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Univalently Induced, Closed Range, Composition Operators on the Bloch-type Spaces

Nina Zorboska

Abstract. While there is a large variety of univalently induced closed range composition operators on the Bloch space, we show that the only univalently induced, closed range, composition operators on the Bloch-type spaces B^α with $\alpha \neq 1$ are the ones induced by a disc automorphism.

Let $\mathbb{D}$ denote the unit disc in the complex plane, and let $H(\mathbb{D})$ be the space of all functions analytic on $\mathbb{D}$. For a non-constant analytic function ϕ that maps the unit disc into itself, the *composition operator* C_ϕ on the Banach space $X \subseteq H(\mathbb{D})$ is defined by $C_\phi f = f \circ \phi$ for f in X. We say that ϕ is the *inducing function* for C_ϕ.

The operator theoretic properties of C_ϕ depend on the function theoretic properties of the inducing function and on the properties of the functions in the space X. Thus, the study of composition operators provides connections between operator theory and complex analysis and leads to a deeper understanding of both. In this paper we will be interested in the closed range composition operators on the so-called Bloch-type spaces. For general results and references on composition operators acting on various other spaces of analytic functions see, for example, [6, 17].

For $\alpha > 0$, the α-*Bloch spaces* B^α (also referred to as Bloch-type spaces) are spaces of functions f in $H(\mathbb{D})$ such that

$$\|f\|_{B^\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|\big(1-|z|^2\big)^{\alpha} < \infty.$$

Each B^α is a Banach space with a norm given by

$$|||f|||_{B^\alpha} = |f(0)| + \|f\|_{B^\alpha}.$$

The family of Bloch-type spaces includes the classical Bloch space $B = B^1$. The spaces B^α with $0 < \alpha < 1$ are analytic Lipschitz spaces $\mathrm{Lip}_{1-\alpha}$. Thus, for $0 < \alpha < 1$, $B^\alpha \subset A(\mathbb{D}) \subset H^\infty$, where $A(\mathbb{D})$ is the disc algebra. In general, for $0 < \alpha < \beta$ we have that $B^\alpha \subset B^\beta$. Note also that the Bloch space B contains H^∞ and is included in all of the Bergman spaces L^p_a, $p \geq 1$, while for large α, such as $\alpha \geq 2$, B^α includes the Bergman space L^2_a. For more details on these general facts about the Bloch-type spaces and for further results and references see [20, 21].

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The boundedness and compactness of composition operators acting on Bloch-type spaces has been established in a series of papers [10, 11, 13, 16, 18]. The classification depends on the behaviour of a determining function, defined as follows: for $\alpha > 0$, and ϕ an analytic self map of $\mathbb{D}$, let

$$\tau_{\phi,\alpha}(z) = \frac{(1-|z|^2)^\alpha |\phi'(z)|}{(1-|\phi(z)|^2)^\alpha}.$$

When $\alpha = 1$, we write τ_ϕ instead of $\tau_{\phi,1}$.

Theorem 1 ([8, 11, 13]) *For $\alpha > 0$ and ϕ an analytic selfmap of $\mathbb{D}$, the composition operator C_ϕ is a bounded operator on B^α if and only if $\sup_{z\in\mathbb{D}} \tau_{\phi,\alpha}(z) < \infty$, and C_ϕ is a compact operator on B^α if and only if $\lim_{|\phi(z)|\to 1} \tau_{\phi,\alpha}(z) = 0$.*

The spaces B^α include the identity function, and so a necessary condition for C_ϕ to be bounded on B^α is that ϕ belongs to B^α. Thus, for example, if $0 < \alpha < 1$, every analytic self-map of $\mathbb{D}$ that is in $H^\infty \setminus B^\alpha$ induces an unbounded composition operator on B^α.

When $\alpha = 1$, we have from the Schwarz–Pick lemma, that

$$\tau_\phi(z) = \frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2} \leq 1.$$

By Theorem 1 we get that every composition operator is bounded on B. Note also that if $\|\phi\|_\infty < 1$ and C_ϕ is bounded on B^α, then C_ϕ must be compact on B^α.

Our main interest is the closed range composition operators on B^α. In the case when ϕ is a disc automorphism with $\phi(0) = a$, we have that C_ϕ is bounded on B^α since

$$\tau_{\phi,\alpha}(z) \leq \Big(\frac{1+|a|}{1-|a|}\Big)^{|1-\alpha|}.$$

It is also an invertible operator, with inverse $C_{\phi^{-1}}$ and thus, trivially, C_ϕ has a closed range.

Since every non-constant ϕ is an open map, the composition operator C_ϕ is always one-to-one. By a basic operator theory result, a one-to-one operator has a closed range if and only if it is bounded below. Thus, C_ϕ has a closed range if and only if it is bounded below, namely if and only if there exists $M > 0$ such that

$$\|C_\phi f\|_{B^\alpha} \geq M\|f\|_{B^\alpha}, \quad \forall f \in B^\alpha.$$

In particular, every isometric C_ϕ has a closed range. The isometric composition operators on B^α have been studied in a number of papers (such as [4, 5, 12, 19, 23]). Since they provide examples of closed range composition operators, we give a theorem that states (a version of) the results on the classification of isometric composition operators on the Bloch-type spaces.

Theorem 2 ([5, 23]) *Let ϕ be an analytic selfmap of $\mathbb{D}$.*

(i) *The composition operator C_ϕ is an isometry on the Bloch space B if and only if $\phi(0) = 0$ and either ϕ is a rotation, or for every a in $\mathbb{D}$ there exists a sequence $\{z_n\}$ in $\mathbb{D}$ such that $|z_n| \to 1$, $\phi(z_n) = a$ and $\tau_\phi(z_n) \to 1$.*

(ii) *If $0 < \alpha$, $\alpha \neq 1$, then the composition operator C_ϕ is an isometry on B^α if and only if ϕ is a rotation.*

On the other hand, recall that the only (closed) subspaces of the range of a compact operator are the finite dimensional ones. A composition operator C_ϕ cannot have a finite rank (because ϕ is an open map), and so a compact composition operator can never have a closed range.

Closed range composition operators on the Bloch-type spaces have been studied in [2, 3, 7, 8, 24]. To state the results obtained, we need the following definitions.

Let $\rho(z, w) = |\psi_z(w)|$ denote the pseudo-hyperbolic distance on $\mathbb{D}$, where ψ_z is a disc automorphism of $\mathbb{D}$ defined by

$$\psi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$

We say that $G \subseteq \mathbb{D}$ is an *r-net* for $\mathbb{D}$ for some $r \in (0, 1)$ if for all $z \in \mathbb{D}$, $\exists w \in G$ such that $\rho(z, w) < r$.

For $\alpha > 0$, $c > 0$ and ϕ a selfmap of $\mathbb{D}$, let $\Omega_{c,\alpha} = \{z \in \mathbb{D} : \tau_{\phi,\alpha}(z) \geq c\}$, and let $G_{c,\alpha} = \phi(\Omega_{c,\alpha})$.

We will use some of the results obtained in the papers cited above, which we state in the following theorem.

Theorem 3 ([3, 24]) *Let ϕ be a selfmap of $\mathbb{D}$ and let $\alpha > 0$.*

(i) *For $\alpha \geq 1$, C_ϕ has a closed range on B^α if and only if there exist $c > 0$, $0 < r < 1$ such that $G_{c,\alpha}$ is an r-net for $\mathbb{D}$.*

(ii) *For $0 < \alpha < 1$, if C_ϕ has a closed range on B^α, then there exist $c > 0$, $0 < r < 1$ such that $G_{c,\alpha}$ is an r-net for $\mathbb{D}$.*

The geometric aspects of the theorem are particularly interesting in the case when ϕ is univalent and $\alpha = 1$, since, as a consequence of the Koebe one-quarter theorem, we have that

$$\tau_\phi(z) \approx \frac{\operatorname{dist}(\phi(z), \partial\phi(\mathbb{D}))}{1 - |\phi(z)|}.$$

Thus, we get that if, for example, ϕ is the Riemann mapping from $\mathbb{D}$ onto the slit disc $\mathbb{D}\backslash[0, 1]$, then C_ϕ has a closed range on B. Or if ϕ is the Riemann mapping onto the simply connected region in $\mathbb{D}$ created by taking away an infinite number of slits and pseudo-hyperbolic discs connected to the slits, one gets either a closed range or a non-closed range composition operator C_ϕ by controlling the size and the placement of the pseudo-hyperbolic discs. For more details on these examples, see [8].

On the other hand, it was shown in [2] that the univalent map ϕ from $\mathbb{D}$ onto the slit disc induces a composition operator C_ϕ that does not have a closed range on any B^α with $\alpha > 1$. Since, moreover, the only isometric composition operators on these spaces are induced by rotations, a natural question arises. Could it be that the only

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univalently induced, closed range, composition operators on B^α with $\alpha > 1$ are the trivial ones, namely, those that are induced by a disc automorphism? As we shall see, the answer is yes, both in this case and in the case when $\alpha < 1$. A similar situation arises on some of the weighted Dirichlet spaces and on the Bergman and Hardy space, where the only univalently induced composition operators are the invertible ones, as shown in [1, 9].

An important part in the proof of our result is the angular derivative behaviour of the inducing function. Recall that if ϕ maps the unit disc into itself, we say that ϕ has an *angular derivative* at $\xi \in \partial\mathbb{D}$ if there is a $\zeta \in \partial\mathbb{D}$ such that $\lim_{r\to 1} \phi(r\xi) = \zeta$ and $\lim_{r\to 1} \phi'(r\xi) = \eta$. In this case we say that $\eta = \phi'(\xi)$ is the angular derivative of ϕ at ξ.

Note that the existence of the radial limits in the definition of the angular derivative is equivalent to the existence of the corresponding nontangential, or angular limits. By the Julia–Caratheodory theorem, their existence is also equivalent to the existence of (a finite) $\liminf_{z\to\xi} \frac{1-|\phi(z)|}{1-|z|}$ with the limit furthermore equal to $|\phi'(\xi)|$. For more details see [6, 17].

It is known that if C_ϕ is bounded on B^α with $0 < \alpha < 1$, then ϕ has to have an angular derivative at every point that is mapped onto the unit circle (see, for example, [6]). This restriction on ϕ is not necessary for the boundedness of C_ϕ on B^α when $1 < \alpha$. But, as we shall see in the proof below, if C_ϕ also has a closed range on B^α, then ϕ has to have an angular derivative over a significant part of the unit circle.

Theorem 4 *Let $\alpha > 0, \alpha \neq 1$, and let ϕ be a univalent selfmap of $\mathbb{D}$. Then C_ϕ has a closed range on B^α if and only if ϕ is a disc automorphism.*

Proof Recall that if ϕ is a disc automorphism, then C_ϕ is an invertible operator, and so it has a closed range.

The other direction of the proof is given in two parts, separating the cases $\alpha < 1$ and $\alpha > 1$.

Case $0 < \alpha < 1$:

Let ϕ be univalent, and let C_ϕ have a closed range on B^α with $0 < \alpha < 1$. Let $E = \{\xi \in \partial\mathbb{D} : |\phi(\xi)| = 1\}$. The set E is not empty, since otherwise $\|\phi\|_\infty < 1$, and C_ϕ is compact, which is not possible since C_ϕ has a closed range. As mentioned above, when C_ϕ is bounded on B^α with $0 < \alpha < 1$, then ϕ is continuous on the closed unit disc and has angular derivative at every point in E. But then ϕ also has to be univalent on E, since if the points ξ_1 and ξ_2 from E are mapped into the same point on $\partial\mathbb{D}$, and ϕ is univalent on $\mathbb{D}$, then ϕ cannot have an angular derivative at both ξ_1 and ξ_2 (see, for example, [9, Lemma 3.3] or [17, p. 74]).

Next we show that $\phi(E) = \partial\mathbb{D}$.

Since C_ϕ has a closed range on B^α, by Theorem 3, there exist $c > 0$ and $0 < r < 1$ such that $G_{c,\alpha}$ is an r-net for $\mathbb{D}$. We will show that $\partial\mathbb{D} \subset \overline{G_{c,\alpha}}$, where the closure of the set is its closure in $\mathbb{C}$. Suppose that there exists $\zeta \in \partial\mathbb{D}$ that is not in $\overline{G_{c,\alpha}}$. Then $\exists t > 0$ such that $D_t(\zeta) = B(\zeta, t) \cap \mathbb{D}$ is in the complement of $\overline{G_{c,\alpha}}$, where $B(\zeta, t)$ is the open ball centered at ζ and with radius t. Let $a_n = (1 - \frac{1}{n^2})^{\frac{1}{2}}\zeta$, and let $r_n = (1 - 1/n)^{1/2}$. Then, eventually, a_n is in $D_t(\zeta)$ and $\rho(a_n, w) > r_n$ for every w in

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$\overline{G_{c,\alpha}}$. Since $r_n \to 1$ as $n \to \infty$, we get a contradiction to the fact that $G_{c,\alpha}$ is an r-net for $\mathbb{D}$. Hence, $\partial\mathbb{D} \subset \overline{G_{c,\alpha}} \subset \overline{\phi(\mathbb{D})} = \phi(\overline{\mathbb{D}})$ and so $\phi(E) = \partial\mathbb{D}$.

Thus, ϕ is a continuous bijection from the compact set E onto $\partial\mathbb{D}$, and so ϕ is a homeomorphism between E and $\partial\mathbb{D}$. But then E must be a connected subset of $\partial\mathbb{D}$, *i.e.*, E is a closed arc $[\xi_1, \xi_2]$. Since $\phi((\xi_1, \xi_2)) = \partial\mathbb{D} \setminus \{\phi(\xi_1), \phi(\xi_2)\}$ is also connected, we have that $\phi(\xi_1) = \phi(\xi_2)$, $\xi_1 = \xi_2$, and $E = \partial\mathbb{D}$. So, ϕ is inner and univalent, which implies that ϕ is a disc automorphism.

Case $\alpha > 1$:

We need to show that if ϕ is univalent and C_ϕ has a closed range on $B^\alpha, \alpha > 1$, then ϕ is a disc automorphism. In the case when $\alpha > 1$, the functions in B^α are not necessarily continuous on the unit circle. Since the inducing function ϕ is in H^∞, it has radial limits almost everywhere on the unit cirle. We denote the radial extension function with the same symbol ϕ.

Note that in general, if ϕ has no angular derivative at any point on $E = \{\xi \in \partial\mathbb{D} : |\phi(\xi)| = 1\}$, then C_ϕ must be compact on $B^\alpha, \alpha > 1$. This follows from the fact that in that case $\limsup_{|z|\to 1} \frac{1-|z|}{1-|\phi(z)|} = 0$, and since $\alpha - 1 > 0$ and $\tau_\phi(z) \leq 1$, we have that

$$\lim_{|\phi(z)|\to 1} \tau_{\phi,\alpha}(z) = \lim_{|\phi(z)|\to 1} \Big(\frac{1-|z|^2}{1-|\phi(z)|^2}\Big)^{\alpha-1} \tau_\phi(z) = 0.$$

Hence, if C_ϕ has a closed range on $B^\alpha, \alpha > 1$, then ϕ has an angular derivative at some $\xi \in E$. As we will see below, even more is true, *i.e.*, ϕ is continuous and has an (uniformly bounded) angular derivative on a subset of $\partial\mathbb{D}$ that is mapped onto $\partial\mathbb{D}$. This, together with the univalence of ϕ, will be enough to imply that ϕ has to be a disc automorphism.

By Theorem 3, there exist $c > 0$ and $0 < r < 1$ such that $G_{c,\alpha}$ is an r-net for $\mathbb{D}$. Let $E_c = E \cap \overline{\Omega_{c,\alpha}}$, where $E = \{\xi \in \partial\mathbb{D} : |\phi(\xi)| = 1\}$. We will show first that $E_c \neq \varnothing$ and then that ϕ is continuous on E_c. (Note that, as before, $E \neq \varnothing$, since C_ϕ is not compact.)

If $\overline{\Omega_{c,\alpha}}$ is contained in $\mathbb{D}$, then $\phi(\overline{\Omega_{c,\alpha}})$ is a compact subset of $\mathbb{D}$ and so $\overline{G_{c,\alpha}}$ is a compact subset of $\mathbb{D}$. But that cannot happen, similarly as in the proof of the case $\alpha < 1$, since $G_{c,\alpha}$ is an r-net of $\mathbb{D}$. Thus $\overline{\Omega_{c,\alpha}} \cap \partial\mathbb{D} \neq \varnothing$. Next we show that

$$\overline{\Omega_{c,\alpha}} \cap \partial\mathbb{D} = \overline{\Omega_{c,\alpha}} \cap E = E_c.$$

Let $\xi \in \overline{\Omega_{c,\alpha}} \cap \partial\mathbb{D}$. Then $\exists\{z_n\}$ in $\Omega_{c,\alpha}$ such that $z_n \to \xi$ and

$$c \leq \tau_{\phi,\alpha}(z_n) = \Big(\frac{1-|z_n|^2}{1-|\phi(z_n)|^2}\Big)^{\alpha-1} \tau_\phi(z_n) \leq \Big(\frac{1-|z_n|^2}{1-|\phi(z_n)|^2}\Big)^{\alpha-1}.$$

Recall that, by the Julia–Caratheodory theorem, ϕ has an angular derivative at ξ, since we have that

$$\liminf_{z\to\xi} \frac{1-|\phi(z)|}{1-|z|} \leq \Big(\frac{1}{c}\Big)^{\frac{1}{\alpha-1}}.$$

Thus, ϕ has an angular (radial) limit ζ of modulus 1 at ξ, *i.e.*, $\xi \in E$, and we have that $E_c = \overline{\Omega_{c,\alpha}} \cap \partial\mathbb{D} \neq \varnothing$. Note that the above also shows that for all $\xi \in E_c$ the

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angular derivative at ξ not only exists, but is also uniformly bounded, since $|\phi'(\xi)| \leq (1/c)^{\frac{1}{\alpha-1}}$.

To prove that ϕ is continuous on E_c, we use Julia's Theorem (see [CM] for more details), which states that whenever ϕ has an angular derivative at ξ,

$$\frac{|\phi(\xi)-\phi(z)|^2}{|\xi - z|^2} \leq |\phi'(\xi)| \frac{1-|\phi(z)|^2}{1-|z|^2}, \quad \forall z \in \mathbb{D}.$$

From the uniform boundedness of the angular derivatives on E_c, we get that for every $\xi \in E_c$ and every $z \in \Omega_{c,\alpha}$,

$$|\phi(\xi)-\phi(z)| \leq \left(\frac{1}{c}\right)^{\frac{1}{\alpha-1}} |\xi - z|.$$

But the points in E_c are boundary points of $\Omega_{c,\alpha}$, and so the extension of ϕ to E_c is continuous.

Since $G_{c,\alpha}$ is an r-net, $\partial\mathbb{D}$ must be contained in $\overline{G_{c,\alpha}}$. If $\{z_n\}$ is a sequence of points in $\Omega_{c,\alpha}$ such that $\phi(z_n) \to \zeta \in \partial\mathbb{D}$, then $\{z_n\}$ (or a subsequence) converges to a point $\xi \in \partial\mathbb{D}$ with $\phi(\xi) = \zeta$, and so

$$\partial\mathbb{D} \subset \overline{\phi(\Omega_{c,\alpha})} \cap \partial\mathbb{D} \subset \phi(\overline{\Omega_{c,\alpha}} \cap \partial\mathbb{D}) = \phi(E_c).$$

Thus, as before, since ϕ is univalent on $\mathbb{D}$, ϕ must be univalent on E_c and so ϕ is a homeomorphism between E_c and $\partial\mathbb{D}$, *i.e.*, $E_c = \partial\mathbb{D}$. Hence ϕ is inner and univalent, and so ϕ is a disc automorphism. ■

There are many non-univalently induced, closed range, composition operators on the Bloch-type spaces. For example, when $\alpha = 1$ and ϕ is not a rotation, the isometric composition operators from Theorem 2(i) are induced by functions with infinite multiplicities. The following proposition provides a large class of non-univalent functions inducing closed range composition operators on all of the Bloch-type spaces. For example, such are the functions $\phi(z) = z^n$, $n \geq 2$, or more generally, all of the finite Blaschke products. A result similar to the one below also holds in the case of the Bergman, Hardy, and some small weighted Dirichlet spaces (for more details see [1, 9, 22]).

Proposition 5 *Let C_ϕ be bounded on B^α. If there exist $c > 0$ and $0 < r_0 < 1$ such that $G_{c,\alpha}$ contains the annulus $A = \{z : r_0 < |z| < 1\}$, then C_ϕ has a closed range on B^α.*

Proof The case $\alpha \geq 1$ follows from Theorem 3(i), since $G_{c,\alpha}$ is an r-net for any $r > r_0$.

Let $0 < \alpha < 1$, and let $G_{c,\alpha}$ contain the annulus $A = \{z : r_0 < |z| < 1\}$. Suppose also that there exists a sequence of functions $\{f_n\}$ with $\|f_n\|_{B^\alpha} = 1$ and such that $\|C_\phi(f_n)\|_{B^\alpha} \to 0$. Since $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f_n'(z)| = 1$, there exists a sequence $\{a_n\}$ in $\mathbb{D}$ such that for all n,

$$(1-|a_n|^2)^\alpha |f_n'(a_n)| \geq \frac{1}{2}.$$

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Let $\varepsilon = \frac{c}{2}$, and let N be such that for all $n > N$, we have $\|C_\phi(f_n)\|_{B^\alpha} < \varepsilon$. Then

$$\sup_{w\in G_{c,\alpha}} \big(1-|w|^2\big)^\alpha |f_n'(w)| \le \frac{1}{c} \sup_{z\in\Omega_{c,\alpha}} \tau_{\phi,\alpha}(z)\big(1-|\phi(z)|^2\big)^\alpha |f_n'(\phi(z))|$$
$$\le \frac{1}{c}\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)^\alpha |f_n'(\phi(z))||\phi'(z)| < \frac{1}{c}\varepsilon = \frac{1}{2},$$

and so each a_n with $n > N$ belongs to the complement of $G_{c,\alpha}$. Thus $|a_n| \le r_0 < 1$ and $a_n \to a$ with $|a| \le r_0$.

On the other hand, since $\|f_n\|_{B^\alpha} = 1$, a normal families argument implies that there exists a subsequence $\{f_{n_m}\}$ that converges uniformly on $\mathbb{D}$ to some function $f \in B^\alpha$. But then $\{f_{n_m}'\}$ converges to f' uniformly on compact subsets of $\mathbb{D}$, and since $\sup_{w\in G_{c,\alpha}}(1-|w|^2)^\alpha|f_n'(w)| \to 0$ and $G_{c,\alpha}$ contains an infinite compact subset of $\mathbb{D}$, we get that $f' \equiv 0$. This contradicts the fact that $|f'(a)|(1-|a|^2)^\alpha \ge \frac{1}{2}$, so C_ϕ must be bounded below, *i.e.,* C_ϕ has a closed range. ∎

The closed range composition operators acting on B^α with $0 < \alpha < 1$ have not yet been fully characterized. Recall that in this case the Bloch-type spaces are subalgebras of $A(\mathbb{D})$. It has been shown that a composition operator C_ϕ has a closed range on $A(\mathbb{D})$, or on H^∞, if and only if $\partial\mathbb{D} \subset \overline{\phi(\mathbb{D})}$ (see [14, 15]). On the other hand, if C_ϕ has a closed range on B^α, by Theorem 3(ii), there exist $c > 0, 0 < r < 1$ such that $G_{c,\alpha}$ is an r-net for $\mathbb{D}$. When $G_{c,\alpha}$ is an r-net , we have that $\partial\mathbb{D} \subset \overline{G_{c,\alpha}} \subset \overline{\phi(\mathbb{D})}$, which is a natural alteration of the characterization in the $A(\mathbb{D})$ and H^∞ case. Thus, we propose the following conjecture.

Conjecture *Let $0 < \alpha < 1$, and let C_ϕ be bounded on B^α. Then C_ϕ has a closed range on B^α if and only if there exists $c > 0$ such that $\partial\mathbb{D} \subset \overline{G_{c,\alpha}}$.*

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Department of Mathematics, University of Manitoba, Winnipeg, MB, R3T 2N2
e-mail: zorbosk@cc.umanitoba.ca

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