multiple fiber, say $F_1 = f^{-1}(1)$. Let t be a coordinate on $\mathbb{C}$ and set

$$\alpha = f^*\left(\frac{dt}{t}\right).$$

Then α is a non-zero logarithmic 1-form on M of Hodge type $(1,1)$ and $c(\alpha) = 1$ since $F_1 \subset Z(\alpha)$. On the other hand, α is not 1-resonant, as this would imply $\alpha \in \mathcal{R}_1(M) = TC_1(\mathcal{V}_1(M))$. This is a contradiction, since there is no irreducible component W_0 of $\mathcal{V}_1(M)$ such that $1 \in W_0$ and W is a translate of W_0 (see [Dim07, Corollary 5.8]).

https://doi.org/10.1112/S0010437X09004461 Published online by Cambridge University Press

Acknowledgement

The research was partially supported by ANR-08-BLAN-0317-02 (SEDIGA). The author is grateful to ASSMS, Government College University, Lahore, Pakistan, where part of the work on this paper was done.

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https://doi.org/10.1112/S0010437X09004461 Published online by Cambridge University Press