20

From 2d bosonized baryons to 4d Skyrmions

20.1 Introduction

Low energy effective actions associated with four-dimensional QCD, and in particular the Skyrme model have been very thoroughly studied with an emphasis on both formal aspects such as anomalies as well as phenomenological ones like the spectrum of baryons. This chapter is devoted to the Skyrme model. We first derive the various terms of the Skyrme action. These include the sigma term, the WZ term, the mass term and the Skyrme term. The first three terms we have encountered already in the two-dimensional analog, the bosonized QCD (Chapter 13) whereas the fourth one, the Skyrme term, shows up as a stabilization term only in the four-dimensional case. We then construct the classical soliton solution, the Skyrmion, of the corresponding equations of motion. Next we determine the classical mass and radius of the baryon. In a similar manner to the procedure taken in the two-dimensional model, we quantize the system semiclassically. This yields mass splitting between the nucleons and the delta particles and the axial coupling of the nucleons. Most of the discussion will be done for $SU(N_f = 2)$ but we will also discuss certain properties of the three-flavor case. The Skyrme model was introduced in [195], [196], [197] and [91].

The topic of baryons as Skyrmions was discussed and reviewed in several books [53], [157] and reviews [22], [186], [231]. In several sections of this chapter we follow the latter review.

20.2 The Skyrme action

In two dimensions using the bosonized version of QCD, we were able to integrate in the strong coupling limit the color degrees of freedom and derive the low energy effective action of the flavor degrees of freedom. The latter took the form of a WZW action for the group $U(N_f)$ of level N_c for massless QCD and modified flavored WZW action with a mass term for massive QCD. The main point there was that the action derived was exact. In four dimensions the situation is quite different. For once we do not have a bosonized version of QCD which enables us (at least in the massless case) to decouple the flavor and color degrees of freedom and then integrate over the latter. However, due to confinement, the low energy degrees of freedom of four-dimensional QCD are also only flavor degrees of freedom and hence it is natural to use an $N_f \times N_f$ group element $g(x^\mu)$ of the flavor group $U(N_f)$. Since we do not have a systematic way to derive the corresponding action, we will now consider various terms that eventually construct the full Skyrme model.

20.2.1 The Sigma term

Recall that the group element $g(x)$ transforms under left and right transformations as follows,

$$g(x) \to Ug(x) \quad g(x) \to g(x)U. \tag{20.1}$$

As we have shown in two dimensions the sigma term is the term with the lowest number of derivatives. In two dimensions it takes the form,

$$S_{2d} = \frac{1}{12\pi} \int \mathrm{d}^2 x \, \mathrm{Tr}\, [\partial_\mu g \partial^\mu g^{-1}]. \tag{20.2}$$

It is easy to see that the analog in four dimensions has the form,

$$S = \frac{1}{16} f_\pi^2 \int \mathrm{d}^4 x \, \mathrm{Tr}\, [\partial_\mu g \partial^\mu g^{-1}], \tag{20.3}$$

where f_π has dimensions of mass (it will be shown below that by comparison to experimental data, for $N_f = 3$, it has to be taken to be $\sim 93 MeV$). This coefficient is needed since our group element still does not carry classically any conformal dimension. The sigma term which is clearly the one with the lowest number of derivatives can also be expressed as,

$$S = \frac{1}{16} f_\pi^2 \int \mathrm{d}^4 x \, \mathrm{Tr}\, [L_\mu L^\mu] = \frac{1}{16} f_\pi^2 \int \mathrm{d}^4 x \, \mathrm{Tr}\, [R_\mu R^\mu], \tag{20.4}$$

where,

$$L_\mu = g^{-1} \partial_\mu g \quad R_\mu = g \partial_\mu g^{-1}. \tag{20.5}$$

It is important to note that by construction the L_μ obey the so-called Maurer–Cartan equation,

$$\partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu] = 0. \tag{20.6}$$

and so does R_μ. Note that they are both like pure gauges in a non-abelian gauge theory, and thus they have $F_{\mu\nu} = 0$.

Before proceeding to the WZ term let us check the symmetries of this action in comparison with the known symmetries of QCD. It is easy to check that it is invariant under global $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ transformations. It is further invariant under the following three discrete transformations,

$$\begin{array}{rll} Transpose: & g \leftrightarrow g^T & \vec{x} \leftrightarrow \vec{x},\ t \leftrightarrow t \\ P_0: & g \leftrightarrow g & \vec{x} \leftrightarrow -\vec{x},\ t \leftrightarrow t \\ (-1)^{N_B}: & g \leftrightarrow g^{-1} & \vec{x} \leftrightarrow \vec{x},\ t \leftrightarrow t. \end{array} \tag{20.7}$$

The second transformation P_0 is a parity transformation and the third is the number of bosons modulo two. To check whether these discrete symmetries

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

are also shared by QCD we first expand $g(x)$ around unity,

$$g(x) = 1 + \frac{2i}{f_\pi} \sum_{1}^{N_f^2-1} T^a \pi^a(x), \tag{20.8}$$

in terms of the Goldstone bosons $\pi(x)$ and then consider $N_f = 3$. It turns out that P_0 and $(-1)^{N_B}$ are not conserved separately but only the combination $P = P_0(-1)^{N_B}$. This is demonstrated by the process $K^+K^- \to \pi^+\pi^0\pi^-$. Obviously the number of bosons modulo two is not conserved as well as the parity P_0 since the π^i are pseudo scalars. It is thus clear that the action (20.3) cannot describe the effective action of QCD and another term that does conserve P and not P_0 and $(-1)^{N_B}$ separately. It is well known that the parity transformation P_0 is violated by a term which is proportional to the Levi Civita tensor which in four dimensions reads $\epsilon_{\mu\nu\rho\sigma}$. However, it is very easy to verify that the only term proportional to $\epsilon_{\mu\nu\rho\sigma}$, namely $\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,[g^{-1}\partial_\mu g g g^{-1}\partial_\nu g g^{-1}\partial_\rho g g^{-1}\partial_\sigma g]$ vanishes due to the antisymmetric nature of $\epsilon_{\mu\nu\rho\sigma}$ and the cyclicity of the trace.

20.2.2 The WZ term

Experienced with the two-dimensional WZ term it is clear that this situation naturally calls for a four-dimensional WZ term: [225], [226]. Recall that the two-dimensional WZ term was written as a three-dimensional integral over a three-dimensional ball or a three disk whose boundary is the two-dimensional space-time. In a similar manner we can construct a term defined on a five-dimensional disk $D5$ whose boundary is the four-dimensional space-time and has the form,

$$S_{WZ} = -i\frac{\lambda_S}{240\pi^2}\int_{D^5} \mathrm{d}^5x\epsilon^{ijklm}\,\mathrm{Tr}\,[g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g g^{-1}\partial_l g g^{-1}\partial_m g], \tag{20.9}$$

where now i, j, k, l, m denote coordinates on $D5$ and λ_S is a coefficient that has to be determined. Extending the map $g(x^\mu)$ from the four-dimensional space-time to the $SU(N)$ group manifold into a map from $D5$ to the group manifold is based on the fact that $\pi_4(SU(N)) = 0$ and $\pi_1(SU(N)) = 0$. Now let us check whether there are any constraints on λ_S. The analogous two-dimensional case tells us that $\lambda_S = N_c$. We will now verify this result in two steps. First we show that on general topological grounds it has to be an integer and then by relating the action to QCD we show that this integer has to be equal to the number of colors. To understand the topological nature of the WZ term it is convenient to use a compactified Euclidean four space of a topology $S^1 \times S^3$ where the S^1 factor corresponds to a compactified time direction. Now the five-dimensional disk D^5 can be taken now to be $S^3 \times D^2$. However as is clear from Fig. 20.1 there are in fact two options of choosing the disk D^2 namely D_n^2 and D_s^2. Requiring that the independence of the physics on choice translates into,

$$\mathrm{e}^{iS^n_{WZ}} = \mathrm{e}^{iS^s_{WZ}} \to \int_{(D_n^2+D_s^2)\times S^3} w_5^0 = \int_{S^2\times S^3} w_5^0 = 2\pi \text{ integer}, \tag{20.10}$$

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

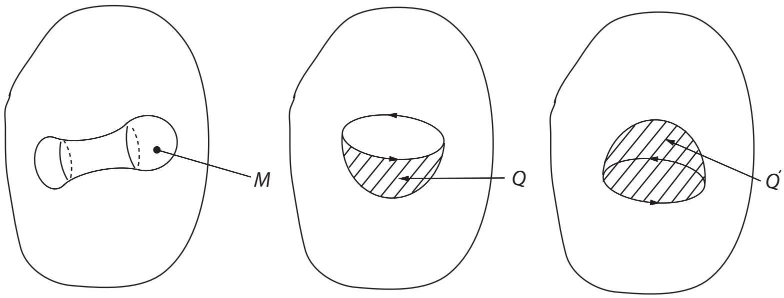

Fig. 20.1. Two options of choosing the disk D^2.

where we have used the fact that the sum of the two disks $(D^2_n + D^2_s)$ is topologically equivalent to S^2, that the five cycles in the group manifold $SU(N)$ of the topology $S^2 \times S^3$ can be represented by the cycles of topology S^5 and that $\pi_5(SU(N)) = \mathcal{Z}$ and hence any S^5 in $SU(N)$ is topologically a multiple of the basic S^5 on which w can be normalized such that $\int_{S^5_0} w^0_5 = 2\pi$.

We thus conclude that the coefficient λ_S has to be an integer. As we mentioned above in two dimensions we have shown that this integer has to be N_c. We will show below when discussing the phenomenology of the Skyrme model that this is the case also in four dimensions. Thus we will take from here on that $\lambda_S = N_c$.

20.2.3 The Skyrme term

Baryons were described in the context of the bosonized theory of two-dimensional QCD in terms of soliton solutions of the WZW theory in flavor space (see Chapter 13). In a similar manner we anticipate that also in four dimensions solitons are intimately related to baryons. However, in Section 5.3 it was shown that according to Derrick's theorem, there are no stable solitons in the space dimension larger that one. To recapitulate this theorem let us analyze the scaling behavior of the energy of a soliton solution in four dimensions. It is easy to realize that in D space dimensions the energy that corresponds to the action (20.3) reads,

$$E = \int d^D x \frac{f_\pi^2}{4} \operatorname{Tr}[L_\mu L^\mu]. \tag{20.11}$$

where a change in normalization was made.

Under a scaling of $x \to \lambda x$, $g(x) \to g(\lambda x)$ since as mentioned above $g(x)$ has a zero scaling dimension, and the energy scales as,

$$E_\lambda = \lambda^{2-D} E. \tag{20.12}$$

Thus for $D = 3$ the energy of the system can shrink to zero by large scaling and hence the solutions are not stable against scale transformations. To avoid this problem we add a term which is quartic in L_μ so that the total Lagrangian is,

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 = \frac{f_\pi^2}{4} \operatorname{Tr}[L_\mu L^\mu] + \frac{1}{4} e^2 Tr([L_\mu, L_\nu]^2). \tag{20.13}$$

Under scale transformation the energy scales as,

$$E_\lambda = \lambda^{2-D} E_{(2)} + \lambda^{4-D} E_{(4)}. \tag{20.14}$$

It is easy to see that for $D \geq 3$ there is a minimum at $\lambda = 1$ and with

$$\begin{aligned} \frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = 0 \rightarrow \quad & \frac{E_{(2)}}{E_{(4)}} = -\frac{4-D}{2-D} \\ \frac{\mathrm{d}^2 E_\lambda}{\mathrm{d}\lambda^2} > 0 \rightarrow \quad & 2(D-2)E_{(2)} > 0. \end{aligned} \tag{20.15}$$

In fact the energy is bounded from below by the topological charge. The energy associated with the action (20.13) takes the following form for a static configuration,

$$E = \int \mathrm{d}^3x \left\{ -\frac{f_\pi^2}{4} \operatorname{Tr}[L_i L_i] - \frac{1}{4} e^2 \operatorname{Tr}([L_i, L_j]^2) \right\}, \tag{20.16}$$

which can be rewritten as,

$$\begin{aligned} E = -\frac{f_\pi^2}{4} \int \mathrm{d}^3x \operatorname{Tr} \left[L_i L^i + \frac{e^2}{f_\pi^2} (\sqrt{2}\epsilon_{ijk} L^j L^k)^2 \right] \geq \\ -\frac{f_\pi^2}{4} \int \mathrm{d}^3x \left| \operatorname{Tr} \left[\left(\frac{2\sqrt{2}e}{f_\pi} \epsilon_{ijk} L^i L^j L^k \right) \right| = 12\sqrt{2}\pi^2 e f_\pi |B|. \end{aligned} \tag{20.17}$$

This is referred to as the Bogomol'ny bound. It is interesting to note that unlike other cases like instantons (see Section 22.1) there is no configuration that saturates the bound. The configuration that does saturates the bound has the form $L_i = \frac{\sqrt{2}e}{f_\pi} e_{ijk} L_j L_k$, however it is easy to see that it does not obey the Maurer–Cartan equation (20.6).

20.2.4 A mass term

In two dimensions the mass term was shown to be a key ingredient to having soliton solutions of strong coupling of QCD_2. In four dimensions this is not the case. Stable soliton solutions do not require a mass term. However, to incorporate the fact that the pions are not massless, one adds a mass term to the full Lagrangian. The mass term has the same form as that of the two-dimensional theory, namely,

$$S_m = \frac{m_\pi^2 f_\pi^2}{16} \int \mathrm{d}^4x \operatorname{Tr}[g + g^{-1} - 2]. \tag{20.18}$$

Upon substitution for $g(x)$ in the ansatz (20.8) this term takes the form of a mass term for the π fields,

$$S_m = \int \mathrm{d}^4x \frac{1}{2} m_\pi^2 (\pi^a)^2. \tag{20.19}$$

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

For $N_f > 2$ one can use a mass term that breaks flavor symmetry by assigning different masses to different flavors. One can also generalize the mass term by using general functions of g which in the limit of $g \to 1$ approach g.

20.2.5 Gauging the Skyrme action

In Section 9.3.1 we discussed the gauging of the WZW action. In that case we were interested in gauging the diagonal $SU_D(N_c) \in SU_L(N_c) \times SU_R(N_c)$. We presented there two methods for the gauging procedure: (i) Noether trial and error method, (ii) covariantization of the associated currents. Here in the case of the Skyrme action we would like to gauge the $U(1)$ diagonal global abelian symmetry that corresponds to electromagnetism, as well as the full $SU(N_f) \times SU(N_f)$ global symmetry of the Skyrme action. Let us first identify the diagonal abelian symmetry that we want to gauge to incorporate EM gauge fields. In the particular case of $N_f = 3$ the EM charge matrix of the u, d, s quarks is given by,

$$Q = \begin{pmatrix} 2/3 & & \\ & -1/3 & \\ & & -1/3 \end{pmatrix}. \tag{20.20}$$

Thus a local transformation that corresponds to the EM gauge transformation is,

$$g(x) \to U g(x) U^{-1} \quad U \sim 1 + i\epsilon(x)[Q, g]. \tag{20.21}$$

As usual the local transformation can be a symmetry of the action only provided we add to the action gauge fields that transform under the EM gauge transformation as $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \epsilon(x)$ where e is a unit EM charge. For the sigma term and the Skyrme term it is obvious that gauge invariance is achieved by replacing the ordinary derivative with covariant ones, namely $\partial_\mu \to D_\mu = \partial_\mu + ie\partial_u$. The gauging of the WZ term $\Gamma \equiv S_{WZ}$ is more subtle and as was done for the two-dimensional case; we use a trial and error method. First we compute the variation of the term under the global $U(1)$ symmetry. We find that,

$$\begin{aligned} \Gamma &\to \Gamma - \int \mathrm{d}^4 x \partial_\mu \epsilon(x) J^\mu \\ J^\mu &= \frac{1}{48\pi^2} \epsilon^{\mu\nu\rho\sigma} \{ \operatorname{Tr}[-Q(R_\nu R_\rho R_\sigma)] \\ &\quad + \operatorname{Tr}[Q(L_\nu L_\rho L_\sigma)] \} \end{aligned} \tag{20.22}$$

where Q is defined in (20.20). The next step in gauging the WZ term is to replace the original term with,

$$\Gamma \to \Gamma - e \int \mathrm{d}^4 x A_\mu J^\mu. \tag{20.23}$$

It turns out that the action after this replacement is still not gauge invariant but it is invariant with the following addition,

$$
\begin{aligned}
S &= \frac{f_\pi^2}{16}\int \mathrm{d}^4x\,\mathrm{Tr}\,[D_\mu g D^\mu g^{-1}] + N_c\tilde{\Gamma},\\
\tilde{\Gamma}(g, A_\mu) &= \Gamma(g) - e\int \mathrm{d}^4x A_\mu J^\mu + \frac{ie}{24\pi^2}\int \mathrm{d}^4x \epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu A_\rho\\
&\quad \times \mathrm{Tr}\,[Q^2(L_\sigma - R_\sigma) - QgQg^{-1}R_\sigma].
\end{aligned}
\tag{20.24}
$$

In a similar manner one can gauge the full global symmetry or its subgroups. Since we will not need it in this chapter we refer the reader to references, for instance [231].

20.3 The baryon as a Skyrmion

The two-dimensional solitonic baryon was analyzed at two levels, firstly the classical configuration and then at the semi-classical level. At both levels properties of the baryon such as its mass and conserved charges were computed. In this section we present the analogous calculations for the 4d Skyrmion and then we compare the four- and two-dimensional results.[1]

20.3.1 The classical Skyrmion

The Skyrmion is by definition a solitonic solution of the Skyrme action. Soliton solutions in two dimensions were discussed in general in Section 5.3 and in particular the solitons of the low energy effective action of QCD_2 in the strong coupling limit in Chapter 13. In fact the classical solitonic baryons were solutions of a sine-Gordon equation that was derived from an action that included a sigma term and a mass term since for static configuration the WZ vanishes. The latter property holds also in four dimensions so the relevant action now includes the sigma term and the Skyrme term. In fact we will describe here the case of two flavors and as mentioned above the WZ term vanishes for the $SU(2)$ group manifold anyhow. The equation of motion derived by computing the variation of the action with respect to $g^{-1}\delta g$ is,

$$
\partial^\mu L_\mu - 2\frac{e^2}{f_\pi^2}\partial^\mu[L_\nu, [L_\mu, L_\nu]] = 0. \tag{20.25}
$$

Obviously, an equivalent equation can be written by replacing $L_\mu \to R_\mu$. Parameterizing the general static configuration as,

$$
g(x) = \mathrm{e}^{i\vec{\tau}\cdot\hat{r}F(r)} = \cos(F(r)) + i\vec{\tau}\cdot\hat{r}\sin(F(r)). \tag{20.26}
$$

[1] The classical properties of the $SU(2)$ baryonic Skyrmion were analyzed in [133], [5] and afterwards in many other papers.

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

For this ansatz the equation of motion reads,

$$F'' + \frac{2}{r}F' - \frac{\sin(2F)}{r^2} + 8\frac{e^2}{f_\pi^2}\left[\frac{\sin(2F)\sin^2 F}{r^4} - \frac{F'^2\sin(2F)}{r^2}\frac{2F''\sin^2 F}{r^2}\right] = 0. \tag{20.27}$$

The boundary conditions are taken to be,

$$F(r=0) = \pi \quad F(r\to\infty) = 0. \tag{20.28}$$

The mass of the classical Skyrmion is derived by substituting a solution of the equation of motion into (20.16) getting,

$$M_s = 4\pi\int_0^\infty r^2\mathrm{d}r\frac{f_\pi^2}{2}\left[F'^2 + \frac{2\sin^2 F}{r^2}\right] + 4e^2\frac{\sin^2 F}{r^2}\left[2F'^2 + \frac{\sin^2 F}{r^2}\right]. \tag{20.29}$$

Using the virial property this reduces to,

$$M_s = 4\pi\sqrt{2}e\int_0^\infty x^2\mathrm{d}x\left[\left(\frac{\mathrm{d}F}{\mathrm{d}x}\right)^2 + \frac{2\sin^2 F}{r^2}\right], \tag{20.30}$$

where $x = \frac{f_\pi}{e\sqrt{2}}r$ is dimensionless. The value of the integral is ~ 11.7. One can either use the mass of the proton combined with the mass of the delta, to determine f_π and the coefficient of the Skyrme term, or use the experimental values of f_π and the axial coupling to be discussed shortly.

Let us now analyze the radial profile of the soliton $F(r)$. Asymptotically for $r\to\infty$ only the terms inside the square brackets can be neglected leading to a solution of the form $F(r)\to\frac{16e^2}{f_\pi^2}\frac{A^2}{r^2}$, where again A can be determined by comparing to experimental data and is found to be $A\sim 1.08$. On the other limit around the origin it is easy to see that the equation is solved by $F(r)\sim n\pi - ar$. The numerical solution of $F(r)$ that interpolates between these two boundary conditions is drawn in Fig. 20.2.

In addition to the mass, we have also extracted in two dimensions from the classical soliton the flavor properties and baryon number. For the Skyrmion we should also be able to determine these properties as well as its spin. When we insert the classical soliton solution in the baryon density we get,

$$B^0 = \frac{i}{24\pi^2}\epsilon^{ijk}L_iL_jL_k = \frac{1}{2\pi^2}\sin^2 F\frac{F'}{r^2}, \tag{20.31}$$

so that the baryonic charge is,

$$B = 4\pi\int_0^\infty \mathrm{d}rr^2B^0(r) = \frac{1}{\pi}(F(0) - F(\infty)) + 12\pi[\sin(2F(\infty)) - \sin(2F(0))] = 1. \tag{20.32}$$

where we have used the boundary conditions of $F(r)$ specified above. Using the distribution of the baryonic charge, one can define the rms radius of the baryon as follows,

$$r_{\text{rms}} = \frac{e}{\pi f_\pi}\left(-\int_0^\infty \mathrm{d}xx^2\sin^2 FF'\right)^{1/2}, \tag{20.33}$$

which is of the order of 0.48 for the $B=1$.

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

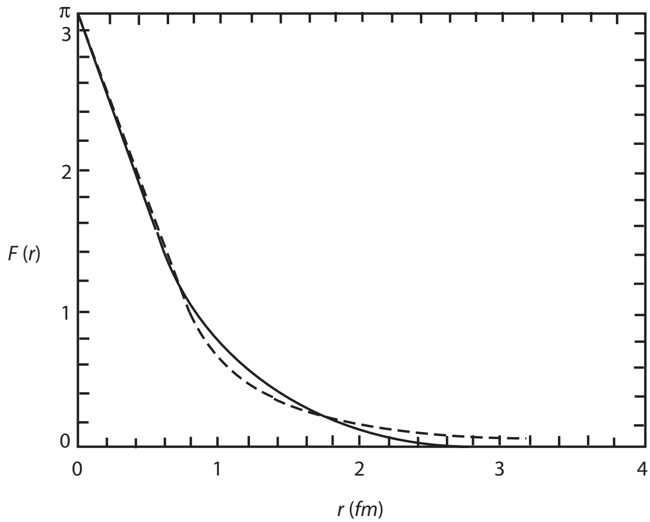

Fig. 20.2. The numerical solution of $F(r)$.

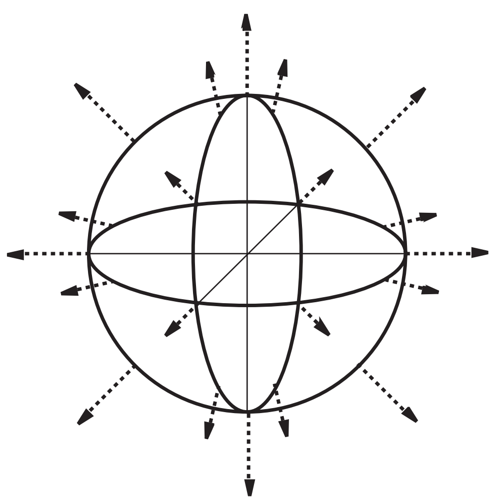

Fig. 20.3. The Skyrmion hedgehog configuration.

The hedgehog configuration, see Fig. 20.3, used as an ansatz for the Skyrmion, is by construction invariant under the operation of,

$$K \equiv J + I = (L + S) + I, \tag{20.34}$$

where L, S and J are the orbital angular momentum, the spin and the total angular momentum, and I is the isospin.

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

It follows from,

$$[K, g(x)] = i \sin F \left[[(r \times -\nabla), \vec{\tau} \cdot \hat{r}] + \left[\frac{\vec{\tau}}{2}, \vec{\tau} \cdot \hat{r} \right] \right] = 0. \tag{20.35}$$

Hence the Skyrmion carries a charge of $K = 0$. It is straightforward to notice that it is also an invariant under the parity operator defined in (20.7), so that altogether it is $K = 0^+$ state.

20.3.2 Semiclassical quantization of the soliton

Recall that the quantization of the collective coordinates of the two-dimensional soliton was performed by elevating the static group element to a space-time dependent one in the following way,

$$g(x) \to g(x,t) = A(t) g(x) A^{-1}(t) \quad A(t) \in U(N_f). \tag{20.36}$$

Nothing in this prescription is two dimensional and hence we now use the same ansatz also for the four-dimensional soliton. In two dimensions we discussed the general N_f case, here we start with the simplest case of $N_f = 2$ and then we discuss the $N_f = 3$ and comment about the general case. As discussed above for $SU(N_f = 2)$ there is no WZ term, thus we have to substitute (20.36) into the action that includes the sigma term and the Skyrme term (for simplicity we do not add the mass term). The collective coordinates $A(t)$ can be parameterized in the following ways either,

$$A(t) = a_0(t) + i\vec{a}(t) \cdot \vec{\sigma} \quad (a_\mu)^2 = 1, \tag{20.37}$$

or,

$$A^{-1}\dot{A} \equiv \frac{i}{2} \vec{\sigma} \cdot \vec{w}. \tag{20.38}$$

It is easy to verify that in terms of $A(t)$ the L_μ defined in (20.5),

$$L_0 = A(t) g^{-1}(x) (A^{-1}\dot{A})(t) g(x) A^{-1}(t) \quad L_i = A(t) g^{-1} \partial_i g(x) A^{-1}(t). \tag{20.39}$$

The result of the substitution of the L_μ expressed in terms of w into the Lagrangian is,

$$\mathcal{L} = \mathcal{L}_{cl} + \frac{1}{2}\alpha^2 w^2, \tag{20.40}$$

where the constant of proportionality α^2 is computed as a spatial integral over the chiral angle to be $\frac{53.3}{e^3 f_\pi}$. In terms of spin and isospin operators the Hamiltonian can be rewritten in terms of a second Casimir operator,

$$\mathcal{H} = E_{cl} + \frac{1}{2\alpha^2} J^2 = E_{cl} + \frac{1}{2\alpha^2} I^2. \tag{20.41}$$

In terms of the a_μ variables the Lagrangian density takes the form,

$$\mathcal{L} = \mathcal{L}_{cl} + \lambda (\dot{a}_\mu)^2, \tag{20.42}$$

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

where λ is the moment of inertia given by,

$$\lambda = \frac{8\pi}{3} f_\pi^2 \int_0^\infty \mathrm{d}r r^2 \sin^2 F \left[1 + \frac{8e^2}{f_\pi^2}\left(F^2 + \frac{\sin^2 F}{r^2}\right)\right]. \tag{20.43}$$

The corresponding Hamiltonian is therefore,

$$\mathcal{H} = \pi_\mu \dot{a}_\mu - \mathcal{L} = E_{cl} + \frac{1}{8\lambda}\pi_\mu^2 = E_{cl} + \frac{1}{8\lambda}\left(-\frac{\partial^2}{\partial a_\mu^2}\right), \tag{20.44}$$

where in the last step we introduced the canonical quantization namely,

$$[a_\mu, \pi_\nu] = i\delta_{\mu\nu} \quad \pi_\mu = -\frac{i\partial}{\partial a_\mu}, \tag{20.45}$$

subjected to the constraint $(a_\mu)^2 = 1$.

The Noether charges associated with the angular momentum and isospin expressed in terms of a_μ take the form,

$$\begin{aligned} I_k &= \frac{i}{2}\left(a_0 \frac{\partial}{\partial a_k} - a_k \frac{\partial}{\partial a_0} - \epsilon_{klm} a_l \frac{\partial}{\partial a_m}\right) \\ J_k &= \frac{i}{2}\left(a_k \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_k} - \epsilon_{klm} a_l \frac{\partial}{\partial a_m}\right). \end{aligned} \tag{20.46}$$

Choosing a fermionic rather than bosonic wave function, namely, odd under changing $A \to -A$, implies that the wave function of baryons of $I = J = \frac{1}{2}$ is linear in a_μ, for instance the proton wave function is $|p\rangle = \frac{1}{\pi}(a_1 + ia_2)$ and those of $I = J = \frac{3}{2}$ cubic in a_μ like $|\Delta^{++}\rangle = \frac{\sqrt{2}}{\pi}(a_1 + ia_2)^2$. It is thus obvious that the mass difference between the Δ and the nucleon is,

$$M_\Delta - M_N = \frac{3}{2\alpha^2}. \tag{20.47}$$

Expectation values of flavor charges can be computed in the semi-classical approximation by expressing the Noether currents and charges in terms of the $a_\mu(t)$ and their corresponding momenta $\pi_\mu = -\frac{i\partial}{\partial a_\mu}$ (20.46) and sandwiching these operators in between quantum states, like N and Δ. For instance the space components of the Baryonic current is given by,

$$B^i = i\frac{e^{ijk}}{2\pi^2}\frac{\sin^2 F}{r}\hat{r}_k \,\mathrm{Tr}\,[\dot{A}^{-1} A \tau_j]. \tag{20.48}$$

We use this expression to compute the isoscalar magnetic moment, defined by,

$$\mu_{I=0} = \frac{1}{2}\int \mathrm{d}^3 r \vec{r} \times \vec{B}, \tag{20.49}$$

of the proton as follows,

$$\begin{aligned} (\mu_{I=0})_3 &= -\frac{2i}{3\pi}\int \mathrm{d}r r^2 F \,\langle p, 1/2|\,\mathrm{Tr}\,(\tau_3 \dot{A}_{-1} A)|p, 1/2\rangle \\ &\frac{i}{3}\,\langle r^2\rangle_{I=0}\langle p, 1/2|\,\mathrm{Tr}\,(\tau_3 \dot{A}_{-1} A)|p, 1/2\rangle = \frac{1}{6I}\,\langle r^2\rangle_{I=0}\,. \end{aligned} \tag{20.50}$$

In a similar manner one can compute the isovector magnetic moment.

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

Another important property of nucleons that can be extracted from the description of baryons as semiclassical solitons is the axial coupling g_A defined via,

$$<N'(p_2)|J_\mu^{A^a}|N(p_1)> = <\bar{u}(p_2)\frac{\tau^a}{2}[g_A(q^2)\gamma_\mu\gamma_5 + h_A(q^2)q_\mu\gamma_5]u(p_1)>, \quad (20.51)$$

where $J_\mu^{A^a}$ is the axial current and $q = p_2 - p_1$. In the chiral limit $h_A(q^2)$ has a pion pole whose residue is $-2f_\pi g_{\pi NN}$ namely,

$$h_A(q^2) = \frac{d_A(q^2)}{q^2} \quad d_A(0) = -2f_\pi g_{\pi NN}, \quad (20.52)$$

where $g_{\pi NN}$ is the pion nucleon coupling. Current conservation implies that,

$$2M_N g_A(q^2) + q^2 h_A(q^2) = 0. \quad (20.53)$$

In the nucleon rest frame the nonrelativistic limit $q \to 0$, taken in a symmetric form $q_i q_j \to \frac{1}{3}\delta_{ij}q^2$ yields,

$$\begin{aligned}
\lim_{q\to 0} <N'(p_2)|J_i^A a|N(p_1)> &= \lim_{q\to 0} < u(p_2)\frac{\tau^a}{2}[g_A(0)\sigma_i \\
&+h_A(q^2)\vec{\sigma}\cdot\vec{\hat{q}}\hat{q}_i]u(p_1)> = \\
&= \lim_{q\to 0} g_A(0)(\delta_{ij} - \hat{q}_i\hat{q}_j) <N'(p_2)|\sigma_i\frac{\tau^a}{2}|N(p_1)> \\
&= \frac{2}{3}g_A(0) <N'(p_2)|\sigma_i\frac{\tau^a}{2}|N(p_1)>,
\end{aligned} \quad (20.54)$$

where we have made use of the Goldberger–Triman relation,

$$g_A(0) = \frac{g_{\pi NN}f_\pi}{M_N}. \quad (20.55)$$

In the Skyrme model we can extract the axial coupling $g_A(0)$ in the following way. First we compute the space integral over the axial current,

$$\int \mathrm{d}^3x {J^A}_i^a(x) = -\frac{1}{2}d\,\mathrm{Tr}\,[\tau_i A^{-1}\tau^a A], \quad (20.56)$$

where d is the space integral over a function that depends only on the classical soliton configuration. We then sandwich this operator in between nucleon states to find,

$$\lim_{q\to 0}\int \mathrm{d}^3x \mathrm{e}^{i\vec{q}\cdot\vec{x}} <N'|{J^A}_i^a(x)|N> = \frac{2}{3}d <N'|\sigma_i\frac{\tau^a}{2}|N> . \quad (20.57)$$

Equating the last expression with (20.54) it was found that the Skyrme model value of $g_A(0) = 0.61$ whereas experientially it is equal to 1.33.

20.3.3 The Skyrme model and large N_C QCD

In Section 19.2 it was shown that the scattering amplitude or quadrilinear coupling of mesons in the large N_c limit behaves like $\frac{1}{N_c}$. Recall that in this limit we

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

take $N_c \to \infty$ while keeping $\lambda_{\mathrm{t'Hooft}} = g^2 N_c$ finite. The amplitude that is proportional to g^2 behaves like $\frac{1}{N_c}$. By comparing this result to the quadrilinear coupling of the Skyrme model we are able to determine the dependence of the Skyrme coefficients on N_c. Let us start by expanding the sigma term in terms of the pion fields,

$$\mathcal{L}_\sigma = \frac{f_\pi^2}{4}\,\mathrm{Tr}\left[\partial_\mu g \partial^\mu g^{-1}\right] \sim \frac{1}{2}\partial_\mu \vec{\pi}\cdot\partial^\mu\vec{\pi} + \frac{1}{6f_\pi^2}\left[(\vec{\pi}\cdot\partial^\mu\vec{\pi})^2 - \pi^2\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}\right] + \mathcal{O}(\pi^6). \tag{20.58}$$

The quadrilinear coupling behaves like $\frac{1}{f_\pi^2}$. If we expand the Skyrme term in a similar manner we find that in that case the coupling behaves like $\frac{1}{e^2 f_\pi^4}$, and hence we conclude in agreement with (19.12) that,

$$f_\pi \sim \sqrt{N_c} \quad e \sim \frac{1}{\sqrt{N_c}}. \tag{20.59}$$

This enables us to check the N_c dependence of the classical Skyrmion mass and its semi-classical extension,

$$M_{cl} \sim \frac{f_\pi}{e} \sim N_c \quad M_{sc} \sim \frac{1}{\alpha^2} \sim e^3 f_\pi \sim \frac{1}{N_c}. \tag{20.60}$$

Recall for comparison that the two-dimensional solitonic baryons were shown to have classical mass which is also order N_c, but the semi-classical correction term behaves like N_c^0 and not $\frac{1}{N_c}$.

20.4 The Skyrme model for $N_f = 3$

Phenomenologically we should obviously be interested in the case of $N_f = 3$ rather than only two flavors. Moreover, to let the WZ term play a role we also have to go beyond $N_f = 2$. So we have two reasons to discuss now the $U(N_f = 3)$ classical solitons and their semi-classical quantization. The action is now the sum of a sigma term (20.3), the Skyrme term (20.13) and the WZ term (20.9). We can further add a mass term (20.18). The latter can be used to introduce an explicit breaking of the flavor symmetry by assigning different masses to the different flavor degrees of freedom. In fact one can add additional flavor symmetry breaking terms which, to simplify the treatment, we would not do.

We first need to choose a parametrization for the $U(N_f = 3)$ group element. The analog of (20.8) including the $U(1)$ factor now reads,

$$g(x) = \mathrm{e}^{i\frac{\sqrt{2}}{\sqrt{3}f_\pi}\eta^0(x)}\mathrm{e}^{i\Phi(x)} = \mathrm{e}^{i\frac{\sqrt{2}}{\sqrt{3}f_\pi}\eta^0}\,\mathrm{e}^{i\sum_{a=1}^{a=8}\frac{\sqrt{2}}{f_\pi}\lambda^a\phi^a}, \tag{20.61}$$

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

where λ^a are the $SU(3)$ Gell–Mann matrices. In terms of the pions, kaons and η we have,

$$\Phi \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^8 & K^0 \\ K^- & \bar{K}^0 & \frac{2}{\sqrt{6}}\eta^8 \end{pmatrix}. \tag{20.62}$$

Next we have to choose an ansatz for the static classical configuration $g_0(x)$. Recall that in the two-dimensional case we took an embedding of the $U(1)$ in $U(N_f)$ of the form $g_0(x) = \mathrm{Diag}(1, 1, \ldots, e^{-i\sqrt{\frac{4\pi}{N_c}}\phi(x)})$. In analogy in the four-dimensional case we embed the $SU(2)$ hedgehog configuration in the $SU(3)$ group element as follows,

$$g_0(x) = \begin{pmatrix} e^{iF(r)\vec{\tau}\cdot\vec{\hat{r}}} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20.63}$$

Since the WZ term vanishes for an $SU(2)$ group the solution for $F(r)$ is identical to that discussed in Section 20.3 and hence the elevation to $N_f = 3$ shows up basically only in the semi-classical quantization of the collective coordinates. Recall that the latter are introduced via $g_0(x) \to A(t)g_0(x)A^{-1}(t)$. In two dimensions we parameterized the quantum fields $A(t)$ in terms of the $Z_i, i = 1, \ldots, N_f$ variables which was adequate for the CP^{N_f-1} that the collective coordinates span in that case. Clearly in the present case since the $g_0(x) \in SU(2)$ a different ansatz is required. The most straightforward one is in terms of the angular velocities w_a that generalize those of (20.38) as follows,

$$A^{-1}(t)\dot{A}(t) = \frac{i}{2}\sum_{a=1}^{8}\lambda^a w_a. \tag{20.64}$$

When substituting this into the Lagrangian one finds,

$$L_{SU(3)} = L_{cl} + \frac{1}{2}\alpha^2\sum_{a=1}^{3} w_a^2 + \frac{1}{2}\beta^2\sum_{a=4}^{7} w_a^2 - N_c\frac{B}{2\sqrt{3}}w_8. \tag{20.65}$$

Note that the WZ term which is proportional to N_c and linear in the angular velocity w_8 associated with λ_8 that commutes with the classical ansatz $[\lambda_8, g_0(x)]$. This is very reminiscent of the structure of the WZ term in two dimensions that is also linear in the angular velocity associated with the hypercharge with N_c as a coefficient.

The quantization of the system is performed as if it is an “$SU(3)$ rigid top” namely one defines the right generators,

$$\begin{aligned} \mathcal{R}_a = -\frac{\partial L_{SU(3)}}{\partial w_a} &= -\alpha^2 w_a = -J_a, && a = 1, 2, 3 \\ &\quad -\beta^2 w_a, && a = 4, .., 7 \\ &\quad \frac{N_C B}{2\sqrt{3}}, && a = 8, \end{aligned} \tag{20.66}$$

https://doi.org/10.1017/9781009401654.021 Published online by Cambridge University Press

and imposes the quantization condition,

$$[\mathcal{R}_a, \mathcal{R}_b] = -i f_{abc} \mathcal{R}_c, \tag{20.67}$$

where f_{abc} are the structure constants of $SU(3)$.

The Hamiltonian of the system takes the form,

$$H = E_{cl} + \frac{1}{2}\left[\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right] J^2 + \frac{1}{2\beta^2} C_2 - \frac{3}{8\beta^2}. \tag{20.68}$$

Again this form of the Hamiltonian is similar to the one we found in two dimensions which is also proportional to the second Casimir operator. One can now apply the Hamiltonian on states associated with representations of the $SU(3)$ and compute the corresponding masses. Using the eigenvalues of the second Casimir operator of the representations 8, 10, $\bar{1}0$, 27 which are given by 3, 6, 6, 8, respectively we can determine the masses of the various Skyrme hadrons. The full analysis of the $N_f = 3$ baryons is beyond the scope of this book. We refer the interested reader to literature. For a review of the topic see for instance [231].