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SECANT SPACES TO CURVES

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Introduction. A classical question in algebraic geometry is whether a given projection of a projective space induces an isomorphism on a given closed subvariety. To answer it, one investigates secant lines to the subvariety. There has been a lot of recent activity in this field ([**12**], [**14**], [**18**], [**21**], [**23**]): see [**14**] and [**12**] for references).

An obvious generalization of the secant lines is provided by the secant r-planes, which intersect a given closed subvariety in $r + 1$ linearly independent points. The closure of the set of these secant r-planes is the secant variety, and the aim of this paper is to determine its rational equivalence class in the case of curves. There is an extensive classical literature about this problem.

In Section 1, we introduce the class of curves that we consider in this paper, namely the "r-twisted" curves, on which any $r + 1$ points are linearly independent. It turns out that this condition is generically satisfied, at least if the degree of the curve and the dimension of the ambient space are large enough. In Section 2, we consider for an r-twisted curve $X \subseteq \mathbf{P}^n$ the natural morphism from X^{r+1} to the Grassmannian mapping an $(r + 1)$-tuple to the corresponding r-plane. This morphism is separable of degree $(r + 1)!$, as expected.

For the computation of the rational equivalence class of the secant variety we consider in Section 3 the inverse image of the Chern class of the tautological bundle under this morphism. Using an exact sequence, similarly as in [**23**], we express this inverse image by diagonal divisors and the hyperplane section. In Section 4, we give a recipe for computing the degree of products of such divisors, and thus obtain an algorithm for determining the rational equivalence class. In particular, it follows that this class depends only on the degree and genus of the curve, a statement that was classically assumed to be true for any curve under consideration. We give an example in Section 5.2 where this statement is not true (involving a curve that is not 2-twisted).

In Section 5, we consider curves in $\mathbf{P}^3$ that are projections of r-twisted curves. Again, this is a generic condition for large enough degree. We

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obtain a host of enumerative formulas for such curves, which were all classically known ([**3**], [**4**], [**5**], [**25**], [**26**], [**27**], [**28**], [**31**], [**32**]).

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1. Twisted curves. We fix some notation. V is a vector space of finite dimension $n + 1$ over an algebraically closed field k, and the corresponding projective space $P = \mathrm{Proj}(V)$ consists of the lines in V. $G(r, P)$ is the Grassmann variety of r-planes ($=$ r-dimensional linear subspaces) in P. A choice of basis for V gives a vector space isomorphism $T:V \rightarrow \mathbf{A}^{n+1}$ and homogeneous coordinates $T_0, \dots, T_n$ on P.

Definition 1.1. Let $X \subseteq P$ be a curve, $x \in X$ smooth, $0 \leqq p \leqq n$ with $T_p(x) \neq 0$, $t \in \mathfrak{O}_{x,X}$ a local parameter, $\sum_{0 \leqq j} f_{ij} t^j$ the Taylor series of $(T_i/T_p \restriction X) \in \mathfrak{O}_{x,X}$, and $W \subseteq A^{n+1}$ the linear subspace spanned by the $r + 1$ vectors $(f_{00}, \dots, f_{n0}), \dots, (f_{0r}, \dots, f_{nr})$. Then

$$\prod\nolimits_{x,X}^{r} = \mathrm{Proj}(T^{-1}(W)) \subseteq P$$

is the *r-th* osculating space of X at x.

Thus $\prod^r_{x,X}$ is a subspace of P of dimension $\leqq r$, and one checks that it is independent of the choices of T, p, and t. Obviously, $\prod^0_{x,X} = \{x\}$ and $\prod^1_{x,X}$ is the projective tangent line of X at x. (Piene [**24**] calls "osculating m-space" the $\prod^r_{x,X}$ with $\dim \prod^r_{x,X} = m$. These spaces have the nice property of being the projective fibers of a vector bundle, if the characteristic is zero or large enough.)

It is not hard to show that $\prod^r_{x,X}$ is the intersection of all hyperplanes that meet X at x with multiplicity $\geqq r + 1$. This property could serve as a definition of osculating spaces also for singular points x on X. Furthermore, it shows that $\prod^d_{x,X} = P$, where $d = \deg X$ and X is non-degenerate.

If $\dim \prod^r_{x,X} = r$ for all but finitely many $x \in X$, one obtains a morphism $\delta_r: X \rightarrow G(r, P)$. This hypothesis is always satisfied in characteristic zero, and we get the usual definition of osculating spaces ([**7**], Chapter 2).

Let $\rho(r, X)(x_0, \dots, x_r)$ denote the linear span of $x_0, \dots, x_r \in X$, taking into account multiplicities, i.e., if

$$R = x_0 + \ldots + x_r = (a_0 + 1)\, y_0 + \ldots + (a_s + 1) y_s \in \mathrm{Div}_{r+1}$$

is an effective divisor with $y_0, \ldots, y_s$ pairwise distinct, then $\rho(r, X)(x_0, \ldots, x_r)$ is the linear span of $\prod_{y_0,X}^{a_0}, \ldots, \prod_{y_s,X}^{a_s}$ ([**8**], p. 243).

Definition 1.2. X is called *twisted at* R (with respect to the given embedding) if $\dim \rho(r, X)(x_0, \ldots, x_r) = r$. If X is twisted at every $R \in \mathrm{Div}_{r+1}$, then X is called *r-twisted.*

Thus an embedded curve is r-twisted if any $r + 1$ distinct (possibly infinitely near) points are linearly independent. We shall write ρ for $\rho(r, X)$ if no confusion is possible. If a linear system S resp. a complete linear system $|D|$ on X induces an r-twisted embedding, we also call S resp. D r-twisted.

Example 1.3. The rational norm curve of degree n in $\mathbf{P}^n$ is n-twisted. In order to see this, we can consider the affine curve given by

$$\begin{aligned} &\boldsymbol{\varphi}: \mathbf{A}^1 \to \mathbf{A}^n \\ &y \mapsto (y, y^2, \ldots, y^n), \end{aligned}$$

and assume $y_0, \ldots, y_s \in \mathbf{A}^1$, $a_0, \ldots, a_s \geqq 0$ and

$$R = \textstyle\sum_{0 \leqq i \leqq s} (a_i + 1) \boldsymbol{\varphi}(y_i) \in \mathrm{Div}_{n+1}.$$

The linear span in $\mathbf{P}^n$ of the corresponding osculating spaces is given by the rows of the $(n + 1) \times (n + 1)$-matrix with entries

$$\binom{i}{j} y_l^{i-j} \quad (0 \leqq l \leqq s, 0 \leqq j \leqq a_l; 0 \leqq i \leqq n).$$

If a linear combination of the columns of this matrix is 0, say with coefficients $u_0, \ldots, u_n \in k$, then $\sum_{0 \leqq i \leqq n} u_i T^i$ is a polynomial with an $(a_l + 1)$-fold zero at y_l for every $0 \leqq l \leqq s$, and hence the zero polynomial. Thus the curve is twisted at R. (We will see in Section 5.4 that this is the only n-twisted curve in $\mathbf{P}^n$.)

The results of Sections 2, 3, 4 are technically independent of the rest of this section, where we now show that a generic embedding $X \hookrightarrow \mathbf{P}^n$ is r-twisted if $2r < n$. The following proposition reduces this question to embeddings given by complete linear systems, and provides a criterion in this case.

PROPOSITION 1.4. *Let* $0 \leqq r \leqq d$, $D \in \mathrm{Div}_d$ *a very ample divisor, and* $2r < m \leqq n = \dim|D|$. *Then*

(i) *there exists a proper closed subvariety* $Z \subseteq G(m, |D|)$ *such that any very ample linear system* $S \in G(m, |D|)$ *is r-twisted if and only if D is r-twisted and* $S \notin Z$,

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(ii) *D is r-twisted if and only if $l(D-R) = l(D)-r-1$ for all $R \in \mathrm{Div}_{r+1}$.*

Proof. Let $S \subseteq |D|$ be a very ample linear system of dimension m, $X \hookrightarrow \mathbf{P}^m$ and $X \hookrightarrow \mathbf{P}^n$ the embeddings given by S and $|D|$, resp. Let

$$R = (a_0 + 1)\, x_0 + \ldots + (a_s + 1)\, x_s \in \mathrm{Div}_{r+1}$$

with pairwise distinct $x_0, \ldots, x_s$, L the linear span in $\mathbf{P}^m$ of

$$\prod\nolimits_{x_0,X}^{a_0}, \ldots, \prod\nolimits_{x_s,X}^{a_s}$$

and

$$Y(R) = \{E \in |D| : E - R \geqq 0\}$$

which is isomorphic to $|D-R|$. Then

$$\begin{aligned}
&\{H \in G(m - 1, \mathbf{P}^m) : L \subseteq H\} \\
&= \{H \in G(m - 1, \mathbf{P}^m) : X.H - R \geqq 0\} \\
&\cong \{E \in S : E - R \geqq 0\} = S \cap Y(R)
\end{aligned}$$

with dimension

$$\geqq m + \dim |D-R| - \dim |D| \geqq m - r - 1.$$

Thus we have

$$\begin{aligned}
&S \text{ is twisted at } R \Leftrightarrow \dim L \geqq r \\
&\Leftrightarrow \dim S \cap Y(R) \leqq m - r - 1 \\
&\Leftrightarrow S \text{ and } Y(R) \text{ intersect properly in } |D|, \\
&\text{and } l(D - R) = l(D) - r - 1.
\end{aligned}$$

This proves (ii) (taking $S = |D|$), and (i) follows from the fact that

$$Z = \{S \in G(m, |D|) : \exists\, R \in \mathrm{Div}_{r+1} \text{ such that } \dim S \cap Y(R) \geqq m - r\}$$

is closed and has only components of dimension

$$\leqq 2r - m + \dim G(m, |D|) < \dim G(m, |D|).$$

THEOREM 1.5. *Let X be an irreducible smooth projective curve of genus g, and $r \geqq 0$.*

(i) *If $2g + r \leqq d$, then any divisor in Div_d is r-twisted.*

(ii) *If $g + 2r < d$, then a generic divisor in Div_d is r-twisted.*

(iii) *Let $g \geqq 2$. The canonical divisor is r-twisted if and only if*

$$\dim |R| = 0 \textit{ for all } R \in \mathrm{Div}_{r+1}.$$

Proof. (i) follows from the fact that for any $D \in \mathrm{Div}_d$, $R \in \mathrm{Div}_{r+1}$ both D and $D-R$ are non-special, and hence $l(D-R) = l(D) - r - 1$. Also

(iii) follows immediately from Proposition 1.4 (ii) and Riemann-Roch.

For (ii), we note that for any $m \geqq 0$ all components of

$$C_m^{m-g+1} = \{D \in \mathrm{Div}_m : \dim |D| \geqq m - g + 1\}$$

have dimension $\leqq g - 1$. For a general curve over $\mathbf{C}$, this would follow from the solution to the Brill-Noether-problem ([**8**]). One can see our simple case directly by taking a canonical divisor K, and

$$\begin{aligned}
&T = \{ (E, F) \in |K| \times \mathrm{Div}_{2g-2-m} : E - F \geqq 0\},\\
&\psi : T \to \mathrm{Div}_m\\
&(E, F) \mapsto E - F.
\end{aligned}$$

Then $C_m^{m-g+1} = \psi(T)$ and $\dim T \leqq g - 1$. In particular, $U = \mathrm{Div}_d \setminus C_d^{d-g+1}$ is open and non-empty, and $C = C_{d-r-1}^{d-r-g}$ has dimension $\leqq g - 1$. For any m, let

$$f_m : \mathrm{Div}_m \to \mathrm{Pic}_m$$

denote the Abel map. Consider the commutative diagram

$$\begin{tikzcd}
C \times \mathrm{Div}_{r+1} \arrow[r, "\alpha"] \arrow[d, "f_{d-r-1} \times \mathrm{id}"'] & \mathrm{Div}_d \arrow[d, "f_d"] \\
\mathrm{Pic}_{d-r-1} \times \mathrm{Div}_{r+1} \arrow[r] & \mathrm{Pic}_d
\end{tikzcd}$$

where the horizontal morphisms are induced by addition of divisors. The fibers of $f_{d-r-1} \upharpoonright C$ have dimension $\geqq d - r - g$, and the fibers of $f_d \upharpoonright U$ have dimension $d - g$. Hence

$$U' = U \setminus (f_d^{-1} \circ f_d \circ \alpha(C \times \mathrm{Div}_{r+1}))$$

is non-empty and open in Div_d. Now one checks that every divisor in U' is r-twisted.

Remarks 1.6. The same proof will show that if $r(s + g - d + 2) < s$, then the r-twisted divisors are dense in C_d^s, provided that C_{d-r-1}^{s-r} has the Brill-Noether-dimension. This is true for a general curve over $\mathbf{C}$ by [**8**].

1.7. Very ample is equivalent to 1-twisted, and our statements generalize well-known facts for this case. Compare e.g. [**9**], Chapter IV, 3.1(b), 3.2(b), 6.1, and 5.2.

1.8. Let $g, r \geqq 0$, $n \geqq 2r + 1$ and $d \geqq \min \{2g + r, g + 2r + 1\}$. We have shown that for any (abstract) curve of genus g, almost all very ample linear systems of degree d and dimension n are r-twisted, and will refer to this fact by saying that "a generic curve is r-twisted". It is not hard to see that $2r + 1$ is the minimal value for n with this property, generalizing the minimal embedding dimension in the case $r = 1$.

For a partial converse, namely a condition on g, d, r, n under which no r-twisted curve in $\mathbf{P}^n$ of degree d and genus g exists, see Theorem 4.2.

2. The secant variety.

Definition 2.1. Let $X \subseteq P$ be a curve. The *secant variety* $\mathrm{Sec}(r, X)$ is the closure in $G(r, P)$ of

$$\{L \in G(r, P): L \cap X \text{ contains } r + 1 \text{ linearly independent points}\}.$$

If X is irreducible and not contained in an r-plane of P, then this is an irreducible variety of dimension $r + 1$. (Throughout the paper, we allow our (reduced) curves to be reducible.) In this section, we start with the computations that lead to the determination of its rational equivalence class in the Chow ring of $G(r, P)$ for an r-twisted curve X.

PROPOSITION 2.2. *If $X \subseteq P$ is r-twisted, then $\rho(r, X): X^{r+1} \to G(r, P)$ is a morphism, and its image is* $\mathrm{Sec}(r, X)$.

Proof. The second statement is clear. For the first one, one reduces inductively to the claim that ρ is a morphism near $y = (x, \dots, x) \in X^{r+1}$. Choose a local parameter $t \in \mathfrak{O}_{x,X}$ and coordinates on P so that $\prod^r_{x,X}$ is given by the first $r + 1$ unit vectors in $\mathbf{A}^{n+1}$, and let

$$t_i = (i\text{-}th \text{ projection})^* (t) \in \mathfrak{O}_{y,X^{r+1}}.$$

The coordinates yield a morphism μ from a neighbourhood of y to $\bigwedge^{r+1} \mathbf{A}^{n+1}$ which, composed with the map to projective space in which $G(r, P)$ is embedded, is equal to ρ for linearly independent tuples in X^{r+1}. Dividing μ by $\prod_{i<j}(t_i - t_j)$, one gets a morphism that also has this property, and moreover is defined at y with image $\rho(y)$. Thus, ρ is a morphism.

Example 2.3. Let $\mathrm{char}\, k = p > 2$ and

$$\begin{aligned} \varphi: \mathbf{A}^1 &\to \mathbf{A}^4 \\ u &\mapsto (u, u^p, u^{p^2}, u^{p^2+1}). \end{aligned}$$

$X = \overline{\mathrm{Im}\varphi} \subseteq \mathbf{P}^4$ is a smooth non-degenerate curve, and

$$\prod\nolimits^2_{\varphi(u),X} = \prod\nolimits^1_{\varphi(u),X}$$

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has dimension 1 for all $u \in \mathbf{A}^1$. The secant 2-planes of X have two limiting positions over every $(\varphi(u), \varphi(u), \varphi(u))$, so there is no morphism $\rho(2, X)$ as for 2-twisted curves.

In the next lemma we state some geometrically obvious facts about the behaviour of osculating spaces and twistedness under linear projections. For simplicity, we only consider a projection $\psi:P \setminus \{z\} \rightarrow H$ from a point $z \in X$, where $H \subseteq P$ is a hyperplane not containing z. Let $Y = \overline{\psi(X \setminus \{z\})}$ and $\varphi:X \rightarrow Y$ be the induced morphism.

LEMMA 2.4. (i) *φ is unramified at $x \in X$ if and only if*

$$\dim \rho(2, X)(x, x, z) = 2.$$

(ii) *Let $r \geqq 1$, X be r-twisted, and $x_0, \ldots, x_{r-1} \in X$. Then*

$$\rho(r-1, Y)(\varphi(X_0), \ldots, \varphi(x_{r-1})) = \rho(r, X)(x_0, \ldots, x_{r-1}, z) \cap H.$$

(iii) *Let $r \geqq 2$ and X be r-twisted. Then φ is an isomorphism, and Y is $(r-1)$-twisted.*

Proof. We first prove (i) in the case $x = z$, the other case being clear. Choose homogeneous coordinates $T_0, \ldots, T_n$ on P such that $z = (0:\ldots:0:1)$ and $H = \{T_n = 0\}$, and a local parameter $t \in \mathfrak{O}_{z,X}$. Let $\sum_{0 \leqq j} f_{ij}t^j$ be the Taylor series of $f_i = (T_i / T_n \upharpoonright X) \in \mathfrak{O}_{z,X}$. Then

$$\varphi(z) = (f_{01}:\ldots:f_{0,n-1}),$$

and we can assume $f_{01} \neq 0$. Also, $e = t/t_0 \in \mathfrak{O}_{z,X}$ is a unit with Taylor series

$$\sum_{0 \leqq k} e_k t^k = 1 / f_{01} - (f_{02} / f_{01}^2)\, t + \ldots$$

and locally around z, φ is given by the morphism $(g_1, \ldots, g_{n-1})$ to $\mathbf{A}^{n-1}$, where

$$g_i = f_i / f_0 = f_{i1}e_0 + (f_{i1}e_1 + f_{i2}e_0)t + \ldots$$

Thus

$$\varphi \text{ is unramified at } z \Leftrightarrow (e_1(f_{11}, \ldots, f_{n-1,1}) + e_0(f_{12}, \ldots, f_{n-1,2}))d_z t \neq 0$$

$$\Leftrightarrow (f_{01}, \ldots, f_{n-1,1}) \text{ and } (f_{02}, \ldots, f_{n-1,2}) \text{ are linearly independent}$$
$$\Leftrightarrow \dim \textstyle\prod^2_{z,X} = 2.$$

This proves (i), and the first statement of (iii) follows e.g. from [**29**], Chapter II, § 5.5. For the rest of the lemma, it suffices to prove (ii) in the case $r \geqq 2$ and $x_0 = \ldots = x_{r-1} = z$. Denote by $S_0, \ldots, S_{n-1}$ the induced coordinates on H, and let $s \in \mathfrak{O}_{\varphi(z),Y}$ be a local parameter with $\varphi^*(s) = t$. Then

$$T_i / T_0 \restriction X = f_i / f_0 = (f_i / t)e \text{ in } \mathfrak{O}_{z,X},$$

and for the Taylor series $\sum g_{ik} s^k$ of $S_i/S_0 \restriction Y$ we have

$$g_{ik} = \sum_{0 \leqq j \leqq k} f_{i,j+1}\, e_{k-j}$$

which implies (ii).

We call a secant space $L \in \mathrm{Sec}(r, X)$ with $\# L \cap X \geqq r + 2$ a *multisecant.*

THEOREM 2.5. *Let $X \subseteq P$ be r-twisted, and such that no irreducible component of X is contained in an $\mathbf{P}^{r+1}$. Then*

(i) *a generic secant space in* $\mathrm{Sec}(r, X)$ *is not a multisecant,*

(ii) *$\rho(r, X)$ induces a finite separable surjective morphism of degree $(r + 1)!$ from X^{r+1} to* $\mathrm{Sec}(r, X)$.

Proof. For (i), we first consider the case $r = 1$, and modify the proof of the statement which is well-known for irreducible curves ([**21**], see e.g. [**9**], IV. 3.8 + 3.9) to include reducible curves. So we can assume that there are three components X_1, X_2, X_3 of X (at least two distinct) such that

$$\forall x_1 \in X_1, x_2 \in X_2 \; \exists \; x_3 \in X_3 \setminus \{x_1, x_2\} : x_3 \in \rho(x_1, x_2).$$

Then the corresponding condition also holds for any permutation of $\{1, 2, 3\}$. For any $x_1 \in X_1$, the images of X_2 and X_3 under the projection φ_{x_1} from x_1 are equal. If two points x_2 and x_3 have the same image and this is smooth on $\varphi_{x_1}(X_2)$, then the tangent lines $\prod^1_{x_2,X_2}$ and $\prod^1_{x_3,X_3}$ intersect.

Since for a generic (x_2, x_3) such an x_1 exists, it follows that any two tangent lines to X_2 and X_3 intersect. By symmetry, this also holds for X_1 and X_2. Now let L_1 resp. L_2 be distinct tangent lines to X_1 resp. X_2. Then every tangent line to X_3 either passes through their intersection point or is contained in their plane. In either case (using [**9**], IV. 3.9, for the first one) it follows that X_3 is a plane curve, contradicting the hypothesis.

Now the general statement follows by induction on r, choosing a projection center $z \in X$ on a multisecant and using Lemma 2.4 (iii).

By (i), it suffices for (ii) to find $y \in X^{r+1}$ such that the differential $d_y\rho$ is injective. The complement U of

$$\rho^{-1}(\mathrm{Sing}(\mathrm{Sec}(r, X)) \cup \rho(\bigcup_{0 \leqq i < j \leqq r} D_{ij}))$$

is open and dense in X^{r+1}, where $D_{ij} = \{x_i = x_j\}$ is a diagonal.

Choose a $y = (x_0, \ldots, x_r) \in U$ and homogeneous coordinates $T_0, \ldots, T_n$ on P such that

$$\forall i \leqq r \quad \prod^1_{x_i, X} \not\subseteq \{T_n = 0\}, \ x_i = (1:0:\ldots:0:1:0:\ldots:0)$$

with a 1 in the i-th position. The Plücker coordinates T_a on $G(r, P)$ are indexed by the nationalities $a = (a_0, \ldots, a_r)$ with $0 \leqq a_0 < \ldots < a_r \leqq n$. Letting $L = \rho(y)$, we consider the following functions

$$\forall j, 0 \leqq j \leqq r, \quad g_j = \left(\frac{T_{0 \ldots \hat{j} \ldots rn}}{T_{0 \ldots r}} \upharpoonright \mathrm{Sec}(r, X)\right) \in \mathfrak{O}_{L, \mathrm{Sec}(r, X)},$$

$$\forall i, j, 0 \leqq i \leqq r, 0 \leqq j \leqq n, \quad f_{ij} = (i\text{-th projection}) * (T_j / T_0 \upharpoonright X) \in \mathfrak{O}_{y, X^{r+1}}.$$

For all i, $(T_n / T_0 \upharpoonright X) \in \mathfrak{O}_{x_i, X}$ is a local parameter, and hence $f_{0n}, \ldots, f_{rn} \in \mathfrak{O}_{y, X^{r+1}}$ is a system of local parameters. The bilinear relations describing the incidence of a point with an r-plane yield in particular the following equations in $\mathfrak{O}_{y, X^{r+1}}$

$$\forall i \leqq r \quad \rho^*(g_0) + \sum_{1 \leqq j \leqq r} (-1)^j f_{ij} \rho^*(g_j) + (-1)^{r+1} f_{in} = 0.$$

Taking differentials gives the following equations in $\Omega_{y, X^{r+1}}$

$$\forall i, 1 \leqq i \leqq r, \quad 0 = \rho^*(d_L(g_0)) + (-1)^i \rho^*(d_L(g_i)) + (-1)^{r+1} d_y(f_{in}),$$

$$0 = \rho^*(d_L(g_0)) + (-1)^{r+1} d_y(f_{0n}).$$

$\Omega_{y, X^{r+1}}$ is generated over k by $d_y(f_{0n}), \ldots, d_y(f_{rn})$, and hence

$$\rho^*: \Omega_{L, \mathrm{Sec}(r, X)} \to \Omega_{y, X^{r+1}}$$

is surjective, and $d_y\rho$ injective (even an isomorphism).

Theorem 2.5 shows that an r-twisted embedding $X \subseteq P$ gives an embedding of the symmetric product $X^{(r+1)} \cong \mathrm{Sec}(r, X) \subseteq G(r, P) \subseteq Q$ into some projective space Q.

3. An exact sequence. Let $1 \leqq r \leqq \dim P$, and

$$Y = \{(x_0, \ldots, x_r, L, M) \in P^{r+1} \times G(r-1, P) \times G(r, P): x_0, \ldots, x_{r-1} \in L \subseteq M \text{ and } x_r \in M\}.$$

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Y is an irreducible smooth closed subvariety. The product projections induce morphisms $\varphi_0, \ldots, \varphi_r, \psi_{r-1}, \psi_r$. For $0 \leqq i \leqq r$, denote by J_i the incidence bundle (tautological bundle) over $G(i, P)$, a subbundle of $G(i, P) \times V$ of rank $i + 1$. Let

$$E = \varphi_r^* (J_0), \quad F_i = \psi_i^*(J_i) \quad \text{for } i = r - 1, r, \quad \text{and}$$
$$W = \{ (x_0, \ldots, x_r, L, M) \in Y : x_r \in L \}.$$

W is an irreducible divisor on Y, and we denote by G the corresponding line bundle. The following lemma generalizes an exact sequence from [**23**].

LEMMA 3.1 *There is an exact sequence on Y*

$$0 \to F_{r-1} \to F_r \to E \otimes G \to 0.$$

Proof. We have a natural exact sequence on $U = Y \setminus W$

$$0 \to F_{r-1} \upharpoonright U \to F_r \upharpoonright U \to E \upharpoonright U \to 0.$$

Since W is irreducible, there exists an $m \in \mathbf{Z}$ such that the cokernel of the embedding $F_{r-1} \hookrightarrow F_r$ is isomorphic to $E \otimes G^m$. Choose $y_0, \ldots, y_l \in P$ linearly independent, let M_0 be their linear span, and

$$Z = \{ (x_0, \ldots, x_r, L, M) \in Y : M = M_0 \text{ and } \forall\, i \leqq r - 1\ x_i = y_i \}.$$

Z is isomorphic to M_0 via φ_r. In the exact sequence

$$0 \to F_{r-1} \upharpoonright Z \to F_r \upharpoonright Z \to (E \upharpoonright Z) \otimes (G \upharpoonright Z)^m \to 0$$

the left and middle terms are trivial, $E \upharpoonright Z$ corresponds to the incidence bundle on M_0, and $G \upharpoonright Z$ corresponds to the hyperplane bundle $(E \upharpoonright Z)^{-1}$. Hence $m = 1$.

In the sequel we denote for a smooth projective variety Z by

$$A(Z) = \bigoplus_{0 \leqq i \leqq \dim Z} A^i(Z)$$

the Chow ring of Z, graduated by the codimension. A morphism $\varphi: Z \to Z'$ induces the inverse image morphism φ^* and the Gysin morphism φ_*. If $E \to Z$ is a vector bundle, then $c(E) = 1 + c_1(E) + \ldots \in A(Z)$ is its Chern class. The projection formula says that for the degree of a 0-dimensional class z we have

$$\int_Z z = \int_{Z'} \varphi_*(z).$$

Let $X \subseteq P$ be an r-twisted curve, and $0 \leqq i < j \leqq r$. We denote by d_{ij} the class in $A^1(X^{r+1})$ of the diagonal $D_{ij} = \{x_i = x_j\}$. For $0 \leqq i \leqq r$, let $\pi_i : X^{r+1} \rightarrow P$ be the i-*th* projection composed with the embedding $X \subseteq P$, and

$$d_{ii} = \pi_i^* \, (c_1(J_0)) = -\pi_i^* \, (\text{hyperplane class}) \in A^1(X^{r+1}).$$

The next theorem describes the inverse image of $c(J_r)$ along $\rho(r, X)$ in terms of the "simple" divisors d_{ij}.

THEOREM 3.2. *Let* $0 \leqq r \leqq \dim P$, *and* $X \subseteq P$ *be an r-twisted curve. Then*

$$\rho(r, X)^* \, (c(J_r)) = (1 + d_{00})(1 + d_{01} + d_{11}) \dots \\ (1 + d_{0r} + \dots + d_{rr}).$$

Proof. Consider the morphism

$$\begin{aligned} &\eta : X^{r+1} \rightarrow Y \\ &(x_0, \dots, x_r) \mapsto (x_0, \dots, x_r, \rho(r-1, X)(x_0, \dots, x_{r-1}), \\ &\qquad\qquad \rho(r, X)(x_0, \dots, x_r)). \end{aligned}$$

We prove the theorem by induction on r, the case $r = 0$ being clear. Let $w \in A^1(Y)$ be the class of W. We claim that

$$\eta^* \, (W) = D_0 + \dots + D_{r-1,r}.$$

Then

$$\begin{aligned} \rho(r, X)^* \, (c(J_r)) &= \eta^* \, (c(F_r)) = \eta^* \, (c(F_{r-1}) \, c \, (E \otimes G)) \\ &= (\psi_{r-1} \circ \eta)_* \, (c(J_{r-1})) \cdot (1 + \eta^* \, (w) + (\varphi_r \circ \eta)_* \, (c_1(J_0))) \\ &= (1 + d_{00}) \dots (1 + d_{0,r-1} + \dots + d_{r-1,r-1})(1 + d_{0r} + \dots \\ &\qquad + d_{r-1,r} + d_{rr}). \end{aligned}$$

Since set-theoretically the claim is clear, it is sufficient to find a $y \notin f(D_{0r})$ and $\omega \in \Omega_{y,Y}$ such that ω vanishes on $T_{y,W}$, but not on $T_{y,\eta(X^{r+1})}$, thus showing that $\eta(X^{r+1})$ and W intersect transversally at $\eta(D_{0r})$. Choose homogeneous coordinates $T_0, \dots, T_n$ such that the unit points $e_0, \dots, e_{r-1}$ are in X, and $e_r \in \prod^1_{x_0,X}$. Let

$$y = \eta(e_0, \dots, e_{r-1}, e_0),$$

and consider the following functions in $\mathfrak{O}_{y,Y}$:

$$f_{ij} = \begin{cases} \varphi_j^* \, (T_i \, / \, T_j) \text{ for } 0 \leqq i \leqq r, \ 0 \leqq j \leqq r-1 \\ \varphi_r^* \, (T_i \, / \, T_0) \text{ for } 0 \leqq i \leqq r, \ j = r \end{cases}$$

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$$f = \det((f_{ij})_{0 \leqq i,j \leqq r}).$$

Then f vanishes on W, $(T_r/T_0 \upharpoonright X) \in \mathfrak{O}_{x_0,X}$ is a local parameter, and $\omega = d_y(f)$ has the required properties.

4. The rational equivalence class of the secant variety. We now want to give a method for computing the rational equivalence class in $A(G(r, P))$ of $\mathrm{Sec}(r, X)$ for an r-twisted curve $X \subseteq P$ of degree d and genus g. That is, we want to determine the integer coefficients (Schubert: "Gradzahlen") of this class, when written as a linear combination of Schubert classes $\Omega(a; P)$, where a runs through the nationalities $a = (a_0, \dots, a_r)$ with $0 \leqq a_0 < \dots < a_r \leqq n = \dim P$. We will call them the *Schubert coefficients* of this class. If we set

$$\sec(r, X) = \frac{1}{(r + 1)!} \rho(r, X)_* (1)$$

then by Theorem 2.5 (ii) this is the class of $\mathrm{Sec}(r, X)$. (Unless some component of X is contained in a $\mathbf{P}^{r+1}$. If X is a rational norm curve of degree r, then $\sec(r, X) = 0$.) By Poincaré duality, we have to compute the degree

$$\int_{G(r,P)} \sec(r, X)\, \Omega(b; P)$$

for all nationalities b. We denote by

$$\zeta_i(r, P) = \Omega(n - r - 1, \dots, n - r + i - 2, n - r + i, \dots, n; P)$$

the "Giambelli class" of codimension i, usually leaving away (r, P) when there is no confusion. Thus

$$\sum_{0 \leqq i \leqq r+1} (-1)^i \zeta_i = c(J_r)$$

is the total Chern class of J_r, and $\zeta_1, \dots, \zeta_{r+1}$ are ring generators for $A(G(r, P))$. (For the facts from Schubert calculus, see [**11**], [**16**], [**17**]). It is sufficient to compute

$$\int_{G(r,P)} \sec(r, X)\, p = \frac{1}{(r + 1)!} \int_{X^{r+1}} \rho(r, X)^* (p)$$

for all products p of $\zeta_1, \dots, \zeta_{r+1}$ with codimension $r + 1$. $\rho^* (p)$ is a sum of products q of the d_{ij} with $r + 1$ factors, and we first determine the degree of such a product q. We associate to q a graph $h(q)$ on $\{0, \dots, r\}$ by connecting node i to node j as many times as d_{ij} appears in q. The

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graphs that we get in this way are undirected, and possibly have loops or multiple edges.

Let G_r be the set of all such graphs with the additional property that every connected component has as many edges as nodes. For $h \in G_r$, we set

$$\int h = (-d)^s(-k)^t,$$

where $s + t$ is the number of connected components of h, s the number of those with a loop, and $k = 2g - 2$ the degree of the canonical divisor class on X.

LEMMA 4.1. *For every product q of the d_{ij} with $r + 1$ factors, we have*

$$\int_{X^{r+1}} q = \begin{cases} \int h(q) & \text{if } h(q) \in G_r \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use induction on r. There are two basic cases:

Case 1: $r = 0$, $q = d_{00}$. Then

$$\int q = -d = \int h(q).$$

Case 2: $r = 1$, $q = d_{01}^2$. Let $\Delta:X \to X^2$ be the diagonal embedding. The normal bundle N_Δ is isomorphic to the tangent bundle T_X, and by the self-intersection formula we have

$$\int_{X^2} q = \int_{X^2} \Delta_* (1)^2 = \int_X \Delta^* (\Delta_* (1))$$
$$= \int c_1(N_\Delta) = \int c_1(T_X) = -k.$$

Now for the induction step, by splitting q into two factors, each corresponding to a similar product on some X^{u+1} with $u < r$, we first reduce to the case where $h(q)$ is in G_r and connected. Then there is either a node with degree (= number of edges issuing from it) one, or else all nodes have degree two. Unless we are in case 1 or case 2 above, we leave away one node (with degree one if the first possibility happens) and there is an obvious product q' on X^r with $\int q = \int q'$ and

$$\int h(q) = \int h(q').$$

THEOREM 4.2. *There are polynomials $f_a \in \mathbf{Z}[D, G]$, where a runs through all $a = (a_0, \ldots, a_r)$ with $0 \leqq a_0 < \ldots < a_r$, such that for any r-twisted curve $X \subseteq P$ of degree d and genus g, and $n = \dim P$*

$$\int \sec(r, X)\,\Omega(n - a_r, \dots, n - a_0; P) = \frac{1}{(r+1)!} f_a(d, g).$$

In particular, this number is an integer, and if $f_a(d, g) \neq 0$ *and* $n < a_r$ *for some* a, *then there is no such curve* X. *If*

$$a_0 + \dots + a_r \neq (r+1)(r+2)/2,$$

then $f_a = 0$.

Proof. Writing out the determinantal formula, one gets integer coefficients $c_{e_1}, \dots, c_{e_{r+1}}$ not depending on n such that for any projective space P we have in $A(G(r, P))$

$$\sum_{0 \leqq e_1, \dots, e_{r+1}} c_{e_1, \dots, e_{r+1}} \zeta_1(r, P)^{e_1} \dots \zeta_{r+1}(r, P)^{e_{r+1}}$$

$$= \begin{cases} \Omega(n - a_r, \dots, n - a_0; P) & \text{if } a_r \leqq n \\ 0 & \text{otherwise.} \end{cases}$$

Let $X \subseteq P$ be an r-twisted curve. By Theorem 3.2 and Lemma 4.1,

$$\int \rho(r, X)^* (\zeta_1(r, P)^{e_1} \dots \zeta_{r+1}(r, P)^{e_{r+1}})$$

is a polynomial in d and g with "universal" integer coefficients. Summing these with coefficients $c_{e_1, \dots, e_{r+1}}$, one gets f_a as in the theorem. The fact that $\mathrm{Sec}(r, X)$ has dimension $r + 1$ implies the last statement.

Example 4.3. For an elliptic curve, there exist smooth embeddings in $\mathbf{P}^3$ of degree 6 (see [**9**], Chapter IV, Corollary 6.2). But no such 2-twisted embedding exists, since $f_{(0,4,5)}(6, 1) = 2 \neq 0$ and $n = 3 < 5 = a_2$.

5. Examples and multisecants.

5.1. The discussion of Section 4 has yielded an algorithm to compute $\sec(r, X)$. For $r = 1, 2$ and an r-twisted curve X of degree d and genus g we have

$$\sec(1, X) = \binom{d}{2} \Omega(1, 2; P) + \frac{1}{2}(d(d-3) - 2g + 2)\,\Omega(0, 3; P)$$

$$\sec(2, X) = \binom{d}{3} \Omega(1, 2, 3; P) + (d-2)\left(\frac{d(d-4)}{3} - g + 1\right) \Omega(0, 2, 4; P)$$

$$+ (d-4)\left(\frac{d(d-5)}{6} - g + 1\right) \Omega(0, 1, 5; P).$$

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5.2. A curve that is not 2-twisted. Let $X \subseteq \mathbf{P}^6$ be an irreducible non-degenerate 2-twisted curve such that

$$\alpha = \int \sec(2, X)\, \Omega(1, 5, 6; \mathbf{P}^6)$$
$$= (d - 4)\left(\frac{d(d - 5)}{6} - g + 1\right) \neq 0.$$

There exists a non-empty open subset $U \subseteq G(1, \mathbf{P}^6)$ such that for all $Z \in U$, $\mathrm{Sec}(2, X)$ and $\sigma(Z), = \Omega(Z, \mathbf{P}^5, \mathbf{P}^6; \mathbf{P}^6)$ intersect properly. Choose $Z \in U$, $L \in \mathrm{Sec}(2, X) \cap \sigma(Z)$, $z \in L \cap Z$ and a hyperplane $H \subseteq \mathbf{P}^6$ not containing z. The projection from z induces an isomorphism $\varphi: X \to Y \subseteq H$. X and Y have the same degree and genus, and Y is not 2-twisted, the image of L being a trisecant line. $\{M \in \mathrm{Sec}(2, Y): M \cap Y$ does not span $M\}$ is contained in a proper closed subset T of $\mathrm{Sec}(2, Y)$, and we have a closed embedding $\psi: H \to G(1, \mathbf{P}^6)$ by mapping y to the line through y and z. We assume that there exists a non-empty open $V \subseteq H$ such that for all $y \in V$ we have:

$\mathrm{Sec}(2, Y) \cap \Omega\,(y, H; H)$ is transversal, and

$T \cap \Omega(y, H; H) = \emptyset.$

In characteristic zero such a V exists by the transversality of a general translate [**13**]. Let

$$y \in V \cap \psi^{-1}(U),\ Z' = \psi(y) \quad \text{and} \quad M \in \mathrm{Sec}(2, Y) \cap \Omega(z, \mathbf{P}^4, H; H).$$

Let

$$M' = \rho(2, X)(x_0, x_1, x_2) \in \mathrm{Sec}(2, X) \cap \sigma(Z'),$$

where $\varphi(x_0)$, $\varphi(x_1)$, $\varphi(x_2)$ span M. Since $L \in \sigma(Z')$, there are at most $\alpha - 1$ many such M'. The association $M \mapsto M'$ is injective, and hence

$$\int \sec(2, Y)\, \Omega(0, 4, 5; H) < \alpha$$

and thus the number given by Theorem 4.2 is not correct for Y. Also, by varying z appropriately, one gets a family of curves which are all 2-twisted with one exception, and our intersection number is not constant in this family.

Put the other way round, it seems that our definition of the intersection number via $\mathrm{Sec}(r, X) \subseteq G(r, P)$ is not appropriate for the general case. See Section 5.7.

5.3. Let $X \subseteq P$ be an r-twisted curve, $e = (e_0, \ldots, e_t)$ with $e_i \geqq 0$ and $e_0 + \ldots + e_t = r + 1$.

$$\Delta_e : X^{t+1} \to X^{r+1}$$

$$(x_0, \ldots, x_t) \mapsto (\underbrace{x_0, \ldots, x_0}_{e_0}, \ldots, \underbrace{x_t, \ldots, x_t}_{e_t})$$

is the diagonal embedding of type e

$$\mathrm{Sec}(r, e, X) = \rho(r, X) \circ \Delta_e(X^{t+1})$$

is the closure of the set of those r-spaces in $\mathrm{Sec}(r, X)$ for which the $(r + 1)$-fold intersection consists of e_i-fold contacts, $0 \leqq i \leqq t$. Let

$$f_i = \# \{j : e_j = i\} \quad \text{and} \quad f = f_1! \ldots f_{r+1}! .$$

Using an appropriate analog of Theorem 2.4, we find for the rational equivalence class

$$(1) \quad \mathrm{sec}(r, e, X) = \frac{1}{f} (\rho(r, X) \circ \Delta_e)_*(1)$$

and one gets integer polynomials in d, g as in Theorem 4.2 which, divided by f, give the Schubert coefficients of $\mathrm{sec}(r, e, X)$.

We can consider the analogous situation for a generic projection Y of X. So let Q resp. Z be generic m-resp. $(n-m-1)$-dimensional subspaces of P which do not meet. Let $\varphi : P \setminus Z \to Q$ be the projection, and $\psi : X \to Y = \overline{\varphi(X)}$ be induced by φ. We assume $m \geqq 3$, so that ψ is an isomorphism, and $0 \leqq s \leqq m - 1$. Since hyperplanes meet a curve in too many points, in the case $s = m - 1$ we also assume that $e_0, \ldots, e_t \geqq 2$. Let

$$W = \{ (L, M) \in G(s, Q) \times G(r, P) : L \subseteq M \text{ and } \varphi(M \setminus Z) \subseteq L \}$$

with projections p resp. q to $G(s, Q)$ resp. $G(r, P)$. p is locally trivial, and q is birational onto its image, which is a Schubert variety of codimension $(r - s)(m - s)$. Thus

$$S = p(q^{-1}(\mathrm{Sec}(r, e, X)))$$

is either empty or has only components of dimension $t + 1 - (r - s)(m - s)$. It contains those $(r + 1)$-secant s-spaces which have e_i-fold contacts with Y, $0 \leqq i \leqq t$. Also, $p \restriction q^{-1}(\mathrm{Sec}(r, e, X))$ is generically (on every component) injective. S is the support of the class

$$\mathrm{sec}(r, s, e, Y) = p_*(q^*(\mathrm{sec}(r, e, X))).$$

For any $\Omega(b; Q) \in A(G(s, Q))$ we have

$$\int_{G(s,Q)} \sec(r, s, e, Y)\, \Omega(b; Q)$$
$$= \int_{G(r,P)} \sec(r, e, X)\, \Omega(n - m - r + s, \dots, n - m - 1,$$
$$n - m + b_0, \dots, n - m + b_s; P).$$

For the sequel, we fix the notation of this section (in particular, Y is a generic projection of the r-twisted curve X for an appropriate r), and will compute some of the above intersection numbers. By remark 1.8, for an abstract curve Y a generic embedding $Y \subsetneq \mathbf{P}^n$ has this property if $n > 2r$ and the degree is $\geqq \min\{2g + r, g + 2r + 1\}$.

5.4. Plücker formulas. Let

$$\delta_r = \rho(r, X) \circ \Delta_{(r+1)}: X \rightarrow G(r, P)$$

be the *r-th* associated map. The degree

$$d_r = \int \delta_{r*}(1)\, \zeta_1\, (r, P)$$

of the *r-th* associated curve is, by the Plücker formulas ([**24**], [**7**]), equal to

$$d_r = (r + 1)(d + r(g - 1)) + \sum_{0 \leqq i \leqq r-1} (r - i)\, \beta_i$$

where β_i is the total ramification of δ_i. We have

$$d_r = \int_{G(r,P)} \sec(r, (r + 1), X)\, \zeta_1\, (r, P)$$
$$= (r + 1)(d + r(g - 1)),$$

i.e., the Plücker formulas hold without ramification. Now if the characteristic is zero or coprime to r, then δ_{r-1} is in fact unramified. But, somewhat unexpectedly, if the characteristic p is positive and p^a divides r, then δ_{r-1} is purely inseparable of degree $\geqq p^a$.

If X is n-twisted, we get

$$0 = d_n = (n + 1)(d + n(g - 1)),$$

hence $g = 0$ and $d = n$, and X is a rational norm curve. Thus these are the only n-twisted curves in $\mathbf{P}^n$.

We get the same Plücker formulas for our generic projection $Y \subseteq Q$ ($r \leqq m - 1$). Canuto [**1**] has proved that for a generic non-special embedding $Y \subseteq \mathbf{P}^m$ over $\mathbf{C}$ the morphisms $\delta_0, \dots, \delta_{m-2}$ are unramified.

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5.5. Hyperosculating points. A point $y \in Y$ is called *hyperosculating* (also "stationary" or "singular") of order r if $\dim \prod^{r}_{y,Y} < r$. These points correspond to $\mathrm{Sec}(r, (r + 1), Y)$ which is non-empty only for $r = m$. Thus there is only one non-zero "stationary index" [**24**], namely

$$\int \sec((m, m - 1, (m + 1), Y) = d_m.$$

5.6. The table below gives an exhaustive list of numerical characters involving a single kind of secant space of a curve $Y \subseteq \mathbf{P}^3$, where Y is a generic projection of an r-twisted curve as in 5.3. These characters are of the form $\int \sec(r, s, e, Y) \cdot z$ where the additional condition z is either some $\Omega(a, \mathbf{P}^3)$ or the class of $\Delta^*_{(1,0,\dots,0)}(p)$ for some $p \in Y$. They are all easily computed using the method of Section 4. For example, the number of "doubly tangent planes" (i.e., planes containing two tangent lines to Y) through a generic point of $\mathbf{P}^3$ is

$$\int_{G(2,\mathbf{P}^3)} \sec(3, 2, (2, 2), Y)\, \Omega(0, 2, 3; \mathbf{P}^3)$$
$$= \int_{G(3,P)} \sec(3, (2, 2), X)\, \Omega(n - 4, n - 3, n - 1, n; P)$$
$$= \frac{1}{2} \int_{X^2} (\rho\,(3, X) \circ \Delta_{(2,2)})^* (\zeta_2(3, P))$$
$$= \frac{1}{2} \int_{X^2} (1 + d_{00})(1 - \pi_0^*(k_X) + d_{00})(1 + 2d_{01} + d_{11})(1 + 2d_{01} - \pi_1^*(k_X) + d_{11})$$
$$= 2((d + 2g - 2)(d - 3) + g(g - 1))$$
$$= \frac{1}{2} (d_1(d_1 - 6) - 8(g - 1)) \quad [\mathbf{3}].$$

Plücker, [**25**], No. 63, gives this as the number of double tangents to a plane curve.

We also include the degrees $\delta(\mu_i)$ of the double point cycles of μ_i, where

$$\mu_1 : Y \to H$$
$$y \mapsto \prod^{1}_{y,Y} \cap H,$$
$$\mu_2 : Y \to G(1, H)$$
$$y \mapsto \prod^{2}_{y,Y} \cap H,$$

TABLE – Numerical characters concerning sec(r, s, e, Y) of a curve $Y \subseteq \boldsymbol{P}^3$ of degree d and genus g which is a generic projection of an r-twisted curve.

r	s	e	dim	additional condition	number	description
0	0	(1)	1	$\Omega(2; \mathbf{P}^3)$	d	points in a plane
1	1	(2)	1	$\Omega(1,3;\mathbf{P}^3)$	$d_1 = 2d + 2g - 2$	tangents meeting a line
1	1	(2)	1	$\delta(\mu_1)$	$2(2(d+g)^2 - 7d - 8g + 6)$	pairs of tangents meeting in a plane
1	1	(1, 1)	2	$\Omega(0,3;\mathbf{P}^3)$	$\frac{1}{2}d(d-3) - g + 1$	apparent double points
1	1	(1, 1)	2	$\Omega(1, 2; \mathbf{P}^3)$	$\frac{1}{2}d(d-1)$	secant lines in a plane
2	1	(1, 1, 1)	1	$\Omega(1, 3; \mathbf{P}^3)$	$(d-2)(\frac{1}{3}(d-1)(d-3) - g)$	trisecants meeting a line
2	1	(1, 1, 1)	1	fix a point	$\frac{1}{2}(d-2)(d-3) - g$	trisecants through a point on the curve
2	1	(2,1)	0		$2(d(d-4) + (g-1)(d-6))$	re-intersecting tangents
2	2	(3)	1	$\Omega(0, 2, 3; \mathbf{P}^3)$	$d_2 = 3d + 6g - 6$	osculating planes through a point
2	2	(3)	1	$\delta(\mu_2)$	$d_2(d_2 - 3) - 2g + 2$	lines in a plane contained in two osculating planes
3	1	(1, 1, 1, 1)	0		$\frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{g}{2}(d^2 - 7d - g + 13)$	quadrisecants
3	2	(2, 2)	1	$\Omega(0, 2, 3; \mathbf{P}^3)$	$2((d + 2g - 2)(d-3) + g(g-1))$	doubly tangent planes through a point
3	2	(2, 2)	1	fix a point	$2d + 2g - 6$	tangents meeting a fixed tangent
3	2	(4)	0		$d_3 = 4d + 12g - 12$	hyperosculating points
4	2	(3, 2)	0		$6((d + g - 7)(d + 2g) + 12)$	osculating planes containing a tangent
5	2	(2, 2, 2)	0		$\frac{4}{3}((d+g)^3 - 12d^2 - 30dg - 18g^2 + 47d + 77g - 60)$	triply tangent planes

and $H \subseteq \mathbf{P}^3$ is a generic plane. These are computed by the double point formula ([**14**], [**18**]).

5.7. The classical results. Hilbert's [**10**] 15th problem asks "to establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially had determined ..." It is generally assumed that the classical numerical results are valid at least "in the general case", but e.g. Giambelli's [**6**] explicit formula for the Schubert coefficients of $\sec(r, X)$ is false for the curve considered in 5.2. The results of this paper prove validity of many formulas, e.g. those of this section, and "general case" meaning "generic projection of an r-twisted curve" for the appropriate r.

All our computations agree with the classical results. With some additional work one can also obtain a proof (valid for r-twisted curves) of Giambelli's formula.

We note that the classical counting methods may sometimes differ from ours. E. g. μ_2 is ramified at the d_3 hyperosculating points, and Cayley [**3**] gives the number of lines in a fixed plane contained in two osculating spaces as

$$\frac{1}{2}\left(d_2\left(d_2 - 3\right) - 2g + 2\right) - d_3.$$

The classical arguments, often involving degeneration, assume a priori that the numbers to be computed depend only on the degree and genus. The authors usually were aware that there was a lacuna; e.g. Castelnuovo [**2**] in a footnote justifies the assumption "... from the fact that in the plane and ordinary space these numbers can be computed by more direct methods as functions of the degree and genus only".

Another ubiquitous (implicit) assumption in the classical work is that intersections are proper and transversal. Using this assumption and correspondences, Macdonald [**22**] gives a nice generalization and simplification of Giambelli's formula, stating a closed formula for the intersection of $\sec(r, s, e, Y)$ with any number of Schubert conditions.

There are some invariants, e.g. the number of hyperosculating points of order $r < m$ for $Y \subseteq \mathbf{P}^m$ (see 5.5), that are always zero for the curves considered here, but may be nonzero for some special curves. It remains an open problem to extend the verification of the classical formulas to more general classes of curves, where also some of these special invariants would be allowed to be nonzero. The most ambitious goal would be to prove them for all curves.

A remarkable result by Le Barz [**19**, **20**] achieves this goal for curves over **C** and three of our formulas (lines 6, 8, 11 of the table). He defines the numbers via the intersection of the two Hilbert schemes of tuples of points on Y and collinear tuples in $\mathbf{P}^3$. Using the Fulton-MacPherson intersection theory, it turns out that the above polynomials in d and g give the correct number, even if the value is negative or there are infinitely many secants. Thus it seems that his definition is more appropriate for the general case than ours. In the same intersection theory, Vainsencher [**30**] proves various classical enumerative results concerning hypersurfaces in projective space.

5.8. In this section we compute some numerical characters defined by imposing conditions on two kinds of secant spaces simultaneously.

The degree of the surface consisting of secant lines that are contained in some osculating space of the curve is

$$\int_{X^3} (\rho(4, X) \circ \Delta_{(1,1,3)})^* (\zeta_2(4, P))(\rho(1, X) \circ \Delta_{(1,1,0)})^* (\zeta_1(1, P))$$
$$= 3(d^3 + 2d^2g - 8d^2 - 11dg - 2g^2 + 19d + 14g - 12).$$

Taking the osculating space to be at one of the two intersection points, the degree of the surface is

$$4d^2 + 6dg - 16d - 12g + 12.$$

Zeuthen [**31**], [**32**] computes these numbers, and most of the other ones of this section. Severi [**28**] gives generalizations to curves in $\mathbf{P}^n$.

The number α of pairs of points, each contained in the osculating space at the other one, is

$$\alpha = \int (\rho(3, X) \circ \Delta_{(3,1)})^* (\zeta_1(3, P))(\rho(3, X) \circ \Delta_{(1,3)})^* (\zeta_1(3, P))$$
$$= 2(5d^2 + 18dg + 18g^2 - 30d - 63g + 45).$$

Here we count the hyperosculating points and the "principal chords", i.e., secant lines contained in the two osculating planes at the intersection points. Severi [**28**] gives this latter number as $\frac{1}{2}(\alpha - d_3)$.

The number of quadruples of points, three of which lie on a line that meets the tangent at the fourth and such that the plane spanned by them contains a fixed generic point of $\mathbf{P}^3$, is

$$\beta = 4(d^4 + d^3 g - 10d^3 - 11d^2g - 6dg^2 + 37d^2 + 39dg + 10g^2$$
$$- 60d - 46g + 36).$$

Taking the tangent to be at one of the three points, we get the number

$$\gamma = 3d^3 + 2d^2g - 21d^2 - 20dg - 4g^2 + 48d + 40g - 36.$$

Here we count the re-intersecting tangents, say δ in number, and the planes through a generic point that contain a trisecant and one of the three tangents at the intersection points. Zeuthen gives the number of these planes as $\frac{1}{2}(\gamma - \delta)$, and the number of planes through a generic point containing a trisecant and a tangent as $\frac{1}{2}(\beta - 2\gamma)$, in agreement with our results.

The number of quadruples of points, three of which are on a line and contained in the osculating plane at the fourth, is

$$6(d^4 + 2d^3g - 12d^3 - 21d^2g - 6dg^2 + 53d^2 + 76dg + 24g^2 - 102d - 96g + 72).$$

Taking the osculating plane to be at one of the three collinear points, the number is

$$2(2d^3 + 3d^2g - 16d^2 - 22dg - 6g^2 + 42d + 42g - 36).$$

5.9. We want to prove six of Cayley's [**4**] enumerative results involving more than one curve. Let $X, X', X'', X''' \subseteq P$ be curves of degrees $d, \dots, d'''$ and genera $g, \dots, g'''$ such that $\bar{X} = X \cup X' \cup X'' \cup X'''$ is 3-twisted, let $Y, \dots, Y''' \subseteq \mathbf{P}^3$ be the images under a generic projection, and

$$\mu: X \times X' \times X'' \times X''' \to \bar{X}^4$$

the embedding. Since the results of Section 4 apply to reducible curves, the number of lines intersecting all four curves $Y, \dots, Y'''$ is

$$\begin{aligned}
&\int (\rho(3, \bar{X}) \circ \mu)^* (\Omega(n-5, n-4, n-1, n; P)) \\
&= \int (((d_{00} + d_{11} + d_{22})(d_{22} + d_{33}) + d_{00}d_{11})^2 \\
&\quad - (d_{00} + d_{11} + d_{22} + d_{33})((d_{00} + d_{11}) d_{22} d_{33} \\
&\qquad + d_{00} d_{11}(d_{22} + d_{33}))) \\
&= 2\, d\, d'\, d''\, d'''
\end{aligned}$$

(see [**16**]).

For the number of lines meeting Y twice, and also intersecting Y' and Y'', consider the embedding

$$\mu': X \times X \times X' \times X'' \to (X \cup X' \cup X'')^4.$$

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Then this number is

$$\int (\rho(3, X \cup X' \cup X'') \circ \mu')^* (\Omega(n - 5, n - 4, n - 1, n; P)) = (d^2 - 2d - g + 1)\, d'd''.$$

Similarly, one finds that the number of lines meeting Y three times and intersecting Y' is

$$(d - 2)\left(\frac{d(d - 4)}{3} - g + 1\right) d',$$

and the number of common chords of Y and Y' is

$$\frac{1}{2}\left((d^2 - 2d - g + 1)(d'^2 - 2d' - g' + 1) + (d + g - 1)(d' + g' - 1)\right).$$

(see [**15**]).

As a particular case, two twisted cubics in general position have 10 common chords [**5**].

For the degree of the surface consisting of lines meeting Y, Y' and Y'', consider the embedding

$$\mu'':X \times X' \times X'' \rightarrow (X \cup X' \cup X'')^3.$$

Then this degree is

$$\int (\rho(2, X \cup X' \cup X'') \circ \mu'')^* (\Omega(n - 4, n - 2, n; P)) = 2\, d\, d'\, d'',$$

and the degree of the surface consisting of secant lines of Y that meet Y' is $(d^2 - 2d - g + 1)d'$.

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