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CLOSED LINEAR MAPS FROM A BARRELLED NORMED SPACE INTO ITSELF NEED NOT BE CONTINUOUS

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Examples of normed barrelled spaces E or quasicomplete barrelled spaces E are given such that there is a non-continuous linear map from the space E into itself with closed graph.

In this note we give a negative answer to a question of Okada and Ricker. In fact, we construct a normed barrelled space E and a linear map $f : E \to E$ which has closed graph but is not continuous. The problem is relevant in connection with spectral measures and automatic continuity. (See [**4, 5, 6**].) Our construction is rather simple, but it seems to have been overlooked in the literature. A variation of our method permits us to construct quasicomplete barrelled spaces E and non-continuous closed linear maps from E to E. This second example was motivated by certain results given in [**2**].

Since the classical work of Banach, the closed graph theorems constitute one of the main tools of functional analysis. Barrelled spaces are those locally convex spaces for which the uniform boundedness principle of Banach Steinhaus holds. We refer the reader to [**3**] and [**7**]. Clearly, if a barrelled space E belongs simultaneously to the domain and the range classes of a closed graph theorem, then every closed linear map from E to E is continuous. This happens for example if E is ultrabornological and has a C-web. In particular, if E is a Banach space, a Fréchet space or an (LF)-space (for example the space of test functions $\mathcal{D}$ of distribution theory) or an ultrabornological projective limit of (LB)-spaces (for example the space of distributions $\mathcal{D}'$) (see [**3**] or [**7**]), then this is the case.

EXAMPLE Let X be an infinite-dimensional Banach space with a norm $\|\cdot\|$. There is a non-continuous linear form u on X. We denote by Y the linear space X endowed with the norm defined by $\|\cdot\| + |u(\cdot)|$. Clearly Y is a normed space, and the identity from Y into X is continuous, but not open. The hyperplane $H := \ker u$ is closed in Y, hence Y is the topological direct sum $Y = H \oplus [x]$, where $x \in X$ satisfies $u(x) = 1$. Moreover the spaces X and Y induce the same topology on H. Since H is a dense hyperplane of X, H is barrelled for the induced topology (see for example, [**7**, 4.3.1]). Accordingly, the normed space Y is barrelled, since the product of barrelled spaces is also barrelled.

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We now define E as the normed barrelled space $X \times Y$ and consider the linear map $f : E \to E$ defined by $f(x, y) := (y, x)$. The map f is linear and has closed graph. Indeed, if we consider on the range space E the coarser locally convex topology τ induced by $X \times X$, then the linear map $f : E \to (E, \tau)$ is continuous. This implies that the graph of f is closed in $E \times E$. However, the map f is not continuous. For if it were, composing with the injection from X into E and with the projection from E onto Y, we could conclude that the linear map $X \to Y$, given by $x \to x$ is continuous, too, a contradiction.

REMARK In the above construction, one can take a barrelled space X not endowed with the finest locally convex topology, select a non-continuous linear form u on X and define Y as the space X, endowed with the Mackey topology of the dual pair $(X, X' + [u])$. This is a barrelled space, called a barrelled countable enlargement, see [**7**, 4.5] for more information about barrelled enlargements. We observe that the space Y, hence E, in the above construction is metrisable or (DF) whenever X is.

It is also possible to use the method of our example above to construct *a quasicomplete barrelled space* E *and a linear map* $f : E \to E$ *with closed graph which is not continuous*. Indeed, let G be the incomplete Montel space of Amemiya and Komura [**1**]. The space $(G, \sigma(G, G'))$ is barrelled, the bounded sets of $(G', \sigma(G', G))$ are finite dimensional and $(G', \sigma(G', G))$ is barrelled and quasicomplete. We denote this space by H. Since G is not endowed with the finest locally convex topology, there is $u \in G^* \setminus G'$. Since $(G, \mu(G, G' \oplus [u]))$ is barrelled see [**7**, 4.5], the space $X := (G' \oplus [u], \sigma(G' \oplus [u], G))$ is quasicomplete and barrelled. Clearly H is a dense hyperplane of X. At this point we repeat the construction of the previous example: define Y as the barrelled space given by the topological direct sum of H and $[u]$. Clearly Y is quasicomplete. The spaces X and Y coincide algebraically but Y has a topology strictly finer than X. Finally we put $E := X \oplus Y$ and define $f : E \to E$, by $f(x, y) := (y, x)$ to reach the conclusion.

It is easier to construct sequentially complete barrelled spaces E as above: it is enough to take the subspace H of the product F of an uncountable number of copies of the scalar field consisting of all the elements of the product with countably many non-vanishing coordinates. The space H is barrelled and sequentially complete for the topology induced by the product F, see [**7**, 4.2.5]. Therefore the space X generated by H and the constant element 1 of F is barrelled, sequentially complete and contains H as a dense hyperplane. Now one proceeds as before.

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