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ON THE CLASSIFICATION OF LIE PSEUDO-ALGEBRAS

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Introduction [5]. For every ($\mathscr{C}^\infty$ differentiable) bundle E over a manifold M, $J_k(E)$ denotes the set of all k-jets of local (differentiable) sections of the bundle E. $J_k(E)$ is a bundle over M such that if X is a section of E, then

$$j^kX\colon M \to J_k(E)$$
$$x \mapsto j_x{}^kX$$

is a (differentiable) section of $J_k(E)$. If E is a vector bundle, $J_k(E)$ is a vector bundle and we have the canonical exact sequence of vector bundles

$$0 \to E \otimes S^k(T^*) \to J_k(E) \xrightarrow{\pi} J_{k-1}(E) \to 0$$

where $S^k(T^*)$ is the symmetric Whitney tensor product of the cotangent vector bundle T^* of M and π is the canonical morphism which associates to each k-jet of section its jet of inferior order. In the case where E is the trivial vector bundle $\mathbf{R}^p$ over an open set M of $\mathbf{R}^n$, for every k the preceding sequence admits a canonical splitting; therefore, there exists an isomorphism

$$J_k(\mathbf{R}^p) \xrightarrow{\simeq} \mathbf{R}^p + \mathbf{R}^p \otimes \mathbf{R}^{n^*} + \ldots + \mathbf{R}^p \otimes S^k(\mathbf{R}^{n^*}),$$

which is the trivial bundle over M. The k-jet $j_x{}^kX$ of a vector-valued function X will be represented under this isomorphism by $X(x) + D_x{}^1X + \ldots + D_x{}^kX$; the last is a polynomial function on $\mathbf{R}^n$ and represents the usual Taylor expansion of the function X at the point x:

$$X(x+h) = X(x) + D_x{}^1X(h) + \ldots + D_x{}^kX(h^k) + O(||h||^{k+1}).$$

The canonical operator, which is a morphism of sheaves of sections,†

$$j^k\colon E \to J_k(E)$$
$$X \mapsto j^kX$$

is universal in the sense that if $\mathscr{D}$ is a differential operator of order k from the bundle E to another bundle F over M, $\mathscr{D}\colon E \to F$, it defines a canonical morphism of bundles $\phi\colon J_k(E) \to F$ such that if X is a section of E, then $\mathscr{D}(X) = \phi(j^kX)$.

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†Without risk of confusion, for every bundle E over M, the same letter E denotes its sheaf of sections over M.

Recall also that a differential system S_k of order k in E is a subset of $J_k(E)$. The sheaf θ of solutions of the differential system S_k is the sheaf of sections X of the bundle E such that J^kX is a section of $J_k(E)$ with values in the subset S_k. If E is a vector bundle and S_k is a subvector bundle of $J_k(E)$, S_k is said to be a linear differential system. In this case, we can define the prolongation differential system of S_k in the following way: for every integer p, $J_p(S_k)$ and $J_{k+p}(E)$ are two subvector bundles of $J_p[J_k(E)]$; hence

$$S_{k(p)} = J_p(S_k) \cap J_{k+p}(E)$$

is defined and it is the prolongation system of order $k + p$ of S_k. The prolongation system $S_{k(p)}$ admits the same sheaf of solutions as S_k.

Infinite Lie pseudo-algebras. Let θ be a subsheaf of the sheaf of vector fields T on a manifold M. We shall denote by $J_k(\theta)$ the subset of $J_k(T)$ composed of the k-jets of all local sections X of θ. Recall that the Lie bracket of vector fields defines on T a structure of sheaf of Lie algebra and let us give the following definition.

Definition. A Lie pseudo-algebra θ over a manifold M is a subsheaf of Lie algebra of T such that for every integer k, $J_k(\theta)$ is a subvector bundle of $J_k(T)$.

The linear differential system $J_k(\theta)$ is by definition completely integrable, i.e. every element of $J_k(\theta)$ is the k-jet of a germ of solution. Furthermore, for every k, $J_{k+1}(\theta)$ is contained in the prolongation system of $J_k(\theta)$; a well-known theorem of Cartan and Kuranishi states then that there is an integer r such that for all $k \geqq r$ the differential system $J_{k+1}(\theta)$ is the prolongation system of $J_k(\theta)$. The smallest of such an integer r will be referred to as the order of the Lie pseudo-algebra. Let θ be a Lie pseudo-algebra of order r. The Lie pseudo-algebra is said to be *complete* if it is the sheaf of solution of the differential system $J_r(\theta)$. In the following, we shall consider only the Lie pseudo-algebra which is complete in this sense.

A Lie pseudo-algebra θ is said to be *infinite* if the stalk θ_x at some point x of M is a vector space (over the real field R) of infinite dimension. If M is connected, it is equivalent to say that the vector bundle G_{k-1}, defined as the kernel of the canonical morphism π

$$0 \longrightarrow G_{k-1} \longrightarrow J_k(\theta) \xrightarrow{\pi} J_{k-1}(\theta) \longrightarrow 0$$

is of positive dimension for every integer k. We shall suppose M to be connected, and denote, for simplicity, the stalk θ_x by L. Let L_0 be the subalgebra of germs of vector fields in $L = \theta_x$ which vanish at the point x. Then the Lie bracket of vector fields induces naturally a linear representation of L_0 into the tangent vector space T_x at x of the manifold M. The Lie pseudo-algebra θ is said to be irreducible if this representation is irreducible. This condition is independent of the choice of the point x. Furthermore, if the Lie pseudo-algebra is irreducible, it is transitive, i.e. we have $J_0(\theta) = T$.

Cartan has given the "local classification", in a sense to be understood later on, of infinite irreducible Lie pseudo-algebras on a manifold M [**1**]. Matsushima [**4**] and Kobayashi and Nagano [**3**] have completed the algebraic part of Cartan's classification, which contains some gap. Singer and Sternberg [**6**] have settled the classification of Cartan, and mainly omitted the analyticity condition of Cartan. This paper is essentially a report on the work of Singer and Sternberg in a simplified setting due to jet theory. We shall also give a complete proof of the classification theorem noted in this paper as Theorem 2a, as it seems to us technically original and is missing in the literature.

Algebraic classification. Consider the filtered Lie algebra

$$L_{-1} = L \supset L_0 \supset L_1 \supset \ldots \supset L_i \supset L_{i+1} \ldots,$$

where

$$L_i = \{X \in L = \theta_x | j_x{}^i X = 0\}.$$

The associated graded Lie algebra will be represented by

$$\mathrm{Gr}(\theta) = G_{-1} + G_0 + G_1 + \ldots + G_i + G_{i+1} + \ldots;$$

this is the associated graded Lie algebra of the Lie pseudo-algebra θ. In the case where $\theta = T$ (the Lie pseudo-algebra of all germs of vector fields on M), the associated graded Lie algebra is

$$\mathrm{Gr}(T) = E + E \otimes E^* + E \otimes S^2(E^*) + \ldots + E \otimes S^k(E^*) + \ldots,$$

where $E = T_x$. In other words, $\mathrm{Gr}(T)$ is the set $\mathbf{R}[E, E] = \mathbf{R} \otimes \mathbf{R}[E]$ of polynomial functions on E with values in E. The graduation of $\mathrm{Gr}(T)$ is a shift of the natural graduation on $\mathbf{R}[E, E]$ and the Lie bracket of $\mathrm{Gr}(T)$ is defined by the following formula

$$[u \otimes P, v \otimes Q] = v \otimes (d_u Q) \cdot P - u \otimes (d_v P) \cdot Q,$$

where $d_u Q$ (respectively, $d_v P$) is the derived polynomial of the polynomial Q by the vector u of E (respectively, P and v).

For every Lie pseudo-algebra θ, $\mathrm{Gr}(\theta)$ is then a subgraded Lie algebra of $\mathrm{Gr}(T)$. In particular, G_0 is a subalgebra of the Lie algebra of the endomorphisms of E: $G_0 \subset E \otimes E^*$; it is the linear representation of L_0 in E. G_0 is the isotropy algebra of θ. Denote by $\mathrm{Gr}(E, G_0)$ the subgraded Lie algebra of $\mathrm{Gr}(T)$ generated by E and G_0; we have the following inclusion of graded Lie algebras

$$\mathrm{Gr}(\theta) \subset \mathrm{Gr}(E, G_0) \subset \mathrm{Gr}(T) = \mathbf{R}[E, E].$$

Hence the Lie pseudo-algebra θ is infinite, only if $\mathrm{Gr}(E, G_0)$ is of infinite dimension. Indeed, for an infinite Lie pseudo-algebra θ, $\mathrm{Gr}(\theta)$ is of infinite dimension.

More generally, if E is a vector space over a field K, the set $K[E, E] = E \otimes K[E]$ of polynomial functions on E with values in E is as before a graded Lie algebra over the field K. Let G be a K-Lie algebra of endomorphisms of E.

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$\mathrm{Gr}(E, G)$ denotes the subgraded Lie algebra generated by E and G in $K[E, E]$:

$$\mathrm{Gr}(E, G) = E + G + G_{(1)} + \ldots + G_{(i)} + \ldots.$$

If $\mathrm{Gr}(E, G)$ is of infinite dimension (i.e. $G_{(i)} \neq 0$ for all i), G will be said to be a K-algebra of infinite type.

In the following, which is due mainly to Guillemin, Quillen, and Sternberg [**2**], we wish to classify the isotropy algebra of irreducible infinite Lie pseudo-algebras, or more precisely irreducible $\mathbf{R}$-algebras of infinite type. If G is an irreducible $\mathbf{R}$-linear Lie algebra of E, it is well known that we have one of the following two cases: its complexification $G^{\mathbf{C}} = G \otimes \mathbf{C}$ is an irreducible $\mathbf{C}$-linear Lie algebra of endomorphisms of $E^{\mathbf{C}} = E \otimes \mathbf{C}$ or the vector space E can be regarded as a complex vector space and G is an irreducible $\mathbf{C}$-linear Lie algebra of endomorphisms of E. It is also immediate that if G is of infinite type, then in the first case, $G^{\mathbf{C}}$ is an irreducible $\mathbf{C}$-algebra of infinite type, and in the second case, G itself is an irreducible $\mathbf{C}$-algebra of infinite type.

Thus, let E be a vector space over the complex field $\mathbf{C}$ and G a $\mathbf{C}$-Lie algebra of endomorphisms of E. Recall that the dual vector space $\mathbf{C}[E, E]^*$ is $\mathbf{C}[E^*, E^*]$. As a subvector space of $\mathbf{C}[E, E]$, $\mathrm{Gr}(E, G)$ has its orthogonal complement denoted by $\mathrm{Gr}(E, G)^{\perp}$ in $\mathbf{C}[E^*, E^*]$. The latter is a module over the ring $\mathbf{C}[E^*]$of polynomials on E^*; $\mathrm{Gr}(E, G)^{\perp}$ is then the sub-$\mathbf{C}[E^*]$-module generated by $G^{\perp}$, the orthogonal complement of G in $(E \otimes E^*)^* = E^* \otimes E$. Consider the conductor I of the sub-module $\mathrm{Gr}(E, G)^{\perp}$:

$$\mathbf{C}[E^*] \supset I = \{P \in \mathbf{C}[E^*] |\quad P \cdot \mathbf{C}[E^*, E^*] \subset \mathrm{Gr}(E, G)^{\perp}\},$$

and we shall denote by $|I|$ the set of zeros of the ideal I in E^*. If $|I| = \{0\}$, by Hilbert's *Nullstellensatz* there is an integer r such that if $k \geqq r$,

$$\mathrm{Gr}(E, G)^{\perp} \supset E^* \otimes S^k(E).$$

Hence we have the following lemma.

LEMMA 1. *The Lie algebra G is a $\mathbf{C}$-algebra of infinite type if and only if $|I| \neq \{0\}$.*

Next we have the following result.

LEMMA 2. *If $\xi \in |I| \subset E^*$, there is a vector $a \neq 0$ in E such that*

$$a \otimes \xi \in G \subset E \otimes E^*.$$

Proof. It is equivalent to proving that ξ, an element of E^*, is in $|I|$ if and only if the linear vector space $\{A(\xi) | A \in G^{\perp} \subset E^* \otimes E\}$ is not equal to E^*.

Let $\xi^1, \ldots, \xi^n$ be a basis of E^*. For any n elements $A^1, \ldots, A^n$ of $G^{\perp}$, with

$$A^i = \sum_{j=1}^{n} \xi^j \otimes a_j{}^i = \sum_{j=1}^{n} a_j{}^i \cdot \xi^j,$$

we have in the $\mathbf{C}[E^*]$-module $\mathbf{C}[E^*, E^*]$,

$$\sum_{\lambda=1}^{n} \Delta_\lambda{}^k \cdot A^\lambda = \det(a_j{}^i) \cdot \xi^k,$$

where (Cramer's rule)

$$\sum_{\lambda=1}^{n} \Delta_\lambda{}^k \cdot a_l{}^\lambda = \delta_l{}^k \det(a_j{}^i) \quad \text{with} \quad \begin{cases} \delta_l{}^k = 1 & \text{if } k = l, \\ \delta_l{}^k = 0 & \text{if } k \neq l, \end{cases}$$

Thus:

(1) Let $\xi \in |I|$. For any n elements $A^1, \ldots, A^n$ of $G^\perp$, by definition we have $\det(a_j{}^i) \in I$ and $\det(a_j{}^i)(\xi) = 0$. Hence

$$\{A(\xi), A \in G^\perp\} \neq E^*.$$

(2) Inversely, if $\xi \notin |I|$, there is a $P \in I$ such that $P(\xi) \neq 0$. Since $P \in I$, for every i we have

$$P \cdot \xi^i = Q^i \cdot A^i \quad \text{with } A^i \in G^\perp.$$

Thus, $A^1(\xi), \ldots, A^n(\xi)$, which are so chosen, are n linearly independent vectors of E^*. Hence

$$\{A(\xi), A \in G^\perp\} = E^*.$$

From these two lemmas, we have the following result.

PROPOSITION [2]. *A $\mathbf{C}$-linear algebra G is of infinite type if and only if it admits an element of rank one (i.e. $a \otimes \xi \in G \subset E \otimes E^*$).*

A vector ξ of E^* such that $a \otimes \xi \in G$ for some $a \neq 0$ of E is said to be a characteristic covector of G. If G is irreducible, the set of characteristic covectors of G is the whole space E^*. Recall then that if G is irreducible, G is reductive. We choose a Cartan subalgebra of its semi-simple part. Let λ be the highest weight of the canonical representation of G into E^* and let ξ_λ be a corresponding weight vector. There is $a \neq 0$ of E such that $a \otimes \xi_\lambda \in G$. Applying to this element the root vectors of G, one sees easily that G contains $a_\mu \otimes \xi_\lambda$ with a_μ as a weight vector corresponding to the highest weight μ of the canonical representation of G into E. Thus $\lambda + \mu$ is a root of G; it is evidently the highest root, or, by the Dynkin diagram, the highest root of a semi-simple Lie algebra is always fundamental except for the cases of A_n and C_n. Hence we have the following result.

PROPOSITION [**1**; **2**; **3**; **6**]. *An irreducible $\mathbf{C}$-linear Lie algebra G of infinite type over the vector space E must be one of the following:*

(i) $\mathrm{gl}(E, \mathbf{C})$,
(ii) $\mathrm{sl}(E, \mathbf{C})$,
(iii) $\mathrm{csp}(E, \mathbf{C})$,
(iv) $\mathrm{sp}(E, \mathbf{C})$.

From this proposition, we have the classification for **R**-linear Lie algebras.

THEOREM 1 [**4**]. *An irreducible* **R**-*linear Lie algebra G of infinite type over a vector space E must be one of the following:*

(I) (*The complexification of G is irreducible*):

(i) $\mathrm{gl}(E, \mathbf{R})$,

(ii) $\mathrm{sl}(E, \mathbf{R})$,

(iii) $\mathrm{csp}(E, \mathbf{R})$,

(iv) $\mathrm{sp}(E, \mathbf{R})$;

(II) (*The complexification of G is not irreducible and E is canonically a vector space over the complex field* **C**):

(i) $\mathrm{gl}(E, \mathbf{C})$,

(ii) $\mathrm{sl}(E, \mathbf{C}) + \mathbf{R}$ (*i.e. with a one-dimensional real centre*),

(iii) $\mathrm{sl}(E, \mathbf{C})$,

(iv) $\mathrm{csp}(E, \mathbf{C})$,

(v) $\mathrm{sp}(E, \mathbf{C}) + \mathbf{R}$,

(vi) $\mathrm{sp}(E, \mathbf{C})$.

Consider now the graded Lie algebras $\mathrm{Gr}(E, G)$ corresponding to these cases. One proves by a remark of Weyl [**3**] that except for the cases (I)(i) and (II) (i), they do not admit any other infinite subgraded Lie algebra containing E and G. For these exceptions, we have:

$$\text{(I)(i) } \mathrm{Gr}(E, \mathrm{gl}(E, \mathbf{R})) \supset E + \mathrm{gl}(E, \mathbf{R}) + \mathrm{sl}(E, \mathbf{R})_{(1)} + \ldots + \mathrm{sl}(E, \mathbf{R})_{(i)} + \ldots,$$

$$\text{(II)(i) } \mathrm{Gr}(E, \mathrm{gl}(E, \mathbf{C})) \supset E + \mathrm{gl}(E, \mathbf{C}) + \mathrm{sl}(E, \mathbf{C})_{(1)} + \ldots + \mathrm{sl}(E, \mathbf{C})_{(i)} + \ldots.$$

Classification of irreducible infinite Lie pseudo-algebras. Let θ and θ' be two Lie pseudo-algebras over the manifolds M and M', respectively. We shall say that they are locally equivalent if and only if there is a diffeomorphism ϕ from an open set U of M to an open set U' of M' such that θ, θ' being restricted to the open sets U and U', respectively, we have:

$$\theta = \phi_*{}^{-1} \circ \theta' \circ \phi,$$

where ϕ_* is the prolongation of ϕ to the tangent bundles

$$\begin{array}{ccc} T(U) & \xrightarrow{\phi_*} & T(U') \\ \big\downarrow & & \big\downarrow \\ U & \xrightarrow{\phi} & U' \end{array}$$

From the preceding theorem of algebraic classification and the remark immediately following it, Singer and Sternberg proved the following theorem.

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THEOREM 2 [**6**]. *If θ is an irreducible infinite Lie pseudo-algebra of order* 1, *then θ is locally equivalent to one of the following Lie pseudo-algebras θ':*

(I) *Over $\mathbf{R}^n$ (with the canonical coordinates $x_1, \ldots, x_n$), θ' is the sheaf of*

(i) *all vector fields,*

(ii) *vector fields leaving invariant the n-form $dx_1 \wedge \ldots \wedge dx_n$,*

(iii) *$(n = 2p)$ vector fields leaving invariant up to a constant factor the 2-form $dx_1 \wedge dx_{p+1} + dx_2 \wedge dx_{p+2} + \ldots + dx_p \wedge dx_{2p}$,*

(iv) *$(n = 2p)$ vector fields leaving the last 2-form invariant.*

(II) *Over $\mathbf{C}^n$ (with the canonical coordinates $z_1, \ldots, z_n$), θ' is the sheaf of*

(i) *all holomorphic vector fields,*

(ii) *holomorphic vector fields leaving invariant up to a real constant factor the n-complex form $dz_1 \wedge \ldots \wedge dz_n$,*

(iii) *holomorphic vector fields leaving invariant the last n-complex form,*

(iv) *$(n = 2p)$ holomorphic vector fields leaving invariant up to a complex constant factor the 2-complex form $dz_1 \wedge dz_{p+1} + \ldots + dz_p \wedge dz_{2p}$,*

(v) *$(n = 2p)$ holomorphic vector fields leaving invariant up to a real constant factor the last 2-complex form,*

(vi) *$(n = 2p)$ holomorphic vector fields leaving invariant the last 2-complex form.*

The proof of this theorem involves a long process. But we intend to prove the following.

THEOREM 2a. *If θ is an irreducible infinite Lie pseudo-algebra which is not of order* 1, *θ is locally equivalent to one of the following Lie pseudo-algebras:*

(I)(i)(a) *Over $\mathbf{R}^n$, the sheaf of vector fields leaving invariant up to a constant factor the n-form $dx_1 \wedge \ldots \wedge dx_n$,*

(II)(i)(a) *Over $\mathbf{C}^n$, the sheaf of holomorphic vector fields leaving invariant up to a (complex) constant factor the n-complex form $dz_1 \wedge \ldots \wedge dz_n$.*

Indeed these are the only two cases where we have

(I)(i)(a) $\mathrm{Gr}(\theta) = E + \mathrm{gl}(E, \mathbf{R}) + \mathrm{sl}(E, \mathbf{R})_{(1)} + \ldots + \mathrm{sl}(E, \mathbf{R})_{(i)} + \ldots,$

(II)(i)(a) $\mathrm{Gr}(\theta) = E + \mathrm{gl}(E, \mathbf{C}) + \mathrm{sl}(E, \mathbf{C})_{(1)} + \ldots + \mathrm{sl}(E, \mathbf{C})_{(i)} + \ldots.$

Using the Newlander-Nirenberg theorem on integrability of almost complex structures, one proves as in [**6**] that in the case (II)(i)(a), the manifold M must be locally complex (we do not suppose that M is orientable) and θ must be a sheaf of holomorphic vector fields. Then the proof for the case (II)(i)(a) is the same (with complex arguments) as for the case (I)(i)(a). Thus, we shall restrict ourselves to the case where $\mathrm{Gr}(\theta)$ is of the form (I)(i)(a) and prove that θ must be locally equivalent to the sheaf of vector fields over $\mathbf{R}^n$ leaving invariant up to a constant factor the n-form $dx_1 \wedge \ldots \wedge dx_n$. Furthermore, as the question is local, without loss of generality, we shall suppose M to be $\mathbf{R}^n$.

Proof of Theorem 2a.

(A) (1) We have the canonical injection (Spencer's δ operator)

$$\mathbf{R}^n \otimes S^2(\mathbf{R}^{n*}) \xrightarrow{\delta} (\mathbf{R}^n \otimes \mathbf{R}^{n*}) \otimes \mathbf{R}^{n*}$$
$$e \otimes \xi^2 \mapsto 2(e \otimes \xi) \otimes \xi$$

Let $\operatorname{tr}(\delta) = (\operatorname{tr} \otimes \operatorname{id}) \circ \delta$, where

$$\operatorname{tr} \otimes \operatorname{id}\colon (\mathbf{R}^n \otimes \mathbf{R}^{n*}) \otimes \mathbf{R}^{n*} \to \mathbf{R}^{n*}$$
$$g \otimes \xi \mapsto \operatorname{trace}(g)\xi$$

and consider the linear complex

$$0 \to \operatorname{sl}(n, \mathbf{R})_{(1)} \to \mathbf{R}^n \otimes S^2(\mathbf{R}^{n*}) \xrightarrow{\operatorname{tr}(\delta)} \mathbf{R}^{n*} \to 0.$$

This complex is $\operatorname{gl}(n, \mathbf{R})$-equivariant relative to the canonical representation of $\operatorname{gl}(n, \mathbf{R})$ into these spaces. Or it has been well known since Weyl that $\operatorname{sl}(n, \mathbf{R})_{(1)}$ and $\mathbf{R}^{n*}$ are irreducible $\operatorname{gl}(n, \mathbf{R})$-modules. Hence, by dimension arguments, this complex is exact.

LEMMA. *We have the direct sum*

$$\mathbf{R}^n \otimes S^2(\mathbf{R}^{n*}) = \operatorname{sl}(n, \mathbf{R})_{(1)} + \mathbf{R}^{n*},$$

where $\mathbf{R}^{n*}$ *is a subspace of* $\mathbf{R}^n \otimes S^2(\mathbf{R}^{n*})$ *such that if* $\xi \in \mathbf{R}^{n*}$, *then*

$$\delta(\xi)(e \otimes e') = \xi(e)e' + \xi(e')e$$

for $e \otimes e' \in \mathbf{R}^n \otimes \mathbf{R}^n$.

Indeed, it is immediate that if $\xi \in \mathbf{R}^{n*} \subset \mathbf{R}^n \otimes S^2(\mathbf{R}^{n*})$, then

$$\operatorname{tr}(\delta)(\xi) = (n + 1)\xi.$$

Thus we have the lemma as $\operatorname{tr}(\delta)\colon \mathbf{R}^{n*} \to \mathbf{R}^{n*}$ is surjective.

(2) Let T be the tangent bundle of $\mathbf{R}^n$. We have the isomorphism of bundles (see introduction)

$$J_2(T) \simeq \mathbf{R}^n + \mathbf{R}^n \otimes \mathbf{R}^{n*} + \mathbf{R}^n \otimes S^2(\mathbf{R}^{n*}),$$

where the second member represents trivial bundles over $\mathbf{R}^n$. By the preceding lemma, we have the isomorphism

$$J_2(T) \simeq \mathbf{R}^n + \mathbf{R}^n \otimes \mathbf{R}^{n*} + \mathbf{R}^{n*} + \operatorname{sl}(n, \mathbf{R})_{(1)}$$

such that every element of $J_2(T)$ will be represented in this direct decomposition by $X + g + \xi + h$.

Consider on $\mathbf{R}^n$ an n-form

$$\omega = f\, dx_1 \wedge \ldots \wedge dx_n$$

with $f \neq 0$ everywhere. If

$$X = X_1 \frac{\partial}{\partial x_1} + \ldots + X_n \frac{\partial}{\partial x_n}$$

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is a vector field on $\mathbf{R}^n$, the Lie derivative of ω by X is

$$\mathscr{L}(X)(\omega) = (X \cdot f + f \operatorname{div} X)\, dx_1 \wedge \ldots \wedge dx_n.$$

Let $\mathscr{D}$ be the differential operator of order 1,

$$\begin{aligned} &T \to \mathbf{R} \text{ (the trivial line bundle over } \mathbf{R}^n\text{)}\\ &X \to X \cdot f + f \operatorname{div} X. \end{aligned}$$

With the universal operator j^1,

$$j^1\colon R \to J_1(R) = \mathbf{R} + \mathbf{R}^{n*},$$

we have

$$j^1 \circ \mathscr{D}\colon T \to \mathbf{R} + \mathbf{R}^{n*}$$

$$X \mapsto (X \cdot f + f \operatorname{div} X) + \left(X_i\, d\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_i} dX_i + \operatorname{div} X\, df + fd \operatorname{div} X\right)$$

with Einstein's convention of summation. Hence $j^1 \circ \mathscr{D}$, as a differential operator of order 2, induces the following morphism ϕ of vector bundles

$$\begin{aligned} \phi\colon J_2(T) &\to \mathbf{R} + \mathbf{R}^{n*}\\ X &\mapsto X \cdot f + X_i\, d\frac{\partial f}{\partial x_i}\\ g &\mapsto f \operatorname{tr}(g) + df \circ g\\ \xi + h &\mapsto (n+1)\xi \end{aligned}$$

If θ is the Lie pseudo-algebra leaving the n-form ω invariant up to a constant factor, its graded Lie algebra $\operatorname{Gr}(\theta)$ is evidently of the form (I)(i)(a) and θ is the sheaf of solutions of the completely integrable linear differential system of order 2, $J_2(\theta)$, which is the kernel of the morphism ϕ,

$$J_2(\theta) \subset J_2(T) = \mathbf{R}^n + \mathbf{R}^n \otimes \mathbf{R}^{n*} + \mathbf{R}^{n*} + \operatorname{sl}(n, \mathbf{R})_{(1)}$$

$$J_2(\theta) = \left\{X + g + \xi + h,\ \xi = -\frac{df \circ g}{(n+1)f}\right\}.$$

(B)(1) Inversely let θ be a Lie pseudo-algebra over $\mathbf{R}^n$ such that its graded Lie algebra is of the form (I)(i)(a). Recall [**5**] that $J_2(T)$ is a sheaf of Lie algebra such that

$$[j^2X, fj^2Y] = (X \cdot f)j^2Y + fj^2[X, Y],$$

where X and Y are vector fields and f is a differentiable function. Then, since θ is a Lie pseudo-algebra, $J_2(\theta)$ is a subsheaf of the Lie algebra of $J_2(T)$. In particular, let $J_2{}^0(\theta)$ be the kernel of the morphism π:

$$0 \to J_2{}^0(\theta) \to J_2(\theta) \xrightarrow{\pi} T \to 0.$$

$J_2{}^0(\theta)$ is a Lie algebra bundle whose fibre is the graded Lie algebra

$$\operatorname{gl}(n, \mathbf{R}) + \operatorname{sl}(n, \mathbf{R})_{(1)} + \ldots + \operatorname{sl}(n, \mathbf{R})_{(i)} + \ldots$$

truncated to the order 1.

By hypothesis on the Lie pseudo-algebra θ, we have

$$J_2(\theta) \subset J_2(T) = \mathbf{R}^n + \mathbf{R}^n \otimes \mathbf{R}^{n*} + \mathbf{R}^{n*} + \mathrm{sl}(n, \mathbf{R})_{(1)},$$
$$J_2(\theta) = \{X + g + \xi + h, \xi = \alpha(g)\},$$

where α is a morphism of vector bundles over $\mathbf{R}^n$,

$$\alpha: \mathbf{R}^n \otimes \mathbf{R}^{n*} \to \mathbf{R}^{n*}.$$

Since $J_2^0(\theta)$ is a Lie algebra bundle, we must have

$$[g + \alpha(g), g' + \alpha(g')] = [g, g'] + \alpha([g, g']) \quad (\mathrm{mod}\ \mathrm{sl}(n, \mathbf{R})_{(1)}).$$

However, the structure of Lie algebra in $J_2^0(\theta)$ is such that

$$[g + \alpha(g), g' + \alpha(g')] = [g, g'] + \alpha(g') \circ g - \alpha(g) \circ g' \quad (\mathrm{mod}\ \mathrm{sl}(n, \mathbf{R})_{(1)}).$$

Hence we have

$$\alpha(g) \circ g' - \alpha(g') \circ g - \alpha([g, g']) = 0.$$

In other words, the morphism α is a 1-cycle of $\mathrm{gl}(n, \mathbf{R})$ with values in the $\mathrm{gl}(n, \mathbf{R})$-module $\mathbf{R}^{n*}$. But since $\mathrm{gl}(n, \mathbf{R})$ is reductive, it must be a coboundary, i.e. there is a 1-form σ such that $\alpha(g) = \sigma \circ g$.

(2) Let X be a section of θ. In the sheaf of $J_2(T)$, it is immediate by a simple calculation that

$$[j^2X, g + \sigma \circ g] = \mathscr{L}(X)(g) - d(\mathrm{div}\, X) \circ g + \mathscr{L}(X)(\sigma \circ g) \qquad (\mathrm{mod}\ \mathrm{sl}(n, \mathbf{R})_{(1)}).$$

On the other hand, it must be a section of $J_2(\theta)$, since $J_2(\theta)$ is a subsheaf of the Lie algebra of $J_2(T)$:

$$[j^2X, g + \sigma \circ g] = \mathscr{L}(X)(g) + \sigma \circ \mathscr{L}(X)(g) \quad (\mathrm{mod}\ \mathrm{sl}(n, \mathbf{R})_{(1)}).$$

Hence

$$\begin{aligned} \sigma \circ \mathscr{L}(X)(g) &= -d(\mathrm{div}\, X) \circ g + \mathscr{L}(X)(\sigma \circ g) \\ &= -d(\mathrm{div}\, X) \circ g + \mathscr{L}(X)(\sigma) \circ g + \sigma \circ \mathscr{L}(X)(g), \\ 0 &= -d(\mathrm{div}\, X) \circ g + (di(X) + i(X)d)(\sigma) \circ g. \end{aligned}$$

Thus, $i(X)d\sigma$ is a closed 2-form; in particular, $\mathscr{L}(X)d\sigma = 0$ for all sections X of θ. It implies that the isotopy algebra $\mathrm{gl}(n, \mathbf{R})$ of θ leaves invariant the 2-form $d\sigma$; this is evidently possible only if $d\sigma = 0$.

By Poincaré's lemma, we have

$$\sigma = dh \quad \text{for some differentiable function } h.$$

Let

$$f = e^{-(n+1)h}, \qquad f \text{ is different from zero everywhere.}$$

The Lie pseudo-algebra θ is then evidently the sheaf of all vector fields on $\mathbf{R}^n$, leaving invariant up to a constant factor the n-form

$$\omega = f\, dx_1 \wedge \ldots \wedge dx_n.$$

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