ON THE NUMBER OF POSITIVE ENTRIES IN THE POWERS OF A NON-NEGATIVE MATRIX

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A real matrix A is said to be _non-negative_ if and only if none of its entries is negative. Suppose A is an r by r non-negative matrix. We want to examine:

(A) The first power of A to maximize the number of positive entries in A^n,

(B) For each $1 \le i \le r$ the first power of A to maximize the number of positive entries in the i-th row of A^n.

We shall call the former first power the _index of_ A and the latter the _i-th row index of_ A (index (i, A)).

More precisely, letting $W(A^n)$ denote the number of positive entries in A^n,

$$\text{index } A = \min \{ n > 0 : W(A^n) = \max_{m>0} W(A^m) \}$$

and letting $W(iA^n)$ denote the number of positive entries in the i-th row of A^n,

$$\text{index } (i, A) = \min \{ m > 0 : W(iA^m) = \max_{n>0} W(iA^n) \} .$$

If A is primitive, i.e. for some N there are r^2 positive entries in A^N, then the largest of the index (i, A) is index A and, as Wielandt stated in [1], index $A \le (r-1)^2 + 1$. Proofs

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were supplied by Rosenblatt [2] and J. C. Holladay and R. S. Varga in [3]. This result is best possible in the sense that for each r there is a matrix whose index is $(r-1)^2 + 1$ (see [1]). Mendelsohn and Dulmage [4] have found bounds for index A for primitive matrices for which index $A < (r-1)^2 + 1$.

The purpose of this paper is to obtain results of a similar character for non-primitive matrices. For example we show that if A is non-primitive and $\det(A) \neq 0$ (more generally if $\operatorname{per}(A) \neq 0$) then

$$\max_{1 \le i \le r} \text{index}(i, A) = \text{index}\, A \le (r-2)^2 + 1 .$$

A natural simplification of the problem is to put into one equivalence class all those matrices whose positive entries occur at the same position. That is, $(a_{ij}) \approx (b_{ij})$ iff for all i, j : $a_{ij} = 0$ when and only when $b_{ij} = 0$. Each equivalence class can then be identified with a matrix over the Boolean algebra of two elements $\{0, 1\}$. That is, the class of $A = (a_{ij})$ is identified with $\Gamma_A = (\gamma_{ij})$ where $\gamma_{ij} = 0$ iff $a_{ij} = 0$. As Wedderburn observed [5], there is a 1 to 1 correspondence between the r by r Boolean matrices Γ and the set of those join-preserving operators f on the family of subsets of $\{1, 2, \ldots, r\}$ which fix $\emptyset$. The correspondence is this:

$$f_\Gamma(x) = \bigcup_{i \in x} \{j : \gamma_{ij} = 1\} \quad \text{for each } x \subseteq \{1, 2, \ldots, r\} .$$

In the sequel we shall reserve the word "operator" for just these. Instead of writing f_{Γ_A} we shall simply write f_A, or we shall say that f is the operator corresponding to A.

If A and B are r by r non-negative matrices, then $f_A(f_B) = f_{BA}$. Let $W(x)$ denote the number of elements in each subset x of $V_r \equiv \{1, 2, \ldots, r\}$; Then $W(f_A^{\,n}(\{i\}))$ is the number of positive entries in the i-th row of A^n, and the number of positive entries in A^n is $\sum_{i=1}^{r} W(f_A{}^n(\{i\}))$. We therefore define

$$\operatorname{index}(x, f) = \min\{n > 0 : W(f^n(x)) = \max_{m>0} (W(f^m(x)))\}$$

and

$$\operatorname{index}(f) = \min\{n > 0 : \sum_{i=1}^{r} W(f^n(\{i\}) = \max_{m>0} (\sum_{i=1}^{r} W(f^m(\{i\})))\}$$

obtaining $\operatorname{index}(\{i\}, f_A) = \operatorname{index}(i, A)$ and $\operatorname{index}(f_A) = \operatorname{index}(A)$. Notice that A is primitive iff there is an N such that for each non-empty subset x of V_r, $f_A{}^N(x) = V_r$. This condition is also equivalent to requiring that for each element i of V_r there be an N_i such that $f^{N_i}(\{i\}) = V_r$. Such operators will, of course, be called primitive.

The connection between the operators and the matrices enabled Holladay and Varga to obtain their results [3]. We shall exploit these connections more fully to obtain ours.

As a first step in this direction, let us say that $x \subseteq V_r$ is repetitive (with respect to f) iff for some $n > 0 : x \subseteq f^n(x)$, and for each such x let $d(x)$ be the first such n. Then for each n:

$$f^{nd(x)}(x) \subseteq f^{(n+1)d(x)}(x) \tag{1}$$

because f preserves join and hence preserves inclusion. The finiteness of 2^{V_r} ensures that equality hold in (1) for some n; let $b(x)$ be the least such n. On the basis of these definitions we have:

$$\operatorname{index}(x, f) \leq (b(x)+1)d(x) - 1 \tag{2}$$

for all repetitive x. Next we find estimates for b and d on the basis of the observation that:

(3) if $x_1, x_2, \ldots, x_M, \ldots$ are finite non-empty sets then for each M either $W(\bigcup_{i=1}^{M'+1} x_i) \geq M' + W(x_1)$ for all $M' \leq M$ or for some $M' \leq M : x_{M'+1} \subseteq \bigcup_{i=1}^{M'} x_i$.

By letting $x_m = f^{(m-1)d(x)}(x)$ and noting that $x_M = \bigcup_{i=1}^{M} x_i$ we obtain:

(4a) $W(f^{b(x)d(x)}(x)) \geq b(x) + W(x)$ for repetitive x

and

(4b) $\max_{n \geq 0} (W(f^n(x))) - W(x) \geq b(x)$.

If $\{i\}$ is repetitive we shall say that i is repetitive, and to simplify the notation from now on we shall write $f^n(i)$, $b(i)$, $d(i)$ for $f^n(\{i\})$, $b(\{i\})$ and $d(\{i\})$ respectively. Again using (3) with $x_m = f^m(i)$ and the definition of d we obtain for each repetitive i :

(5) $$d(i) \leq W(\bigcup_{m \geq 1} f^m(i)) - W(f(i)) + 1$$

and hence $d(i) \leq r$.

One immediate consequence is that if i is repetitive then index $(i, f) \leq r^2 - 1$. In matrix theoretic terms, if $a_{ii}^{(n)} > 0$ for some $n > 0$ then index $(i, A) \leq r^2 - 1$. For example, if A is a stochastic matrix, then the first time n at which it is possible to reach the largest number of states from state i is at most $r^2 - 1$.

We can also use these observations to obtain a short proof of Wielandt's result which is similar to that given in [3].

THEOREM 1 (Wielandt). If f is primitive then

$$\max_i \text{ index } (i, f) = \text{index } (f) \leq (r-1)^2 + 1 .$$

Proof. The primitivity of f ensures that each subset x is repetitive, that index $(x, f) \leq b(x)d(x)$ and (assuming $r > 1$) that $W(f(j)) \geq 2$ for some $j \in V_r$. Applying the definition of d we see that $d(f(j)) \leq d(j)$. But $d(j) \leq r - 1$ by (5) and $b(f(j)) \leq r - 2$ by (4a); hence index $(f(j), f) \leq (r - 1)(r - 2)$. The primitivity of f and (3) imply that $V_r = \bigcup_{m=0}^{r-1} f^n(i)$. Therefore $j \in f^t(i)$ for some $0 \leq t \leq r-1$ and hence $f^{(r-1)(r-2)}(j) \subseteq f^{(r-2)(r-1)+t}(i)$; but the left member is V_r by the definition of index and the sentence before last. Thus $V_r = f^m(i)$ for all $m \geq (r-2)(r-1)+(r-1)$. Consequently index $(i, f) \leq (r-1)^2 + 1$. As we remarked at the beginning, $\max_{1 \leq i \leq r}$ index $(i, f) =$ index (f), so we have proven Wielandt's result. We also remark that for each $x \subseteq V_r$, if f is primitive then index $(x, f) \leq (r-1)^2 + 1$. It would seem from the proof that equality might be achieved if $d(j) = r-1$ and $W(f(j)) = 2$ for some j. This is indeed the case if $W(f(i)) = 1$ for all $i \neq j$. The corresponding matrix is exhibited by Wielandt in [1].

Many of the convenient features of primitive operators are enjoyed by another class of operators. These are the ones which do not reduce the size of subsets. We shall say that an operator is non-singular if and only if for each $x \subseteq V_r$, $W(x) \leq W(f(x))$. It then follows that $W(f^n(x))$ is monotone non-decreasing in n for fixed x and hence that for all non-singular f :

(6) $$\max_{1 \leq i \leq r} \{\text{index } (i, f)\} = \text{index } (f)$$

and

(7) each $x \subseteq V_r$ is repetitive.

To prove (7), we let $T = \bigcup_{n>0} f^n(i)$. $f(T \cup \{i\}) = T$; hence $W(T) \geq W(T \cup \{i\})$ because f is non-singular. But $T \subset T \cup \{i\}$; hence $T = T \cup \{i\}$ and therefore $i \in \bigcup_{n>0} f^n(x)$. Thus i is repetitive for each $i \in V_r$. Consequently each subset of V_r is repetitive. We also have:

(8) If f is non-singular then for each x : index $(x, f) \leq b(x)d(x)$.

To see why this is so, we observe first that, from the definition of b, $f^{b(x)d(x)}(x) = f^{(b(x)+n)d(x)}(x)$ for all n. Secondly, we use the monotonicity of $W(f^n(x))$ to see that for all $m \leq b(x)d(x) \leq n$:

$$W(f^m(x)) \leq W(f^{b(x)d(x)}(x)) \leq W(f^n(x)) .$$

Consequently, by the definition of index, index $(x, f) \leq b(x)d(x)$. In fact

$$(b(x)-1)d(x) < \text{index } (x, f) \leq b(x)d(x) .$$

We shall call a matrix <u>essentially non-singular</u> iff the operator corresponding to it is non-singular. The following lemma states some alternative characterizations of this property.

LEMMA 1. If A is an r by r non-negative matrix, and Γ and f are the Boolean matrix and operator corresponding to A, then the following conditions are equivalent:

a) f is non-singular;

b) Γ contains a permutation matrix, i.e. for some permutation q of V_r, $\gamma_{iq(i)} = 1$ for each i ;

c) each r by m matrix formed by extracting $1 \leq m \leq r$ rows from A (respectively Γ) has at least m non-zero columns;

d) the permanent of A is not zero. (per (A) is the sum over all permutations g of V_r of $\prod_{i=1}^{r} a_{ig(i)}$).

e) There is a matrix $B \approx A$ whose determinant is not zero.

Proof. a) implies b): The union of any m of the $f(i)$ $(1 \leq i \leq r)$ has at least m elements because f preserves join and f is non-singular. A theorem due to P. Hall (see e.g. [6]) implies that if the union of any m of r sets has at least m elements $(1 \leq m \leq r)$ then there exist r distinct points, one in each of the r sets, no two of which are in the same set. Consequently there exist r distinct points j_i such that $j_i \in f(i)$ $(1 \leq i \leq r)$. Now let $q(i) = j_i$ $(1 \leq i \leq r)$.

b) is evidently equivalent to both d) and e).

b) implies a): Let $g(x) = \{q(i) : i \in x\}$. Then $W(g(x)) = W(x)$ because q is a permutation; but $g(x) \subseteq f(x)$ for all x and hence $W(f(x)) \geq W(x)$ for all subsets x of V_r.

a) iff c): The i-th row of Γ corresponds to the image of $\{i\}$ under f, i.e., $\gamma_{ij} = 1$ iff $j \in f(i)$, whose non-singularity means that the join

$$((\gamma_{i_1 1}, \ldots, \gamma_{i_1 r}) \cup (\gamma_{i_2 1}, \ldots, \gamma_{i_2 r}) = (\gamma_{i_1 1} \cup \gamma_{i_2 1}, \ldots, \gamma_{i_1 r} \cup \gamma_{i_2 r}))$$

of any m rows of Γ has at least m non-zero entries. Thus any m by r submatrix of Γ has at least m non-zero columns. This completes the proof of lemma 1. Alternative proofs of d) iff c) and b) iff c) can be obtained from Ore's results on term rank in [7] and [8].

In order to state and obtain the results in the sequel we must turn our attention for a moment to the set V_r. If the matrix A is stochastic (i.e., each row sum is 1) or if A is $\approx$-equivalent to such a matrix (i.e., no row of A is 0), then using the classical results (e.g. Doob [9]) on stochastic

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matrices we may decompose V_r into two disjoint subsets T and E, $(E \neq \emptyset)$. E in turn is the disjoint union of ν subsets E_k each of which is the disjoint union of t_k subsets C_{kj}. These sets have the following properties, letting f be the operator corresponding to A: if $i \in T$ then for some $m > 0$, $f^m(i) \cap E \neq \emptyset$; $f(E_k) = E_k$ for each $1 \leq k \leq \nu$ and, in fact, $f(C_{kj}) = C_{kj+1}$ (the second subscript is read modulo t_k). Moreover, the restriction of f^{t_k} to C_{kj} is primitive. In probabilistic terminology: V_r is the set of <u>states</u>, E the <u>ergodic states</u>, T the <u>transient</u> states, E_k is called an <u>ergodic class</u> and C_{kj} is called a <u>cyclically moving class</u>. Borrowing this terminology we shall say that A is <u>ergodic</u> iff $T = \emptyset$.

LEMMA 2. If A is ergodic and non-primitive then

$$\max_{1 \leq i \leq r} \operatorname{index}(i, A) \leq (r-2)^2 + 1 .$$

<u>Proof.</u> In the terminology of the last paragraph, let E_k be the ergodic class to which i belongs and $r_k = W(E_k)$. Let C' be a cyclically moving class of minimal size c', and C'' be one of maximal size c'', both in E_k. If $c' = c''$, choose $C'' = C'$. For some $0 \leq t$, $s < t_k$:

$$f^s(i) \subseteq C' \quad \text{and} \quad f^t(C') = C'' .$$

Using Theorem 1 we obtain: $f^{t_k[(c'-1)^2 + 1] + s + t}(i) = C''$, because f^{t_k} restricted to C' is primitive. Consequently

$$\begin{aligned} \text{(9)} \qquad \operatorname{index}(i, f) &\leq t_k[(c'-1)^2 + 1] + s + t \\ &\leq t_k[(c'-1)^2 + 3] - 2 . \end{aligned}$$

But $c' t_k \le r_k$; thus

(9a) $$\text{index}\,(i,f) \le \frac{r_k}{c'}[(c'-1)^2 + 3] - 2 .$$

(10) If $c' = c''$, then we may choose C' so that $s = t = 0$ and hence index $(i,f) \le t_k[(c'-1)^2 + 1]$.

Case 1: $c' = r_k$. In this case $C' = C''$ and $t_k = 1$ so, by (10), index $(i,f) \le (r_k - 1)^2 + 1$. The non-primitivity of f ensures that $r_k < r$ when $t_k = 1$, and so the lemma follows in this case.

Case 2: $c' = r_k - 1$. In this case $c' = c''$; but then $t_k c' = r_k$ and hence $r_k - 1$ divides r_k. Therefore $r_k = 2 = t_k$ and so lemma 2 follows from (10) in case 2, since we may assume $r > 2$.

Case 3: $c' \le r_k - 2$. In this case $\frac{r_k}{c'}[(c'-1)^2 + 3] - 2$ is maximized relative to $1 \le c' \le r_k - 2$ at $r_k - 2$. Therefore by (9a), index $(i,f) \le (r_k - 2)^2 + \frac{8}{r_k - 2} - 2$. Lemma 2 then follows from this inequality when $r_k > 4$. If $r_k = 3$ or 4 the lemma is obtained by considering the possible values of c', c'', s and t and by applying (10).

As we noted above, each transient state leads eventually to an ergodic state i.e., for each $i \in T$ there is some n for which $f^n(i) \cap E \ne \emptyset$. In the sequel we shall need to know how soon this occurs. To answer this we have:

LEMMA 3. If f is non-singular and $T \ne \emptyset$ then

a) g, defined by: $g(x) = f(x) \cap T$ for all $x \subseteq T$, is also non-singular and for each $i \in T$: $d(i) \le W(T)$; and

b) for each $\emptyset \neq x \subseteq T : f^{W(T)}(x) \cap E \neq \emptyset$.

Proof. a) is most readily seen by looking at Γ, the Boolean matrix corresponding to f. $\Gamma = \begin{vmatrix} \Sigma & 0 \\ \psi & \Delta \end{vmatrix}$ where Δ corresponds to g. g's non-singularity follows from lemma 1c. Consequently Δ contains a $W(T)$ by $W(T)$ permutation matrix. This establishes part a). Now suppose $\emptyset \neq x \subseteq T$. According to (3), either $W(\bigcup_{k=0}^{W(T)} f^k(x)) \geq W(T) + 1$ or $f^{W(T)}(x) \subseteq \bigcup_{k=0}^{W(T)-1} f^k(x)$. If the latter is true then, by induction on n : $f^{W(T)+n}(x) \subseteq \bigcup_{k=0}^{W(T)-1} f^k(x)$ for all n and hence $\bigcup_{k=0}^{W(T)-1} f^k(x) \cap E \neq \emptyset$. If the former is true then $T \not\supseteq \bigcup_{k=0}^{W(T)} f^k(x)$. This establishes part b.

THEOREM 2. If A is essentially non-singular and non-primitive then

$$\max_i \text{index}\,(i, A) = \text{index}\, A \leq (r-2)^2 + 1 .$$

Proof. In view of (6) we need only prove that index $(i, f) \leq (r-2)^2 + 1$ for arbitrary i in V_r. If A is ergodic the result follows from lemma 2. We now assume that $T \neq \emptyset$. If $i \in E$ then, since the restriction $f|E$ of f to E is an ergodic operator on the subsets of a set of fewer than r elements, index $(i, f|E) \leq (r-2)^2 + 1$ by lemma 2 or by theorem 1, depending on whether $f|E$ is non-primitive or not. The desired inequality then follows from the fact that $f(E) = E$.

Now suppose $i \in T$. By lemma 3, $d(i) \leq W(T)$. Then, by (4b) and (8), index $(i, f) \leq d(i)b(i) \leq W(T)(r-1)$. So we may assume that $W(T) \geq r-2$. If $W(T) = r-2$ we may assume that $d(i) = r-2$ and $b(i) = r-1$; then, as we shall now show, g (in the notation of lemma 3) is primitive. According to (4b) $\max_{n>0} W(f^n(i)) = r$ when $b(i) = r-1$, so $W(f^N(i)) = r$ for some N. Let $j \in T$. According to 1c applied to g, there is a permutation q of T for which $q(j) \in g(j)$; but $d(i) = r-2$ therefore $T = \{q^n(i) : 1 \leq n \leq r-2\}$ and hence $T = \{q^n(j) : 1 \leq n \leq r-2\}$. Therefore $i = q^n(j) \subseteq g^n(j)$ for some n and hence $V_r = f^{n+N}(j)$. Consequently g is primitive.

According to theorem 1, $T \subseteq f^{(r-3)^2+1}(i)$. Letting y be the right member we have $d(y) \leq d(i)$ and hence $d(y) \leq r-2$. Now $y \neq T$ by lemma 3b. But $T \subseteq y$; therefore $W(y) \geq r-1$ and hence $b(y) \leq 1$, by (4b). Consequently index $(y, f) \leq r-2$ by (8). Now $\max_n W(f^n(y)) = \max_n W(f^n(i))$ and hence $W(f^{(r-2)+(r-3)^2+1}(i)) = \max_n W(f^n(i))$. Therefore index $(i, f) \leq r^2 - 8r + 8$.

If $W(T) = r-1$ then $W(f^n(i)) = 1 + W(g^n(i))$ for each $n \geq r-1$ by lemma 3b and the fact that $f(E) \subseteq E$. Therefore

$$(11) \qquad \text{index}\,(i, f) = \max\{\text{index}\,(i, g),\ r-1\}\ .$$

If g is primitive then index $(i, g) \leq (r-2)^2 + 1$ by theorem 1, and hence index $(i, f) \leq (r-2)^2 + 1$. Suppose g is non-primitive. Theorem 2 holds vacuously for 1 by 1 matrices. Assume it is true for R by R matrices $(1 \leq R \leq r-1)$. g is non-singular by lemma 3a; consequently index $(i, g) \leq (r-3)^2 + 1$, and hence index $(i, f) \leq (r-2)^2 + 1$ by (11).

$W(T) \leq r-1$ because E is not empty. This completes the proof of theorem 2.

In [1] Wielandt provided, for each r, a primitive r by r matrix whose index is $(r-1)^2 + 1$. It is

$$(w_{ij}) = \begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 \end{vmatrix}$$

where $w_{r1} = w_{r2} = w_{ii+1} = 1$ $(1 \leq i \leq r-1)$ are the only non-zero w_{ij}.

That theorem 2 is best possible may be seen by observing that index $A = (r-2)^2 + 1$ when $A = \begin{vmatrix} W & 0 \\ 0 & 1 \end{vmatrix}$, where W is a primitive $(r-1)$ by $(r-1)$ matrix whose index is $(r-2)^2 + 1$ as furnished by Wielandt.

Finally we exhibit an r by r matrix A for which $\max_i$ index $(i, A) <$ index A and for which index $A > (r-1)^2 + 1$.

Let

$$A_n = \begin{vmatrix} 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{vmatrix}$$

for each $n \geq 3$. That is, the only non-zero entries in $(b_{ij}) = A_n$ are $b_{13} = b_{n1} = b_{n2} = b_{i,i+1} = 1$ $(2 \leq i \leq n-1)$. One can then show that index $(i, A) \leq n-1$, and that $W(A_n^m) \leq n+2$ with equality iff m is a positive multiple of $n-1$. We now let A be the 41 by 41 matrix

$$\begin{vmatrix} A_{12} & 0 & 0 \\ 0 & A_{14} & 0 \\ 0 & 0 & A_{15} \end{vmatrix}$$

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and obtain: $\max_i$ index $(i, A) = 14$, and index $(A) = \text{lcm}(11, 13, 14)$ so that A is a matrix of the required type.

REFERENCES

1. H. Wielandt, Unzerlegbare nicht negative Matrizen, Math. Zeit. 52 (1950) pp. 642-648.

2. D. Rosenblatt, Asymptotic Forms of Two Classes of Matrices, Naval Logistics Quarterly 4 (1957) pp. 151-165.

3. J. C. Holladay and R. S. Varga, On Powers of Non-Negative Matrices, A. M. S. Proceedings 9 (1958) pp. 631-634.

4. A. L. Dulmage and N. S. Mendelsohn, The Exponent of a Primitive Matrix, Can. Math. Bulletin 5 (1962) pp. 241-244.

5. J. H. M. Wedderburn, Boolean Linear Associative Algebra, Ann. Math. 35 (1934) pp. 185-194.

6. H. B. Mann and H. J. Ryser, Systems of Distinct Representatives, Amer. Math. Mon. 60 (1953) pp. 397-401.

7. O. Ore, Graphs and Matching Theorems, Duke Math. J. 22 (1955) p. 635.

8. O. Ore, Theory of Graphs, A. M. S. Colloquium Publ. No. 38 (1962).

9. J. L. Doob, Stochastic Processes, Wiley, N. Y. 1953.

10. V. Pták, On a Combinatorial Theorem and Its Application to Non-negative Matrices, Czechoslovakian Math. J., vol. 8 (83), (1958) pp. 487-492.

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