COMPOSITIO MATHEMATICA

The André–Oort conjecture for Drinfeld modular varieties

Patrik Hubschmid

Compositio Math. **149** (2013), 507–567.

doi:10.1112/S0010437X12000681

FOUNDATION
COMPOSITIO
MATHEMATICA

LONDON
MATHEMATICAL
SOCIETY

Compositio Math. 149 (2013) 507–567
doi:10.1112/S0010437X12000681

The André–Oort conjecture for Drinfeld modular varieties

Patrik Hubschmid

Abstract

We consider the analogue of the André–Oort conjecture for Drinfeld modular varieties which was formulated by Breuer. We prove this analogue for special points with separable reflex field over the base field by adapting methods which were used by Klingler and Yafaev to prove the André–Oort conjecture under the generalized Riemann hypothesis in the classical case. Our result extends results of Breuer showing the correctness of the analogue for special points lying in a curve and for special points having a certain behaviour at a fixed set of primes.

Introduction

The André–Oort conjecture

The André–Oort conjecture asserts that every irreducible component of the Zariski closure of a set of special points in a Shimura variety is a special subvariety. There has been remarkable progress on this conjecture recently.

Edixhoven proved the conjecture for products of modular curves and Hilbert modular surfaces assuming the generalized Riemann hypothesis (GRH) in [Edi01, Edi05, Edi98]. Both proofs exploit the Galois action on special points and use geometric properties of Hecke correspondences. In the special case of a product of two modular curves, André [And98] gave a proof without assuming GRH using transcendence theory. Recently, Pila [Pil11] found an unconditional proof of the conjecture for products of modular curves using techniques from model theory.

Edixhoven and Yafaev extended their Galois-theoretic and geometric methods in [EY03] to prove the conjecture for curves in general Shimura varieties containing infinitely many special points all lying in the same Hecke orbit. Subsequently, Yafaev [Yaf06] also proved the conjecture for general curves assuming GRH.

Recently, Klingler and Yafaev [KY12] and Ullmo and Yafaev [UY12] have announced a proof of the full André–Oort conjecture assuming GRH. Their methods use a combination of the methods of Edixhoven and Yafaev and equidistribution results of Clozel and Ullmo [CU05] established by methods from ergodic theory.

For a more detailed exposition of results concerning the André–Oort conjecture for Shimura varieties, we refer to the survey article of Noot [Noo06].

Received 27 January 2012, accepted in final form 25 June 2012, published online 14 February 2013.
2010 Mathematics Subject Classification 11G09, 14G35.
Keywords: Drinfeld modular varieties, André–Oort conjecture.
The work presented in this paper has been partially funded by the SNSF, Switzerland (project no. 117638).

Drinfeld modular varieties

Drinfeld modular varieties are a natural analogue of Shimura varieties in the function field case. They can be interpreted as moduli spaces for Drinfeld A-modules over a global function field F of a given rank r with $\mathcal{K}$-level structure, where A is the ring of elements of F that are regular outside of a fixed place ∞ and $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ is a compact open subgroup with $\mathbb{A}_F^f$ the ring of adeles of F outside ∞.

One can define *special subvarieties* of a Drinfeld modular variety $S = S_{F,\mathcal{K}}^r$ parametrizing Drinfeld A-modules of rank r in analogy to the case of Shimura varieties. For each finite extension F' of F of degree r/r' with only one place above ∞ and integral closure A' of A in F', the restriction of Drinfeld A'-modules to A gives a morphism from the moduli space of Drinfeld A'-modules of rank r' (with a certain level structure) to S defined over F'. These morphisms are analogues of morphisms induced by a Shimura subdatum. A special subvariety V is defined to be a geometrically irreducible component of a Hecke translate of the image of such a morphism. A *special point* is a special subvariety of dimension 0.

In fact, we can interpret each special subvariety V as a geometrically irreducible component of a *Drinfeld modular subvariety* which is the union of Galois conjugates of V over the corresponding extension F' of F. A Drinfeld modular subvariety X is the image of the composition of an above morphism defined by the restriction of Drinfeld A'-modules to A with a morphism given by a Hecke correspondence. Such a composition, called *inclusion morphism*, is associated to an extension F'/F of the above type and an $\mathbb{A}_F^f$-linear isomorphism $b : (\mathbb{A}_F^f)^r \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$ encoding the involved Hecke correspondence. We say that F' is the *reflex field* of X and its geometrically irreducible components.

In [Pin12], Pink constructs the *Satake compactification* $\overline{S}_{F,\mathcal{K}}^r$ of a Drinfeld modular variety $S_{F,\mathcal{K}}^r$. It is characterized up to unique isomorphism by a certain universal property. If $\mathcal{K}$ is sufficiently small in a certain sense, there is a natural ample invertible sheaf $\mathcal{L}_{F,\mathcal{K}}^r$ on $\overline{S}_{F,\mathcal{K}}^r$. This allows us to define the *degree* of a subvariety of $S_{F,\mathcal{K}}^r$ as the degree of its Zariski closure in $\overline{S}_{F,\mathcal{K}}^r$ with respect to $\mathcal{L}_{F,\mathcal{K}}^r$. The degree of a subvariety can be seen as a measure for the 'complexity' of the subvariety.

André–Oort conjecture for Drinfeld modular varieties

The following analogue of the André–Oort conjecture was formulated by Breuer in [Bre12].

CONJECTURE 1. Let $S = S_{F,\mathcal{K}}^r$ be a Drinfeld modular variety and Σ a set of special points in S. Then each irreducible component over $\mathbb{C}_\infty$ of the Zariski closure of Σ is a special subvariety of S.

Breuer [Bre12] proved this analogue in two cases. Firstly, when the Zariski closure of Σ is a curve, and secondly when all special points in Σ have a certain behaviour at a fixed set of primes. Earlier in [Bre05, Bre07], he proved an analogue of the André–Oort conjecture for products of modular curves in odd characteristic. These proofs use an adaptation of the methods of Edixhoven and Yafaev in [Edi05, EY03, Yaf06]. The results are unconditional because GRH holds over function fields.

In this thesis, we extend the arguments of Breuer using an adaptation of the methods of Klingler and Yafaev in [KY12]. Our main result is the following theorem.

THEOREM 2. *Conjecture 1 is true if the reflex fields of all special points in Σ are separable over F.*

Since the reflex field of a special point in a Drinfeld modular variety $S^r_{F,\mathcal{K}}$ is of degree r over F, special points with inseparable reflex field over F can only occur if r is divisible by $p = \mathrm{char}(F)$. So Theorem 2 implies the following theorem.

THEOREM 3. *Conjecture 1 is true if r is not a multiple of $p = \mathrm{char}(F)$.*

Sketch of the proof of Theorem 2

First reductions. We need to show that a geometrically irreducible subvariety Z of $S^r_{F,\mathcal{K}}$ containing a Zariski dense subset of special points with separable reflex field over F is a special subvariety. An induction argument shows that it is enough to show the following crucial statement.

THEOREM 4. *Let Σ be a set of Drinfeld modular subvarieties of S of dimension d whose union is Zariski dense in a subvariety $Z \subset S$ of dimension greater than d which is defined and irreducible over F. Then, for almost all $X \in \Sigma$, there is a Drinfeld modular subvariety X' of S with $X \subsetneq X' \subset Z$.*

In [KY12], Klingler and Yafaev perform the same induction; however, they work with special subvarieties instead of certain unions of their Galois conjugates (Drinfeld modular subvarieties in our case).

In the proof of this statement, we can assume without loss of generality that the following hold.

- The subgroup $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}^f_F)$ is sufficiently small such that the degree of subvarieties of $S = S^r_{F,\mathcal{K}}$ is defined.
- Also, Z is *Hodge generic*, i.e., no geometrically irreducible component of Z is contained in a proper Drinfeld modular subvariety of S.

Degree of Drinfeld modular subvarieties. We give a classification of the Drinfeld modular subvarieties of S and then use it to show the following unboundedness result.

THEOREM 5. *If Σ is an infinite set of Drinfeld modular subvarieties of S, then $\deg X$ is unbounded as X varies over Σ.*

Note that, for a special subvariety V which is a geometrically irreducible component of a Drinfeld modular subvariety X, the union of the Galois conjugates of V over its reflex field is equal to X. Therefore, $\deg X$ measures both the degree of V and the number of Galois conjugates of V. So our unboundedness statement tells us that it is not possible that, in an infinite family of special subvarieties V, the degrees and the number of Galois conjugates of V are both bounded. Since we can exclude this case, we only need an adaptation of the Galois-theoretic and geometric methods in [KY12] and do not need equidistribution results as in [CU05].

Geometric criterion. We deduce a geometric criterion for Z being equal to S. It is a key ingredient of our proof of Theorem 4 and says that Z is equal to the whole of S provided that Z is contained in a suitable Hecke translate $T_{g_\mathfrak{p}}Z$ of itself. A similar geometric criterion appears in the proof of Klingler and Yafaev in the classical case.

THEOREM 6. *Suppose that $\mathcal{K} = \mathcal{K}_\mathfrak{p} \times \mathcal{K}^{(\mathfrak{p})}$ with $\mathcal{K}_\mathfrak{p} \subset \mathrm{GL}_r(F_\mathfrak{p})$, and assume that $Z \subset T_{g_\mathfrak{p}}Z$ for some $g_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$ and Z Hodge-generic and irreducible over F. If, for all $k_1, k_2 \in \mathcal{K}_\mathfrak{p}$, the cyclic subgroup of $\mathrm{PGL}_r(F_\mathfrak{p})$ generated by the image of $k_1 \cdot g_\mathfrak{p} \cdot k_2$ is unbounded, then $Z = S$.*

The proof of this theorem is based on two results.

(i) (Zariski density) We define the $(T_{h_{\mathfrak{p}}}+T_{h_{\mathfrak{p}}^{-1}})$-orbit of a geometric point $x \in S(\mathbb{C}_\infty)$ to be the smallest subset of $S(\mathbb{C}_\infty)$ containing x which is invariant under $T_{h_{\mathfrak{p}}}$ and $T_{h_{\mathfrak{p}}^{-1}}$. We show that the $(T_{h_{\mathfrak{p}}}+T_{h_{\mathfrak{p}}^{-1}})$-orbit of an arbitrary point $x \in S(\mathbb{C}_\infty)$ is Zariski dense in the geometrically irreducible component of S containing x provided that $h_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$ is chosen such that the cyclic subgroup of $\mathrm{PGL}_r(F_{\mathfrak{p}})$ generated by the image of $h_{\mathfrak{p}}$ is unbounded.

(ii) A result of Pink [Pin97, Theorem 0.1] on the Galois representations associated to Drinfeld modules implies that the image of the arithmetic étale fundamental group of a geometrically irreducible component of Z is open in $\mathrm{GL}_r(F_{\mathfrak{p}})$, see [BP05, Theorem 4]. Here we need our assumption that Z is Hodge-generic.

Induction. Our final step of the proof of Theorem 4 consists of an induction which uses a Hecke correspondence with specific properties. By induction we show the following statement.

THEOREM 7. *Let X be a Drinfeld modular subvariety of S associated to F'/F and $b:(\mathbb{A}_F^f)^r \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$ and assume that X is contained in a Hodge-generic subvariety $Z \subset S$ which is irreducible over F.*

Suppose that $T_{g_{\mathfrak{p}}}$ is a Hecke correspondence localized at a prime $\mathfrak{p}$ with the following properties.

(i) *The element $g_{\mathfrak{p}}$ is defined by some $g'_{\mathfrak{p}'} \in \mathrm{GL}_{r'}(F'_{\mathfrak{p}'})$ where $\mathfrak{p}'$ is a prime of F' lying over $\mathfrak{p}$, i.e., $g_{\mathfrak{p}} = b^{-1} \circ g'_{\mathfrak{p}'} \circ b$.*

(ii) *The element $g_{\mathfrak{p}}$ satisfies the unboundedness condition in Theorem 6, i.e., $\mathcal{K} = \mathcal{K}_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}$ with $\mathcal{K}_{\mathfrak{p}} \subset \mathrm{GL}_r(F_{\mathfrak{p}})$ and, for all $k_1, k_2 \in \mathcal{K}_{\mathfrak{p}}$, the cyclic subgroup of $\mathrm{PGL}_r(F_{\mathfrak{p}})$ generated by the image of $k_1 \cdot g_{\mathfrak{p}} \cdot k_2$ is unbounded.*

(iii) *If $\iota: S' \to S$ is an inclusion morphism with $X \subset \iota(S')$, then the Hecke correspondence T' on S' defined by $g'_{\mathfrak{p}'}$ satisfies (ii) and $\deg T' = \deg T_{g_{\mathfrak{p}}}$.*

(iv) *The inequality $\deg X > \deg(T_{g_{\mathfrak{p}}})^{2^s-1} \cdot (\deg Z)^{2^s}$ holds for $s := \dim Z - \dim X$.*

Then there is a Drinfeld modular subvariety X' of S with $X \subsetneq X' \subset Z$.

We perform an induction over $s := \dim Z - \dim X$. Property (i) implies that $X \subset T_{g_{\mathfrak{p}}}X$; in particular, we therefore have

$$X \subset Z \cap T_{g_{\mathfrak{p}}}Z.$$

The lower bound (iv) for $\deg X$ now says that X cannot be a union of geometrically irreducible components of $Z \cap T_{g_{\mathfrak{p}}}Z$. Therefore we find an irreducible component Z' over F of $Z \cap T_{g_{\mathfrak{p}}}Z$ with $X \subset Z'$ and $\dim Z' > \dim X$. There are two cases.

If $Z' = Z$, we have $Z \subset T_{g_{\mathfrak{p}}}Z$ and conclude by Theorem 6 that $Z = S$, so the conclusion of Theorem 4 is true with $X' = S$.

If $Z' \subsetneq Z$, then $\dim Z' < \dim Z$ because Z is irreducible over F. We replace Z by Z' and apply the induction hypothesis. In this step, it is possible that Z' is not Hodge-generic any more. In this case, we replace S by a smaller Drinfeld modular variety S' and show that properties (i)–(iv) from Theorem 7 are still valid in S' using our property (iii).

Choice of a suitable Hecke correspondence. To finish the proof of Theorem 4, by Theorem 7 we need to show that, for almost all $X \in \Sigma$, there is a Hecke correspondence $T_{g_\mathfrak{p}}$ localized at a prime $\mathfrak{p}$ with the properties (i)–(iv) from Theorem 7. To construct such a $T_{g_\mathfrak{p}}$ for a $X \in \Sigma$, we need the prime $\mathfrak{p}$ to satisfy specific conditions under which we call the prime *good* for X.

Definition 8. Let X be a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$ associated to F'/F and $b:(\mathbb{A}^f_F)^r \xrightarrow{\sim} (\mathbb{A}^f_{F'})^{r'}$. A prime $\mathfrak{p}$ of F is called *good* for $X \subset S^r_{F,\mathcal{K}}$ if there is an $s_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$ such that the following hold for the $A_\mathfrak{p}$-lattice $\Lambda_\mathfrak{p} := s_\mathfrak{p} \cdot A^r_\mathfrak{p}$:

(a) $\mathcal{K} = \mathcal{K}_\mathfrak{p} \times \mathcal{K}^{(\mathfrak{p})}$ where $\mathcal{K}_\mathfrak{p} = s_\mathfrak{p}\mathcal{K}(\mathfrak{p})s_\mathfrak{p}^{-1}$ for the principal congruence subgroup $\mathcal{K}(\mathfrak{p})$ of $\mathrm{GL}_r(A_\mathfrak{p})$;

(b) $b_\mathfrak{p}(\Lambda_\mathfrak{p})$ is an $A' \otimes_A A_\mathfrak{p}$-submodule of $(A' \otimes_A A_\mathfrak{p})^{r'}$;

(c) there exists a prime $\mathfrak{p}'$ of F' above $\mathfrak{p}$ with local degree 1 over F.

Theorem 9. *If $\mathfrak{p}$ is a good prime for a Drinfeld modular subvariety $X \subset S^r_{F,\mathcal{K}}$, then there is a Hecke correspondence $T_{g_\mathfrak{p}}$ localized at $\mathfrak{p}$ satisfying properties (i)–(iii) from Theorem 7 with*

$$\deg T_{g_\mathfrak{p}} = |k(\mathfrak{p})|^{r-1},$$

where $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$.

We show this theorem by defining

$$g_\mathfrak{p} := s_\mathfrak{p}\,\mathrm{diag}(\pi_\mathfrak{p}^{-1}, 1, \dots, 1)s_\mathfrak{p}^{-1}$$

for a uniformizer $\pi_\mathfrak{p}$ at $\mathfrak{p}$. In the proof, it is crucial that $\mathcal{K}_\mathfrak{p}$ is *not* a maximal compact subgroup of $\mathrm{GL}_r(F_\mathfrak{p})$, which is guaranteed by condition (a) in Definition 8 of good prime. Otherwise we are not able to satisfy the unboundedness condition (ii) from Theorem 7.

However, condition (a) in Definition 8 is a very strict condition on the prime $\mathfrak{p}$: for a fixed level $\mathcal{K}$ it can only be satisfied at most at a finite set of primes because $\mathcal{K}$ is maximal compact at almost all primes. Since conditions (b) and (c) in Definition 8 are both satisfied only for an infinite set of primes of density smaller than one, for a fixed level $\mathcal{K}$, in general we cannot find a prime $\mathfrak{p}$ satisfying conditions (a)–(c). We get rid of this problem by starting with a prime $\mathfrak{p}$ for which there is an $s_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$ such that

$$(\mathrm{a}') \quad \mathcal{K} = s_\mathfrak{p}\mathrm{GL}_r(A_\mathfrak{p})s_\mathfrak{p}^{-1} \times \mathcal{K}^{(\mathfrak{p})}$$

and also conditions (b) and (c) in Definition 8 are satisfied. With an effective version of Čebotarev's theorem which relies on the correctness of GRH for function fields we can show that such a prime satisfying an upper bound for $|k(\mathfrak{p})|$ exists provided that $\deg X$ is large enough.

In this situation we consider the Drinfeld modular variety $\tilde{S} := S^r_{F,\tilde{\mathcal{K}}}$ with $\tilde{\mathcal{K}} = s_\mathfrak{p}\mathcal{K}(\mathfrak{p})s_\mathfrak{p}^{-1} \times \mathcal{K}^{(\mathfrak{p})}$ which is a finite cover of $S = S^r_{F,\mathcal{K}}$. The conditions (a)–(c) from Definition 8 are satisfied for some Drinfeld modular subvariety $\tilde{X}$ of $\tilde{S}$ lying over X, i.e., $\mathfrak{p}$ is a good prime for $\tilde{X} \subset \tilde{S}$. By Theorem 9, we then find a Hecke correspondence $T_{g_\mathfrak{p}}$ on $\tilde{S}$ localized at $\mathfrak{p}$ satisfying properties (i)–(iv) from Theorem 7 for $\tilde{X} \subset \tilde{S}$ where property (iv) is ensured by the above upper bound for $|k(\mathfrak{p})|$.

Since $\deg X$ is unbounded as X ranges over Σ by Theorem 5, this works for almost all $X \in \Sigma$. For these X, Theorem 7 gives a Drinfeld modular subvariety $\tilde{X}'$ of $\tilde{S}$ with $\tilde{X} \subsetneq \tilde{X}' \subsetneq \tilde{Z}$. The image $X' \subset S$ of $\tilde{X}'$ under the covering map $\tilde{S} \to S$ then satisfies the conclusion of Theorem 4.

Difficulties in the inseparable case

Unfortunately, the above methods do not work in the inseparable case, i.e., if Σ in Theorem 4 contains Drinfeld modular subvarieties of S with inseparable reflex field. This is caused by the fact that every prime ramifies in an inseparable field extension. Therefore, for a Drinfeld modular subvariety with inseparable reflex field, there is no prime for which condition (c) in Definition 8 is satisfied. So we cannot apply Theorem 9 to find a Hecke correspondence satisfying properties (i)–(iii) from Theorem 7.

Also, other approaches to find such Hecke correspondences fail. For example, if X is a Drinfeld modular subvariety of dimension 0 with purely inseparable reflex field F'/F and $\mathfrak{p}$ any prime of F, then a Hecke correspondence $T_{g_{\mathfrak{p}}}$ localized at $\mathfrak{p}$ satisfying property (i) of Theorem 7 does not satisfy the unboundedness condition (ii) in Theorem 7. Indeed, in this case there is exactly one prime $\mathfrak{p}'$ of F' above $\mathfrak{p}$ with ramification index r and, if $\pi_{\mathfrak{p}'} \in F'_{\mathfrak{p}'}$ is a uniformizer, then $1, \pi_{\mathfrak{p}'}, \ldots, \pi_{\mathfrak{p}'}^{r-1}$ is an $F_{\mathfrak{p}}$-basis of $F'_{\mathfrak{p}'}$. Therefore, if $g_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$ is defined by $g'_{\mathfrak{p}'} = \pi_{\mathfrak{p}'}^k \in \mathrm{GL}_1(F'_{\mathfrak{p}'})$ as in property (i) of Theorem 7, then $g_{\mathfrak{p}}$ is a conjugate of the matrix

$$\begin{pmatrix} & & & \pi_{\mathfrak{p}} \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}^{k} \in \mathrm{GL}_r(F_{\mathfrak{p}})$$

for $\pi_{\mathfrak{p}} := \pi_{\mathfrak{p}'}^r$. Its rth power is a scalar matrix, and hence the cyclic subgroup of $\mathrm{PGL}_r(F_{\mathfrak{p}})$ generated by the image of $g_{\mathfrak{p}}$ is bounded and we cannot apply our geometric criterion (Theorem 6) for the Hecke correspondence $T_{g_{\mathfrak{p}}}$.

Organization of the paper

After discussing preliminaries in § 1, we define *Drinfeld modular varieties* for arbitrary level $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ as quotients of fine moduli schemes of Drinfeld modules in § 2.

In § 3, we first define *projection morphisms* and *Hecke correspondences* on Drinfeld modular varieties. Then we define *inclusion morphisms* of Drinfeld modular varieties which allow us to define *Drinfeld modular subvarieties* and *special subvarieties* of a Drinfeld modular variety S. Subsequently, we show various properties of these morphisms, give a classification of the Drinfeld modular subvarieties of S, and describe the Galois action on the sets of Drinfeld modular subvarieties and irreducible components of S.

In § 4, we define the *degree* of subvarieties of a Drinfeld modular variety using the Satake compactification constructed in [Pin12] and discuss some of its properties. We then show our unboundedness statement for the degree of Drinfeld modular subvarieties (Theorem 5).

The next two sections are devoted to the proof of our geometric criterion for being a Drinfeld modular subvariety (Theorem 6). Section 5 deals with Zariski density of $(T_g + T_{g^{-1}})$-orbits and in § 6 we give the proof of the actual criterion.

In § 7, we first define *good primes* for Drinfeld modular subvarieties. We then explain, for a fixed Drinfeld modular subvariety, how we can find a suitable Hecke correspondence at a good prime as in Theorem 9. The last subsection of § 7 is devoted to finding a good prime $\mathfrak{p}$ satisfying an upper bound for $|k(\mathfrak{p})|$ for a given Drinfeld modular subvariety after passing to a finite cover of S.

In § 8, we finally conclude the proof of Theorem 4 by proving Theorem 7 and applying the results of the previous sections. Here we also explain why Theorem 4 implies our main result (Theorem 2).

1. Preliminaries

1.1 Notation and conventions

The following notation and conventions will be used throughout this paper.

- The symbol $\mathbb{F}_q$ denotes a fixed finite field with q elements.
- For an $\mathbb{F}_q$-algebra R, we denote by $R\{\tau\}$ the ring of non-commutative polynomials in the variable τ with coefficients in R and the commutator rule $\tau\lambda = \lambda^q\tau$ for $\lambda \in R$.
- The symbol F always denotes a global function field of characteristic p with field of constants $\mathbb{F}_q$ and ∞ a fixed place of F.
- For a pair (F, ∞), we use the following notation:

A	ring of elements of F regular outside ∞;
$F_\mathfrak{p}$	completion of F at a place $\mathfrak{p}$;
$A_\mathfrak{p}$	discrete valuation ring of $F_\mathfrak{p}$;
$k(\mathfrak{p})$	residue field of $\mathfrak{p}$;
$\mathbb{C}_\infty$	completion of an algebraic closure of F_∞;
$\mathbb{A}_F^f$	ring of finite adeles of F (i.e., adeles outside ∞);
$\mathbb{A}_F^{f,\mathfrak{p}}$	ring of finite adeles of F outside $\mathfrak{p}$ (i.e., adeles outside $\mathfrak{p}$ and ∞);
$\hat{A}$	profinite completion $\prod_{\mathfrak{p}\neq\infty} A_\mathfrak{p}$ of A;
$\mathrm{Cl}(F)$	class group of A.

- A place $\mathfrak{p} \neq \infty$ of F is said to be a *prime* of F. We identify a prime $\mathfrak{p}$ of F with a prime ideal of A.
- For a place $\mathfrak{p}$ and a finite extension F' of F, we set $F'_\mathfrak{p} := F' \otimes_F F_\mathfrak{p}$ and $A'_\mathfrak{p} := A' \otimes_A A_\mathfrak{p}$. We identify $F'_\mathfrak{p}$ with $\prod_{\mathfrak{p}'|\mathfrak{p}} F'_{\mathfrak{p}'}$ and $A'_\mathfrak{p}$ with $\prod_{\mathfrak{p}'|\mathfrak{p}} A'_{\mathfrak{p}'}$ via the canonical isomorphisms. For a second finite extension F'' of F, we use the analogous conventions and notations.
- For a subfield $K \subset \mathbb{C}_\infty$ we denote by K^{sep} the separable and by $\overline{K}$ the algebraic closure of K in $\mathbb{C}_\infty$. Each K-automorphism of K^{sep} has a unique continuation to a K-automorphism of $\overline{K}$. Therefore, we can and do identify the absolute Galois group $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ with the automorphism group $\mathrm{Aut}_K(\overline{K})$.

For the formulation of algebro-geometric results, we use the following conventions.

- Unless otherwise stated, *variety* means a reduced separated scheme of finite type over $\mathbb{C}_\infty$ and *subvariety* means a reduced closed subscheme of a variety. We identify the set $X(\mathbb{C}_\infty)$ of $\mathbb{C}_\infty$-valued points of a variety X with the set of its closed points.
- For a subfield $K \subset \mathbb{C}_\infty$, a variety X together with a scheme X_0 of finite type over K and an isomorphism of schemes $\alpha_X : X_{0,\mathbb{C}_\infty} \xrightarrow{\sim} X$ is called a *variety over* K. We often write X in place of (X, X_0, α_X) and identify $X_{0,\mathbb{C}_\infty}$ with X via α_X if this leads to no confusion. Such a variety X is called K*-irreducible* if X_0 is irreducible. Note that a variety over K is also a variety over K' if $K \subset K' \subset \mathbb{C}_\infty$.
- Let $X' = X'_{0,\mathbb{C}_\infty}$ and $X = X_{0,\mathbb{C}_\infty}$ be two varieties over K. A morphism $X' \to X$ is said to be *defined over* K if it is the base extension to $\mathbb{C}_\infty$ of a morphism $X'_0 \to X_0$ of schemes over K.
- For a variety X over K, a subvariety $X' \hookrightarrow X$ is said to be *defined over* K if X' is a variety over K and the closed immersion $X' \hookrightarrow X$ is defined over K.
- For a variety $X = X_{0,\mathbb{C}_\infty}$ over K and a subfield $K' \subset \mathbb{C}_\infty$ containing K, we denote by $X(K')$ the set of K'-valued points of X_0. Note that $X(K')$ is naturally a subset of the set

of closed points of X; in fact it is equal to the set of closed points of X defined over K', see, e.g., [Bor91, p. 26].

- The *degree* of a finite surjective morphism $X \to Y$ of irreducible varieties is defined to be the degree of the extension of the function fields $\mathbb{C}_\infty(X)/\mathbb{C}_\infty(Y)$. We say that a general finite morphism $f : X \to Y$ of (not necessarily irreducible) varieties is of degree d if for each irreducible component Z of $f(X)$

$$\sum_{\substack{\text{irreducible components}\\ X_i \text{ of } f^{-1}(Z)}} \deg(f|X_i : X_i \to Z) = d. \tag{1.1.1}$$

1.2 Galois action on subvarieties

Let $X = X_{0,\mathbb{C}_\infty}$ be a variety over $K \subset \mathbb{C}_\infty$. Then there is a natural action of the absolute Galois group G_K on $X_{0,\overline{K}}$ which induces an action of G_K on the set of subvarieties of X which are defined over $\overline{K}$.

Proposition 1.2.1. *A subvariety of X which is defined over $\overline{K}$ is already defined over K if and only if it is defined over K^{sep} and G_K-stable.*

Proof. This follows from [Bor91, Theorem AG. 14.4]. □

Proposition 1.2.2. *Let $X = X_{0,\mathbb{C}_\infty}$ be a variety over $K \subset \mathbb{C}_\infty$. Then the irreducible components of X are defined over K^{sep}. The absolute Galois group acts transitively on the set of irreducible components of X if and only if X is K-irreducible.*

Proof. Corollary 5.56(2) in [GW10] implies that the irreducible components of X are defined over K^{sep}. The second statement is a direct consequence of Proposition 1.2.1. □

2. Drinfeld modular varieties

2.1 Analytic description and modular interpretation

We consider the following datum:

- a global function field F together with a fixed place ∞;
- a positive integer r, called *rank*; and
- a compact open subgroup $\mathcal{K}$ of $\mathrm{GL}_r(\mathbb{A}_F^f)$, called *level*.

We define *Drinfeld's upper half-space* over F of dimension $r-1$ by

$$\Omega_F^r := \mathbb{P}^{r-1}(\mathbb{C}_\infty)\backslash\{F_\infty\text{-rational hyperplanes}\}.$$

Proposition 2.1.1. *The points of Drinfeld's upper half-space Ω_F^r are in bijective correspondence with the set of injective F_∞-linear maps $F_\infty^r \hookrightarrow \mathbb{C}_\infty$ up to multiplication by a constant in $\mathbb{C}_\infty^*$ via the assignment*

$$[\omega_1 : \cdots : \omega_r] \longmapsto [(a_1, \ldots, a_r) \mapsto a_1\omega_1 + \cdots + a_r\omega_r].$$

Proof. We have the canonical bijection

$$\begin{array}{ccc} \mathbb{C}_\infty^r & \longrightarrow & \{F_\infty\text{-linear maps } F_\infty^r \to \mathbb{C}_\infty\} \\ (\omega_1, \ldots, \omega_r) & \longmapsto & (a_1, \ldots, a_r) \mapsto a_1\omega_1 + \cdots + a_r\omega_r. \end{array}$$

The F_∞-linear map $(a_1, \ldots, a_r) \mapsto a_1\omega_1 + \cdots + a_r\omega_r$ is injective if and only if $\omega_1, \ldots, \omega_r$ are F_∞-linearly independent, i.e., if and only if $(\omega_1, \ldots, \omega_r)$ does not lie in a F_∞-rational hyperplane.

Hence, factoring out the action of $\mathbb{C}_\infty^*$ on both sides, we get the desired bijection of Drinfeld's upper half-space with the set of injective F_∞-linear maps $F_\infty^r \hookrightarrow \mathbb{C}_\infty$ up to multiplication by a constant in $\mathbb{C}_\infty^*$. □

In the following, we use the identification given by Proposition 2.1.1 and denote the element of Ω_F^r associated to an injective F_∞-linear map $\omega : F_\infty^r \hookrightarrow \mathbb{C}_\infty$ by $\overline{\omega}$.

Using this notation, one sees that $\mathrm{GL}_r(F)$ acts on Ω_F^r from the left by

$$T \cdot \overline{\omega} := \overline{\omega \circ T^{-1}} \tag{2.1.1}$$

for $T \in \mathrm{GL}_r(F)$ considered as automorphism of F_∞^r.

Remark. This action can also be described by regarding Ω_F^r as a subset of $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. A short calculation shows that, for $\omega = [\omega_1 : \cdots : \omega_r] \in \Omega_F^r \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty)$ and $T \in \mathrm{GL}_r(F)$ with $T^{-1} = (s_{ij})$, we have

$$T \cdot \omega = [s_{11}\omega_1 + \cdots + s_{r1}\omega_r : \cdots : s_{1r}\omega_1 + \cdots + s_{rr}\omega_r]. \tag{2.1.2}$$

In other words, the action of a $T \in \mathrm{GL}_r(F)$ on $\Omega_F^r \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty)$ is the restriction to Ω_F^r of the natural action of $(T^{-1})^T \in \mathrm{GL}_r(\mathbb{C}_\infty)$ on $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$.

Theorem 2.1.2. *There is a normal affine variety $S_{F,\mathcal{K}}^r$ of dimension $r-1$ over F together with an isomorphism*

$$S_{F,\mathcal{K}}^r(\mathbb{C}_\infty) \cong \mathrm{GL}_r(F)\backslash(\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}) \tag{2.1.3}$$

of rigid-analytic spaces, where $\mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$ is viewed as a discrete set.

Remarks.

- In the proof, we define a variety $S_{F,\mathcal{K}}^r$ over F together with a rigid-analytic isomorphism of the form (2.1.3) up to isomorphism over F. This variety is called the *Drinfeld modular variety* associated to the datum $(F, r, \mathcal{K})$. We will identify its $\mathbb{C}_\infty$-valued points with double cosets in $\mathrm{GL}_r(F)\backslash(\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K})$ via the rigid-analytic isomorphism given in the proof.
- Later (Corollary 3.1.4), we will show that $S_{F,\mathcal{K}}^r$ is a non-singular variety if $\mathcal{K}$ is sufficiently small in a certain sense.

Proof. The proof consists of several steps.

(i) We use Drinfeld's construction of Drinfeld moduli schemes in [Dri74] to define $S_{F,\mathcal{K}}^r$ and a rigid-analytic isomorphism of the form (2.1.3) for $\mathcal{K} = \mathcal{K}(I) \subset \mathrm{GL}_r(\hat{A})$ a principal congruence subgroup modulo a proper ideal I of A.

(ii) For $g \in \mathrm{GL}_r(\mathbb{A}_F^f) \cap \mathrm{Mat}_r(\hat{A})$ and proper ideals I, J of A with $J\hat{A}^r \subset gI\hat{A}^r$, we define morphisms

$$\pi_g : S_{F,\mathcal{K}(J)}^r \longrightarrow S_{F,\mathcal{K}(I)}^r,$$

which are defined over F and satisfy the compatibility relation

$$\pi_g \circ \pi_{g'} = \pi_{gg'}.$$

In particular, these morphisms define an action of $\mathrm{GL}_r(\hat{A})$ on $S_{F,\mathcal{K}(I)}^r$.

(iii) We use this action to extend the definition in (i) to all compact open subgroups $\mathcal{K} \subset \mathrm{GL}_r(\hat{A})$.

(iv) We extend the definition in (ii) to get morphisms

$$\pi_g : S_{F,\mathcal{K}'}^r \longrightarrow S_{F,\mathcal{K}}^r$$

for arbitrary $\mathcal{K}, \mathcal{K}' \subset \mathrm{GL}_r(\hat{A})$ and $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ with $\mathcal{K}' \subset g^{-1}\mathcal{K}g$.

(v) We define $S^r_{F,\mathcal{K}}$ and a rigid-analytic isomorphism β of the form (2.1.3) for arbitrary levels $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}^f_F)$. We use the morphisms π_g from (iv) to show the well-definedness of $(S^r_{F,\mathcal{K}}, \beta)$ up to isomorphism over F.

Step (i). Recall that a *Drinfeld* A*-module of rank* r over an F-scheme S is a line bundle $\mathcal{L}$ over S together with a ring homomorphism φ from A to the ring $\mathrm{End}_{\mathbb{F}_q}(\mathcal{L})$ of $\mathbb{F}_q$-linear endomorphisms of $\mathcal{L}$ (as a group scheme over S) such that, over any trivializing affine open subset $\mathrm{Spec}(B) \subset S$, the homomorphism φ is given by

$$\varphi : \begin{array}{rcl} A & \longrightarrow & \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,\mathrm{Spec}(B)}) = B\{\tau\} \\ a & \longmapsto & \varphi_a = \displaystyle\sum_{i=0}^{m(a)} b_i(a)\tau^i \end{array}$$

where τ denotes the q-power Frobenius endomorphism and, for all $a \in A$, we have:

(a) $q^{m(a)} = |A/(a)|^r$;

(b) $b_{m(a)}(a) \in B^*$;

(c) $b_0(a) = \gamma(a)$ where γ is the ring homomorphism $F \to B$ corresponding to the morphism of affine schemes $\mathrm{Spec}(B) \hookrightarrow S \to \mathrm{Spec}(F)$.

For a proper ideal I of A, an I*-level structure* on a Drinfeld module $\mathcal{L}/S$ of rank r is an A-linear isomorphism of group schemes over S

$$\alpha : \underline{(I^{-1}/A)^r} \longrightarrow \mathcal{L}_I := \bigcap_{a \in I} \ker(\mathcal{L} \xrightarrow{a} \mathcal{L}),$$

where $\underline{(I^{-1}/A)^r}$ denotes the constant group scheme over S with fibers $(I^{-1}/A)^r$.

Remark. In general, one can also define Drinfeld A-modules together with level structures over A-schemes instead of F-schemes. In this case, one uses a different definition of I-level structure to deal smoothly with the fibers over $\mathfrak{p} \in \mathrm{Spec}(A)$ dividing I; see, for example, [DH87, § I.6].

By [Dri74, § 5], the functor

$$\mathcal{F}^r_{F,I} : \begin{array}{rcl} F\text{-schemes} & \longrightarrow & \text{Sets} \\ S & \longmapsto & \{\text{Isomorphism classes of Drinfeld } A\text{-modules} \\ & & \quad \text{of rank } r \text{ over } S \text{ with } I\text{-level structure}\} \end{array}$$

is representable by a non-singular affine scheme of finite type over F of dimension $r-1$. Note that, in [Dri74], it is actually shown that the corresponding functor from the category of schemes over Spec A to the category of sets is representable if I is contained in two distinct maximal ideals of A. The argument in the proof shows that it is enough that I is a proper ideal of A if we work with schemes over Spec F.

By our conventions in § 1.1, the base extension to $\mathbb{C}_\infty$ of the above representing scheme is a non-singular variety of dimension $r-1$ defined over F. We denote it by $S^r_{F,\mathcal{K}(I)}$, where $\mathcal{K}(I)$ denotes the principal congruence subgroup modulo I. By [Dri74, Proposition 6.6], there is a natural isomorphism

$$S^r_{F,\mathcal{K}(I)}(\mathbb{C}_\infty) \cong \mathrm{GL}_r(F)\backslash(\Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}(I)) \tag{2.1.4}$$

of rigid-analytic spaces. Under this isomorphism, the equivalence class of an element $(\overline{\omega}, h) \in \Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)$ is mapped to the $\mathbb{C}_\infty$-valued point of $S^r_{F,\mathcal{K}(I)}$ corresponding to the Drinfeld module over $\mathbb{C}_\infty$ associated to the lattice

$$\Lambda := \omega(F^r \cap h\hat{A}^r)$$

with I-level structure given by the composition of the isomorphisms

$$(I^{-1}/A)^r \xrightarrow{h} I^{-1} \cdot (F^r \cap h\hat{A}^r)/(F^r \cap h\hat{A}^r) \xrightarrow{\omega} I^{-1} \cdot \Lambda/\Lambda,$$

where the first isomorphism is given by the multiplication by h on $(\mathbb{A}_F^f)^r$ via the natural identifications

$$\begin{gathered}(I^{-1}/A)^r \cong I^{-1}\hat{A}^r/\hat{A}^r,\\ I^{-1} \cdot (F^r \cap h\hat{A}^r)/(F^r \cap h\hat{A}^r) \cong I^{-1} \cdot h\hat{A}^r/h\hat{A}^r\end{gathered}$$

by the inclusion maps. For a detailed survey of this modular interpretation, we refer to the explanations in [DH87, § II.5].

Step (ii). Let I, J be proper ideals of A and $g \in \mathrm{GL}_r(\mathbb{A}_F^f) \cap \mathrm{Mat}_r(\hat{A}^r)$ such that $J\hat{A}^r \subset gI\hat{A}^r$. For such a datum, we construct a morphism of functors

$$\mathcal{F}^r_{F,J} \longrightarrow \mathcal{F}^r_{F,I}.$$

The given $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ with matrix entries in $\hat{A}$ induces a surjective endomorphism of $(\mathbb{A}_F^f)^r/\hat{A}^r$ with kernel $g^{-1}\hat{A}^r/\hat{A}^r$. Since there is a natural isomorphism $(F/A)^r \cong (\mathbb{A}_F^f/\hat{A})^r$ induced by the inclusion maps, we therefore get a surjective homomorphism of A-modules

$$(F/A)^r \xrightarrow{g} (F/A)^r.$$

The kernel $U := \ker g$ of this homomorphism is contained in $(J^{-1}/A)^r$ because we have $g^{-1}\hat{A}^r \subset J^{-1}I\hat{A}^r \subset J^{-1}\hat{A}^r$ by our assumption $J\hat{A}^r \subset gI\hat{A}^r$.

For any Drinfeld module $\mathcal{L}$ over an F-scheme S with J-level structure $\alpha : \underline{(J^{-1}/A)^r} \xrightarrow{\sim} \mathcal{L}_J$, the image of $\underline{U} \subset \underline{(J^{-1}/A)^r}$ under α is a finite A-invariant subgroup scheme of $\mathcal{L}$ over S. Hence, the quotient $\mathcal{L}' := \mathcal{L}/\alpha(\underline{U})$ is also a Drinfeld A-module over S and contains the finite subgroup scheme $\mathcal{L}_J/\alpha(\underline{U})$. Since $g(J^{-1}/A)^r \cong (J^{-1}/A)^r/U$, there is a unique A-linear isomorphism α' of group schemes over S such that the diagram

$$\begin{tikzcd}
\underline{(J^{-1}/A)^r} \arrow[r, "\sim", "\alpha"'] \arrow[d, "g"] & \mathcal{L}_J \arrow[d, "\pi"] \\
\underline{g(J^{-1}/A)^r} \arrow[r, "\sim", "\alpha'"'] & \mathcal{L}_J/\alpha(\underline{U})
\end{tikzcd}$$

commutes, where $\pi : \mathcal{L}_J \to \mathcal{L}_J/\alpha(\underline{U})$ is the canonical projection. By the assumption $J\hat{A}^r \subset gI\hat{A}^r$, we have $(I^{-1}/A)^r \subset g(J^{-1}/A)^r$. Restricting the isomorphism α' to the I-torsion gives, therefore, an I-level structure

$$\underline{(I^{-1}/A)^r} \xrightarrow{\sim} \mathcal{L}'_I$$

of $\mathcal{L}'$.

The assignment $(\mathcal{L}, \alpha) \to (\mathcal{L}', \alpha'|_{\underline{(I^{-1}/A)^r}})$ induces a morphism of functors $\mathcal{F}^r_{F,J} \to \mathcal{F}^r_{F,I}$ and therefore a morphism $\pi_g : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}(I)}$ defined over F.

A simple verification shows that π_g is given by

$$[(\omega, h)] \longmapsto [(\omega, hg^{-1})] \tag{2.1.5}$$

on $\mathbb{C}_\infty$-valued points identified with double cosets via the isomorphisms (2.1.4).

This description implies that we have the relation

$$\pi_g \circ \pi_{g'} = \pi_{gg'}$$

for two such morphisms:

$$\begin{aligned} \pi_g &: S^r_{F,\mathcal{K}(I')} \longrightarrow S^r_{F,\mathcal{K}(I)}, \\ \pi_{g'} &: S^r_{F,\mathcal{K}(I'')} \longrightarrow S^r_{F,\mathcal{K}(I')}. \end{aligned}$$

In particular, we have an action of $\mathrm{GL}_r(\hat{A})$ on $S^r_{F,\mathcal{K}(I)}$ by morphisms defined over F and hence also on isomorphism classes of Drinfeld A-modules with I-level structure.

Step (iii). Using the action of $\mathrm{GL}_r(\hat{A})$ on $S^r_{F,\mathcal{K}(I)}$ by the morphisms π_g, we define, for a compact open subgroup $\mathcal{K} \subset \mathrm{GL}_r(\hat{A})$,

$$S^r_{F,\mathcal{K}} := S^r_{F,\mathcal{K}(I)}/\mathcal{K},$$

where $\mathcal{K}(I)$ is a principal congruence subgroup contained in $\mathcal{K}$. Since $\mathcal{K}(I)$ acts trivially on $S^r_{F,\mathcal{K}(I)}$, this quotient can be viewed as a quotient under the action of the finite group $\mathcal{K}/\mathcal{K}(I)$ by morphisms defined over F. Hence, it is an affine variety defined over F of dimension $r-1 = \dim S^r_{F,\mathcal{K}(I)}$ which is normal because $S^r_{F,\mathcal{K}(I)}$ is normal (see, e.g., [Ser88, § III.12]). By the description (2.1.5) of the above action on $\mathbb{C}_\infty$-valued points, the rigid-analytic isomorphism (2.1.4) induces one of the form

$$\beta_I : (S^r_{F,\mathcal{K}(I)}/\mathcal{K})(\mathbb{C}_\infty) \cong \mathrm{GL}_r(F)\backslash(\Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}). \tag{2.1.6}$$

It remains to show that, up to F-isomorphism, $(S^r_{F,\mathcal{K}(I)}/\mathcal{K}, \beta_I)$ is independent of the choice of I. For this, note that, for two ideals I, J with $I \subset J$, the functors

$$\begin{aligned} S &\longmapsto \mathcal{F}^r_{F,I}(S)/\mathcal{K}(J), \\ S &\longmapsto \mathcal{F}^r_{F,J}(S) \end{aligned}$$

are isomorphic, where the quotient is taken with respect to the action of $\mathrm{GL}_r(\hat{A})$ on $\mathcal{F}^r_{F,I}(S)$. The isomorphism is given by restricting I-level structures to $(J^{-1}/A)^r$.

Therefore, we have a natural isomorphism

$$S^r_{F,\mathcal{K}(I)}/\mathcal{K}(J) \cong S^r_{F,\mathcal{K}(J)}$$

defined over F, which is compatible with the isomorphisms (2.1.6) and (2.1.4).

So for two ideals J, I with $\mathcal{K}(I) \subset \mathcal{K}$ and $\mathcal{K}(J) \subset \mathcal{K}$ we have

$$S^r_{F,\mathcal{K}(I)}/\mathcal{K} \cong S^r_{F,\mathcal{K}(I\cap J)}/\mathcal{K} \cong S^r_{F,\mathcal{K}(J)}/\mathcal{K},$$

and these isomorphisms are compatible with the isomorphisms (2.1.6). Therefore, we can well define $S^r_{F,\mathcal{K}}$ up to isomorphism over F by $S^r_{F,\mathcal{K}(I)}$ together with the rigid-analytic isomorphism β_I.

Step (iv). Let $g \in \mathrm{GL}_r(\mathbb{A}^f_F) \cap \mathrm{Mat}_r(\hat{A})$ and $\mathcal{K}, \mathcal{K}' \subset \mathrm{GL}_r(\hat{A})$ with $\mathcal{K}' \subset g^{-1}\mathcal{K}g$ be given. Choose proper ideals I and J of A such that $\mathcal{K}(I) \subset \mathcal{K}$, $\mathcal{K}(J) \subset \mathcal{K}'$ and $J\hat{A}^r \subset gI\hat{A}^r$. Then, by Step (iii),

$$\begin{aligned} S^r_{F,\mathcal{K}'} &:= S^r_{F,\mathcal{K}(J)}/\mathcal{K}', \\ S^r_{F,\mathcal{K}} &:= S^r_{F,\mathcal{K}(I)}/\mathcal{K}, \end{aligned}$$

and, by Step (ii), there is a morphism

$$\pi_g : S^r_{F,\mathcal{K}(J)} \longrightarrow S^r_{F,\mathcal{K}(I)}.$$

Since $g\mathcal{K}'g^{-1} \subset \mathcal{K}$, for each $k' \in \mathcal{K}'$, there is a $k \in \mathcal{K}$ such that $gk' = kg$ and

$$\pi_g \circ \pi_{k'} = \pi_k \circ \pi_g$$

as morphisms $S^r_{F,\mathcal{K}(J)} \longrightarrow S^r_{F,\mathcal{K}(I)}$. So the composition of π_g with the canonical projection $S^r_{F,\mathcal{K}(I)} \to S^r_{F,\mathcal{K}}$ is $\mathcal{K}'$-invariant and induces therefore a morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ such that the diagram

$$\begin{tikzcd}
S^r_{F,\mathcal{K}(J)} \arrow[r, "\pi_g"] \arrow[d] & S^r_{F,\mathcal{K}(I)} \arrow[d] \\
S^r_{F,\mathcal{K}'} \arrow[r, "\pi_g"] & S^r_{F,\mathcal{K}}
\end{tikzcd}$$

commutes, where the vertical maps are the canonical projections. By (2.1.5), using the identifications $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ and $S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty)$ with double coset spaces given by (2.1.6), this morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is given by

$$[(\omega, h)] \longmapsto [(\omega, hg^{-1})] \tag{2.1.7}$$

on $\mathbb{C}_\infty$-valued points. Therefore, we have defined π_g independently of the choice of I and J if all matrix entries of g lie in $\hat{A}$.

If $g \in \mathrm{GL}_r(\mathbb{A}^f_F)$ is arbitrary, there is a $\lambda \in A\backslash\{0\}$ such that $\lambda \cdot g \in \mathrm{GL}_r(\mathbb{A}^f_F) \cap \mathrm{Mat}_r(\hat{A})$. We then define $\pi_g := \pi_{\lambda \cdot g}$. This morphism is independent of the choice of λ because we have

$$[(\omega, h(\lambda g)^{-1})] = [(\omega, hg^{-1})]$$

in $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ for all $\lambda \in A\backslash\{0\}$ and $[(\omega, h)] \in S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty)$. In particular, π_g is still described by (2.1.7) on $\mathbb{C}_\infty$-valued points.

The latter implies the relation

$$\pi_g \circ \pi_{g'} = \pi_{gg'} \tag{2.1.8}$$

for two such morphisms $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ and $\pi_{g'} : S^r_{F,\mathcal{K}''} \to S^r_{F,\mathcal{K}'}$.

Step (v). For an arbitrary compact open subgroup $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}^f_F)$, we choose a $g \in \mathrm{GL}_r(\mathbb{A}^f_F)$ such that $g\mathcal{K}g^{-1} \subset \mathrm{GL}_r(\hat{A})$. The composition of the rigid-analytic isomorphism (2.1.6)

$$S^r_{F,g\mathcal{K}g^{-1}}(\mathbb{C}_\infty) \cong \mathrm{GL}_r(F)\backslash(\Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)/g\mathcal{K}g^{-1})$$

and $[(\overline{\omega}, h)] \mapsto [(\overline{\omega}, hg)]$ gives a rigid-analytic isomorphism

$$\beta_g : S^r_{F,g\mathcal{K}g^{-1}}(\mathbb{C}_\infty) \cong \mathrm{GL}_r(F)\backslash(\Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}).$$

For another $g' \in \mathrm{GL}_r(\mathbb{A}^f_F)$ with $g'\mathcal{K}g'^{-1} \subset \mathrm{GL}_r(\hat{A})$, the diagram

$$\begin{tikzcd}
S^r_{F,g\mathcal{K}g^{-1}}(\mathbb{C}_\infty) \arrow[dr, "\beta_g", "\sim"'] \arrow[dd, "\pi_{g'g^{-1}}", "\sim"'] & \\
& \mathrm{GL}_r(F)\backslash(\Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}) \\
S^r_{F,g'\mathcal{K}g'^{-1}}(\mathbb{C}_\infty) \arrow[ur, "\beta_{g'}", "\sim"'] &
\end{tikzcd}$$

commutes. By the relation (2.1.8), the vertical arrow $\pi_{g'g^{-1}}$ is an isomorphism over F with inverse $\pi_{gg'^{-1}}$.

Therefore, we can well define $S^r_{F,\mathcal{K}}$ up to F-isomorphism as $S^r_{F,g\mathcal{K}g^{-1}}$ together with the rigid-analytic isomorphism β_g. Since we have seen in Step (iii) that $S^r_{F,g\mathcal{K}g^{-1}}$ is a normal affine variety of dimension $r-1$ defined over F, the same holds for $S^r_{F,\mathcal{K}}$. □

Proposition 2.1.3. *Let C be a set of representatives in $\mathrm{GL}_r(\mathbb{A}^f_F)$ for $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}$, and set $\Gamma_g := g\mathcal{K}g^{-1}\cap\mathrm{GL}_r(F)$ for $g\in C$. Then the map*

$$\begin{aligned}\coprod_{g\in C}\Gamma_g\backslash\Omega^r_F &\longrightarrow \mathrm{GL}_r(F)\backslash(\Omega^r_F\times\mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K})\\ [\overline{\omega}]_g &\longmapsto [(\overline{\omega},g)]\end{aligned}$$

is a rigid-analytic isomorphism which maps for each $g\in C$ the quotient space $\Gamma_g\backslash\Omega^r_F$ to the $\mathbb{C}_\infty$-valued points of an irreducible component Y_g of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$.

This theorem implies that the irreducible components of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$ are disjoint and that C is in bijective correspondence with the set of irreducible components of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$ where $g\in C$ corresponds to the irreducible component Y_g with $Y_g(\mathbb{C}_\infty)\cong\Gamma_g\backslash\Omega^r_F$ via the isomorphism given in the theorem.

Proof. A direct calculation shows that the considered map is well defined and bijective. Since $\mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}$ is viewed as a discrete set, the map is also an isomorphism of rigid-analytic spaces.

Therefore, it only remains to show that the quotient spaces $\Gamma_g\backslash\Omega^r_F$, $g\in C$, are irreducible as rigid-analytic spaces because the irreducible components of the rigid analytification of $S^r_{F,\mathcal{K}}$ coincide with the rigid analytification of the irreducible components of $S^r_{F,\mathcal{K}}$ (see, e.g., [Con99, Theorem 2.3.1]). Since $S^r_{F,\mathcal{K}}$ is a normal variety, and therefore its rigid analytification is a normal rigid-analytic space, this is equivalent to the connectedness of the quotient spaces $\Gamma_g\backslash\Omega^r_F$. The latter follows because Ω^r_F is a connected rigid-analytic space by [Koh11, Theorem 2.4]. □

Definition 2.1.4. For a $\mathbb{C}_\infty$-valued point $p=[(\overline{\omega},h)]\in S(\mathbb{C}_\infty)$ of a Drinfeld modular variety $S=S^r_{F,\mathcal{K}}$ with $h\in\mathrm{GL}_r(\mathbb{A}^f_F)$ and $\overline{\omega}\in\Omega^r_F$ associated to $\omega:F^r_\infty\hookrightarrow\mathbb{C}_\infty$, the elements of

$$\mathrm{End}(p) := \{u\in\mathbb{C}_\infty : u\cdot\omega(F^r)\subset\omega(F^r)\}$$

are called *endomorphisms* of p.

Note that $\mathrm{End}(p)$ is well defined because the homothety class of $\omega(F^r)\subset\mathbb{C}_\infty$ does not depend on the chosen representatives ω and h.

Remark. If $\mathcal{K}=\mathcal{K}(I)$ and $p\in S^r_{F,\mathcal{K}(I)}(\mathbb{C}_\infty)$ is corresponding to the Drinfeld module φ over $\mathbb{C}_\infty$ associated to the lattice $\Lambda\subset\mathbb{C}_\infty$, then $\omega(F^r)=F\cdot\Lambda$, and therefore

$$\mathrm{End}(p)=F\cdot\mathrm{End}(\varphi)$$

for the endomorphism ring $\mathrm{End}(\varphi)\subset\mathbb{C}_\infty$ of φ.

Lemma 2.1.5. *The set $\mathrm{End}(p)$ of endomorphisms of p is a field extension of F contained in $\mathbb{C}_\infty$ of finite degree dividing r with only one place above ∞.*

Proof. This follows from the argumentation in the proof of Proposition 4.7.17 in [Gos98] noting that the endomorphism ring of a Drinfeld module in generic characteristic is commutative. □

Lemma 2.1.6. *Each irreducible component X of a Drinfeld modular variety $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$ contains a point $p \in X(\mathbb{C}_\infty)$ with $\mathrm{End}(p) = F$.*

Proof. Choose $\overline{\omega} \in \Omega^r_F$ such that $\omega(F^r) = F \oplus F \cdot \xi_2 \oplus \cdots \oplus F \cdot \xi_r$ with $\xi_2, \ldots, \xi_r \in \mathbb{C}_\infty$ algebraically independent over F. This is possible because $\mathbb{C}_\infty$ as uncountable field is of infinite transcendence degree over the countable field F.

Now choose $h \in \mathrm{GL}_r(\mathbb{A}^f_F)$ such that $p := [(\overline{\omega}, h)] \in X(\mathbb{C}_\infty)$ (use the description of the irreducible components of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$ given in Proposition 2.1.3). Since $1 \in \omega(F^r)$, we have on the one hand $\mathrm{End}(p) \subseteq \omega(F^r)$. On the other hand, all elements of $\mathrm{End}(p)$ are algebraic over F because the extension $\mathrm{End}(p)/F$ is finite. However, by the choice of $\xi_2, \ldots, \xi_r$, every element of $\omega(F^r)$ which is algebraic over F lies in F. Hence, $\mathrm{End}(p) = F$. □

2.2 Rank one case

In the case $r = 1$ the variety $S^r_{F,\mathcal{K}}$ is zero-dimensional and defined over F for any compact open subgroup $\mathcal{K} \subset \mathrm{GL}_1(\mathbb{A}^f_F) = (\mathbb{A}^f_F)^*$. Hence, $S^1_{F,\mathcal{K}}$ consists only of finitely many closed points and it can be set-theoretically identified with $S^1_{F,\mathcal{K}}(\mathbb{C}_\infty)$. By Proposition 1.2.2, the closed points are all defined over F^{sep} and the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on $S^1_{F,\mathcal{K}}$.

Drinfeld's upper half-space Ω^1_F just consists of one point. Therefore, we have

$$S^1_{F,\mathcal{K}} = F^* \backslash (\mathbb{A}^f_F)^* / \mathcal{K}$$

as a set. Since $(\mathbb{A}^f_F)^*$ is abelian, this set can be identified with the abelian group $(\mathbb{A}^f_F)^*/(F^* \cdot \mathcal{K})$.

Since $F^* \cdot \mathcal{K}$ is a closed subgroup of finite index of $(\mathbb{A}^f_F)^*$, by class field theory, there is a finite abelian extension H/F totally split at ∞ such that the Artin map

$$\psi_{H/F} : (\mathbb{A}^f_F)^* \longrightarrow \mathrm{Gal}(H/F)$$

induces an isomorphism $(\mathbb{A}^f_F)^*/(F^* \cdot \mathcal{K}) \cong \mathrm{Gal}(H/F)$. In particular we have

$$|S^1_{F,\mathcal{K}}| = [H : F].$$

Theorem 2.2.1. *If $\psi_{H/F}(g) = \sigma|_H$ for a $g \in (\mathbb{A}^f_F)^*$ and a $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$, then the action of σ on $S^1_{F,\mathcal{K}} = F^* \backslash (\mathbb{A}^f_F)^* / \mathcal{K}$ is given by multiplication with g^{-1}.*

Proof. This follows from Theorem 1 in § 8 of Drinfeld's article [Dri74]. Note that in this article the action of an element $g \in (\mathbb{A}^f_F)^*$ on $S^1_{F,\mathcal{K}} = F^* \backslash (\mathbb{A}^f_F)^* / \mathcal{K}$ is given by the morphism π_g, which is given by multiplication with g^{-1}. □

Corollary 2.2.2. *The absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts transitively on $S^1_{F,\mathcal{K}}$.*

3. Morphisms and Drinfeld modular subvarieties

3.1 Projection morphisms and Hecke correspondences

Let $S^r_{F,\mathcal{K}}$ be a fixed Drinfeld modular variety. For each $g \in \mathrm{GL}_r(\mathbb{A}^f_F)$ and all compact open subgroups $\mathcal{K}' \subset g^{-1}\mathcal{K}g$ of $\mathrm{GL}_r(\mathbb{A}^f_F)$, we have a well-defined map

$$\begin{array}{rcl} S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty) & \to & S^r_{F,\mathcal{K}}(\mathbb{C}_\infty) \\ [(\overline{\omega}, h)] & \mapsto & [(\overline{\omega}, hg^{-1})]. \end{array} \tag{3.1.1}$$

Theorem 3.1.1. *This map is induced by a unique finite morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ defined over F.*

Proof. In the case that $\mathcal{K}$ and $\mathcal{K}'$ are contained in $\mathrm{GL}_r(\hat{A})$, we already showed the existence of a morphism π_g which is described by (3.1.1) on $\mathbb{C}_\infty$-valued points in Step (iv) of the proof of Theorem 2.1.2. If $\mathcal{K}$ and $\mathcal{K}'$ are arbitrary with $\mathcal{K}' \subset g^{-1}\mathcal{K}g$, there is an $s \in \mathrm{GL}_r(\mathbb{A}^f_F)$ with

$$s\mathcal{K}'s^{-1} \subset sg^{-1}\mathcal{K}gs^{-1} \subset \mathrm{GL}_r(\hat{A}).$$

By our definition in the proof of Theorem 2.1.2, we have $S^r_{F,\mathcal{K}'} = S^r_{F,s\mathcal{K}'s^{-1}}$, where under the identifications of $\mathbb{C}_\infty$-valued points introduced in Step (v) of the proof of Theorem 2.1.2

$$[(\omega, h)] \in S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty) \longleftrightarrow [(\omega, hs^{-1})] \in S^r_{F,s\mathcal{K}'s^{-1}}(\mathbb{C}_\infty).$$

Similarly, we have $S^r_{F,\mathcal{K}} = S^r_{F,sg^{-1}\mathcal{K}gs^{-1}}$ with

$$[(\omega, h)] \in S^r_{F,\mathcal{K}}(\mathbb{C}_\infty) \longleftrightarrow [(\omega, hgs^{-1})] \in S^r_{F,sg^{-1}\mathcal{K}gs^{-1}}(\mathbb{C}_\infty).$$

Using these identifications, we can define the morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ as $\pi_1 : S^r_{F,s\mathcal{K}'s^{-1}} \to S^r_{F,sg^{-1}\mathcal{K}gs^{-1}}(\mathbb{C}_\infty)$. Since the latter morphism π_1 is given by $[(\omega, h)] \mapsto [(\omega, h)]$ on $\mathbb{C}_\infty$-valued points, by the above identifications π_g is indeed described by (3.1.1) on $\mathbb{C}_\infty$-valued points. So we have shown the existence of the morphism π_g defined over F. It is uniquely determined by (3.1.1) because $\mathbb{C}_\infty$ is algebraically closed.

It remains to show finiteness of the morphism π_g. By the above definition of a general morphism π_g, it is enough to show it for morphisms of the form $\pi_1 : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ with $\mathcal{K}' \subset \mathcal{K} \subset \mathrm{GL}_r(\hat{A})$.

We first assume that $\mathcal{K}' = \mathcal{K}(I)$ is a principal congruence subgroup. Then π_1 is the canonical projection

$$S^r_{F,\mathcal{K}(I)} \to S^r_{F,\mathcal{K}(I)}/\mathcal{K}$$

by the construction in the proof of Theorem 2.1.2. Since $\mathcal{K}(I) \subset \mathcal{K}$ acts trivially on $S^r_{F,\mathcal{K}(I)}$, the quotient $S^r_{F,\mathcal{K}(I)}/\mathcal{K}$ can be viewed as a quotient under the action of the finite group $\mathcal{K}/\mathcal{K}(I)$. Therefore, π_1 is finite.

For general subgroups $\mathcal{K}' \subset \mathcal{K} \subset \mathrm{GL}_r(\hat{A})$, choose a proper ideal I of A with $\mathcal{K}(I) \subset \mathcal{K}'$. Then we have the following commutative diagram of projection maps.

$$\begin{tikzcd}
S^r_{F,\mathcal{K}(I)} \arrow[dr, "\pi_1"] \arrow[dd, "\pi_1"'] & \\
& S^r_{F,\mathcal{K}'} \arrow[dl, "\pi_1"] \\
S^r_{F,\mathcal{K}} &
\end{tikzcd}$$

We have already shown that the morphisms $\pi_1 : S^r_{F,\mathcal{K}(I)} \to S^r_{F,\mathcal{K}'}$ and $\pi_1 : S^r_{F,\mathcal{K}(I)} \to S^r_{F,\mathcal{K}}$ are finite. Therefore $\pi_1 : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is also finite. □

In the following, we call the morphisms π_g *projection morphisms* of Drinfeld modular varieties. In the case $g = 1$ we also call them *canonical projections* of Drinfeld modular varieties. For two elements $g, g' \in \mathrm{GL}_r(\mathbb{A}^f_F)$ and two subgroups $\mathcal{K}' \subset g^{-1}\mathcal{K}g$, $\mathcal{K}'' \subset g'^{-1}\mathcal{K}'g'$, by the description on $\mathbb{C}_\infty$-valued points, we have

$$\pi_{gg'} = \pi_g \circ \pi_{g'}. \tag{3.1.2}$$

DEFINITION 3.1.2. A compact open subgroup $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ is called *amply small* if there is a proper ideal I of A and a $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ such that $g\mathcal{K}g^{-1}$ is contained in the principal congruence subgroup $\mathcal{K}(I) \subset \mathrm{GL}_r(\hat{A})$.

PROPOSITION 3.1.3. *Let $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ be amply small, $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and $\mathcal{K}' \subset g^{-1}\mathcal{K}g$. Then the finite morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is étale of degree $[g^{-1}\mathcal{K}g : \mathcal{K}']$. Furthermore, if $\mathcal{K}'$ is a normal subgroup of $g^{-1}\mathcal{K}g$, it is an étale Galois cover over F with group $g^{-1}\mathcal{K}g/\mathcal{K}'$ where the automorphism of the cover corresponding to a coset $[x] \in g^{-1}\mathcal{K}g/\mathcal{K}'$ is given by $\pi_x : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}'}$.*

Remark. In fact, the condition that some conjugate of $\mathcal{K}$ is contained in a principal congruence subgroup of $\mathrm{GL}_r(\hat{A})$ could be weakened. Indeed it is enough that there is a prime $\mathfrak{p}$ such that the image of some conjugate of $\mathcal{K}$ in $\mathrm{GL}_r(A/\mathfrak{p})$ is unipotent (cf. [Pin12, Proposition 1.5]).

Proof. Since $\mathcal{K}$ is amply small, there is an $h \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and a proper ideal I of A such that $h^{-1}\mathcal{K}h \subset \mathcal{K}(I) \subset \mathrm{GL}_r(\hat{A})$. By the relation (3.1.2), we have the commutative diagram

$$\begin{tikzcd}
S^r_{F,h^{-1}g\mathcal{K}'g^{-1}h} \arrow[r, "\pi_{g^{-1}h}", "\sim"'] \arrow[d, "\pi_1"] & S^r_{F,\mathcal{K}'} \arrow[d, "\pi_g"] \\
S^r_{F,h^{-1}\mathcal{K}h} \arrow[r, "\pi_h", "\sim"'] & S^r_{F,\mathcal{K}}
\end{tikzcd}$$

where the horizontal morphisms are isomorphisms with $(\pi_h)^{-1} = \pi_{h^{-1}}$ and $(\pi_{g^{-1}h})^{-1} = \pi_{h^{-1}g}$. Therefore, we can assume without loss of generality that $g = 1$ and $\mathcal{K}' \subset \mathcal{K} \subset \mathcal{K}(I) \subset \mathrm{GL}_r(\hat{A})$.

Case (i). Let $\mathcal{K}'$ be a principal congruence subgroup $\mathcal{K}(J)$ modulo a proper ideal J of A, i.e., $\mathcal{K}' = \mathcal{K}(J) \lhd \mathcal{K} \subset \mathcal{K}(I)$.

Then, by our definition in the proof of Theorem 2.1.2, $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}}$ is the canonical projection

$$S^r_{F,\mathcal{K}(J)} \longrightarrow S^r_{F,\mathcal{K}(J)}/\mathcal{K}.$$

We show that $\mathcal{K}/\mathcal{K}(J)$ acts freely on the closed points of $S^r_{F,\mathcal{K}(J)}$. This implies that this projection is a finite étale morphism of degree $[\mathcal{K} : \mathcal{K}(J)]$ (see, e.g., [Mum70, § II.7]). By the modular interpretation of $S^r_{F,\mathcal{K}(J)}$ given in the proof of Theorem 2.1.2, it is enough to show that the action of $\mathcal{K}/\mathcal{K}(J)$ on isomorphism classes of Drinfeld A-modules over $\mathbb{C}_\infty$ together with J-level structure is free.

Indeed, assume that a coset $[k] \in \mathcal{K}/\mathcal{K}(J)$ stabilizes the isomorphism class of the Drinfeld module φ over $\mathbb{C}_\infty$ associated to a lattice $\Lambda \subset \mathbb{C}_\infty$ together with J-level structure $\alpha : (J^{-1}/A)^r \xrightarrow{\sim} J^{-1} \cdot \Lambda/\Lambda$. By our definition of the action of $\mathrm{GL}_r(\hat{A})$ on Drinfeld modules with J-level structure in the proof of Theorem 2.1.2, this means that there is an automorphism c of φ under which the J-level structure α passes into $\alpha \circ k^{-1}$. Note that the restrictions of α and $\alpha \circ k^{-1}$ to $(I^{-1}/A)^r$ coincide because $k \in \mathcal{K}(I)$. Rigidity of Drinfeld modules with I-level structure (see, e.g., [Leh09, p. 30]) therefore implies that c is the identity. This is only possible if $k \in \mathcal{K}(J)$, i.e. if $[k]$ is trivial in $\mathcal{K}/\mathcal{K}(J)$.

So we have shown that $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}} = S^r_{F,\mathcal{K}(J)}/\mathcal{K}$ is an étale cover of degree $[\mathcal{K} : \mathcal{K}(J)]$. The group $\mathcal{K}/\mathcal{K}(J)$ injects into the automorphism group over F of this cover via $[k] \mapsto \pi_k$.

Since the degree of the cover is equal to $[\mathcal{K} : \mathcal{K}(J)]$ and $S^r_{F,\mathcal{K}(J)}$ is F-irreducible, the automorphism group (over F) is therefore equal to $\mathcal{K}/\mathcal{K}(J)$. Furthermore, the cover is Galois because this group acts simply transitively on the geometric fibers.

Case (ii). Let $\mathcal{K}'$ be an arbitrary normal subgroup of $\mathcal{K}$, i.e., $\mathcal{K}' \lhd \mathcal{K} \subset \mathcal{K}(I)$.

Choose a proper ideal J of A such that $\mathcal{K}(J) \subset \mathcal{K}'$ and note that the diagram

$$\begin{tikzcd}
S^r_{F,\mathcal{K}(J)} \arrow[dd, "\pi_1"'] \arrow[dr, "\pi_1"] & \\
& S^r_{F,\mathcal{K}'} = S^r_{F,\mathcal{K}(J)}/\mathcal{K}' \arrow[dl, "\pi_1"] \\
S^r_{F,\mathcal{K}} = S^r_{F,\mathcal{K}(J)}/\mathcal{K} &
\end{tikzcd} \tag{3.1.3}$$

commutes. Since $\mathcal{K}'$ is normal in $\mathcal{K}$, the action of $\mathcal{K}$ on $S^r_{F,\mathcal{K}(J)}$ induces an action of $\mathcal{K}/\mathcal{K}'$ on the quotient $S^r_{F,\mathcal{K}'} = S^r_{F,\mathcal{K}(J)}/\mathcal{K}'$. By the commutativity of the diagram, the variety $S^r_{F,\mathcal{K}}$ is the quotient of $S^r_{F,\mathcal{K}'}$ under this action. Furthermore, this action is free on the closed points of $S^r_{F,\mathcal{K}'}$ because $\mathcal{K}/\mathcal{K}(J)$ acts freely on the closed points of $S^r_{F,\mathcal{K}(J)}$. Therefore, we conclude by the same arguments as above that $\pi_1 : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is an étale Galois cover of degree $[\mathcal{K} : \mathcal{K}']$ with group $\mathcal{K}/\mathcal{K}'$ where the automorphism of the cover corresponding to a coset $[k] \in \mathcal{K}/\mathcal{K}'$ is given by π_k.

Case (iii). Let $\mathcal{K}'$ be an arbitrary subgroup of $\mathcal{K}$, i.e., $\mathcal{K}' \subset \mathcal{K} \subset \mathcal{K}(I)$.

As in Case (ii) above, choose a proper ideal J of A such that $\mathcal{K}(J) \subset \mathcal{K}'$. The diagram above then also commutes and $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}'}$ and $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}}$ are surjective étale morphisms by Case (i). Furthermore, $S^r_{F,\mathcal{K}(J)}$ is a non-singular variety as explained in Step (i) of the proof of Theorem 2.1.2.

Proposition 17.3.3.1 in EGA IV [Gro64] says that if $X \to Y$ is a flat, surjective morphism of schemes and X is regular, then Y is also regular. Therefore, $S^r_{F,\mathcal{K}}$ and $S^r_{F,\mathcal{K}'}$ are both non-singular varieties.

By [Har77, Proposition 10.4], a morphism $f : X \to Y$ of non-singular varieties of the same dimension over an algebraically closed field is étale if and only if, for every closed point $x \in X$, the induced map $T_x \to T_{f(x)}$ on Zariski tangent spaces is an isomorphism. We can apply this criterion because $S^r_{F,\mathcal{K}(J)}$, $S^r_{F,\mathcal{K}}$ and $S^r_{F,\mathcal{K}'}$ are all non-singular. Since the morphisms $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}'}$ and $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}}$ are étale, the commutativity of the above diagram therefore implies that $\pi_1 : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is étale and finite of degree $[\mathcal{K} : \mathcal{K}(J)]/[\mathcal{K}' : \mathcal{K}(J)] = [\mathcal{K} : \mathcal{K}']$. ☐

COROLLARY 3.1.4. *If $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}^f_F)$ is amply small, then the Drinfeld modular variety $S^r_{F,\mathcal{K}}$ is non-singular.*

Proof. See Case (iii) of the above proof of Proposition 3.1.3. ☐

DEFINITION 3.1.5 (Hecke correspondence). For $g \in \mathrm{GL}_r(\mathbb{A}^f_F)$ and $\mathcal{K}_g := \mathcal{K} \cap g^{-1}\mathcal{K}g$ the diagram

$$\begin{tikzcd}
& S^r_{F,\mathcal{K}_g} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_g"] & \\
S^r_{F,\mathcal{K}} & & S^r_{F,\mathcal{K}}
\end{tikzcd}$$

is called the *Hecke correspondence* T_g associated to g. For subvarieties $Z \subset S^r_{F,\mathcal{K}}$ we define

$$T_g(Z) := \pi_g(\pi_1^{-1}(Z)).$$

Note that $T_g(Z)$ is a subvariety of $S^r_{F,\mathcal{K}}$ for any subvariety $Z \subset S^r_{F,\mathcal{K}}$ because π_g is finite and hence proper. The integer

$$\deg(T_g) := [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g]$$

is called the *degree* of the Hecke correspondence T_g. If $\mathcal{K}$ is amply small, by Proposition 3.1.3, it is equal to $\deg \pi_1$.

3.2 Inclusions of Drinfeld modular varieties

Let $S = S^r_{F,\mathcal{K}}$ be a given Drinfeld modular variety. We consider the following datum:

- a finite extension $F' \subset \mathbb{C}_\infty$ of F of degree r/r' for some integer $r' \geqslant 1$ with only one place ∞' lying over ∞; and
- an $\mathbb{A}^f_F$-linear isomorphism $b : (\mathbb{A}^f_F)^r \xrightarrow{\sim} (\mathbb{A}^f_{F'})^{r'}$.

Note that the integral closure A' of A in F' is equal to the ring of elements of F' regular away from ∞' because ∞' is the only place of F' lying over ∞.

The above datum defines a subgroup

$$\mathcal{K}' = (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$$

of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$.

LEMMA 3.2.1. *The subgroup $\mathcal{K}'$ is compact and open in $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. If $\mathcal{K}$ is amply small, it is also amply small.*

Proof. We fix an $\mathbb{A}^f_F$-linear isomorphism $b' : (\mathbb{A}^f_F)^r \xrightarrow{\sim} (\mathbb{A}^f_{F'})^{r'}$ with $b'(\widehat{A}^r) = \widehat{A'}^{r'}$ and set $g := b'^{-1} \circ b \in \mathrm{GL}_r(\mathbb{A}^f_F)$. Since $\mathcal{K}$ is compact and open in $\mathrm{GL}_r(\mathbb{A}^f_F)$, there is a proper ideal I of A such that the principal congruence subgroup $\mathcal{K}(I)$ is contained in $g\mathcal{K}g^{-1}$ with finite index. Therefore, $\mathcal{K}' = (b'g\mathcal{K}g^{-1}b'^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ contains the subgroup $\mathcal{K}'' = (b'\mathcal{K}(I)b'^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ with finite index. The latter subgroup exactly consists of the elements of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ which stabilize $b'(\widehat{A}^r) = \widehat{A'}^{r'}$ and induce the identity on the quotient $\widehat{A'}^{r'}/I \cdot \widehat{A'}^{r'} \cong (A'/IA')^{r'}$. Hence, $\mathcal{K}''$ is the principal congruence subgroup modulo IA' and $\mathcal{K}'$ is compact and open in $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$.

If $\mathcal{K}$ is amply small, there is a proper ideal I of A and an $h \in \mathrm{GL}_r(\mathbb{A}^f_F)$ such that $h\mathcal{K}h^{-1} \subset \mathcal{K}(I)$. Therefore $\mathcal{K}'$ is contained in the subgroup

$$(bh^{-1}\mathcal{K}(I)hb^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$$

of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. This subgroup exactly consists of all elements of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ which stabilize $\Lambda := bh^{-1}(\hat{A}^r) \subset (\mathbb{A}^f_{F'})^{r'}$ and induce the identity on $\Lambda/I \cdot \Lambda$. Since these elements are $\mathbb{A}^f_{F'}$-linear, they also stabilize the $\widehat{A'}$-lattice $\Lambda' := \widehat{A'} \cdot \Lambda$ and induce the identity on $\Lambda'/I \cdot \Lambda'$. Since Λ' is a finitely generated $\widehat{A'}$-submodule of $(\mathbb{A}^f_{F'})^{r'}$ with $\mathbb{A}^f_{F'} \cdot \Lambda' = (\mathbb{A}^f_{F'})^{r'}$ and $\widehat{A'}$ is a direct product of principal ideal domains, Λ' is a free $\widehat{A'}$-module of rank r'. Hence, there is a $g' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ such that $\Lambda' = g'\widehat{A'}^{r'}$ and

$$\mathcal{K}' \subset (bh^{-1}\mathcal{K}(I)hb^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'}) \subset g'\mathcal{K}(I')g'^{-1}$$

for $I' := IA'$. This implies that $\mathcal{K}'$ is amply small. □

We choose an isomorphism

$$\varphi : F^r \xrightarrow{\sim} F'^{r'}$$

of vector spaces over F. By scalar extension to F_∞ and $\mathbb{A}_F^f$ it induces isomorphisms

$$\begin{array}{ccc} F_\infty^r & \xrightarrow{\varphi} & F'^{r'}_{\infty'} \\ (\mathbb{A}_F^f)^r & \xrightarrow{\varphi} & (\mathbb{A}_{F'}^f)^{r'} \end{array}$$

which we also denote by φ. We now define a morphism from the lower-rank Drinfeld modular variety $S' = S^{r'}_{F',\mathcal{K}'}$ into S.

THEOREM 3.2.2. *There is a finite morphism $\iota^{F'}_{F,b} : S' \to S$ defined over F' which on $\mathbb{C}_\infty$-valued points is given by the injective map*

$$\begin{array}{ccc} S'(\mathbb{C}_\infty) & \longrightarrow & S(\mathbb{C}_\infty) \\ [(\overline{\omega'}, h')] & \longmapsto & [(\overline{\omega' \circ \varphi}, \varphi^{-1} \circ h' \circ b)], \end{array} \tag{3.2.1}$$

where $\overline{\omega'} \in \Omega^{r'}_{F'}$ and $h' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. The morphism $\iota^{F'}_{F,b}$ is independent of the choice of $\varphi : F^r \xrightarrow{\sim} F'^{r'}$.

Proof. Case (i). We first consider the case where $b(\widehat{A}^r) = \widehat{A'}^{r'}$ and $\mathcal{K} = \mathcal{K}(I)$ is a principal congruence subgroup modulo a proper ideal I of A. In this case, $\mathcal{K}' = (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ is the principal congruence subgroup modulo $I' := IA'$ (see the proof of Lemma 3.2.1) and b induces an A-linear isomorphism $(I^{-1}/A)^r \xrightarrow{\sim} (I'^{-1}/A')^{r'}$, which we again denote by b. Therefore, for a Drinfeld A'-module $(\mathcal{L}, \psi)$ of rank r' over an F'-scheme with I'-level structure

$$\alpha : \underline{(I'^{-1}/A')^{r'}} \xrightarrow{\sim} \mathcal{L}_{I'},$$

the restriction $(\mathcal{L}, \psi|_A)$ to $A \subset A'$ is a Drinfeld A-module of rank $r = r' \cdot [F'/F]$ over S and the composition

$$\underline{(I^{-1}/A)^r} \xrightarrow{b} \underline{(I'^{-1}/A')^{r'}} \xrightarrow{\alpha} \mathcal{L}_I$$

is an I-level structure on $(\mathcal{L}, \psi|_A)$ (note that the I-torsion subgroup scheme $\mathcal{L}_I$ of $\mathcal{L}$ coincides with the I'-torsion subgroup scheme $\mathcal{L}_{I'}$ because I generates I' as an ideal of A'). The assignment

$$(\mathcal{L}, \psi, \alpha) \longmapsto (\mathcal{L}, \psi|_A, \alpha \circ b) \tag{3.2.2}$$

defines a morphism of functors from $\mathcal{F}^{r'}_{F',I'}$ to the restriction of $\mathcal{F}^r_{F,I}$ to the subcategory of F'-schemes (see Step (i) of the proof of Theorem 2.1.2 for the definition of these functors). Therefore, we have a morphism

$$\iota^{F'}_{F,b} : S^{r'}_{F',\mathcal{K}(I')} \longrightarrow S^r_{F,\mathcal{K}(I)}$$

defined over F'. By [Bre12, Lemma 3.1 and Proposition 3.2], it is a proper morphism which is injective on $\mathbb{C}_\infty$-valued points. Since $S^{r'}_{F',\mathcal{K}(I')}$ and $S^r_{F,\mathcal{K}(I)}$ are both affine schemes of finite type over $\mathbb{C}_\infty$, the morphism $\iota^{F'}_{F,b}$ is therefore a proper morphism of finite presentation with finite fibers. This implies that $\iota^{F'}_{F,b}$ is finite by Theorem 8.11.1 of EGA IV [Gro66].

Using

$$\omega'(F'^{r'} \cap h'\widehat{A'}^{r'}) = (\omega' \circ \varphi)(F^r \cap (\varphi^{-1} \circ h' \circ b)\widehat{A}^r)$$

one sees that $\iota^{F'}_{F,b}$ is given by (3.2.1) on $\mathbb{C}_\infty$-valued points.

Case (ii). For $b : (\mathbb{A}_F^f)^r \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$ and $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ arbitrary, we choose:

- a $g' \in \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ with $g'\mathcal{K}'g'^{-1} \subset \mathrm{GL}_{r'}(\widehat{A'})$;
- an $\mathbb{A}_F^f$-linear isomorphism $b' : (\mathbb{A}_F^f)^r \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$ with $b'(\widehat{A}^r) = \widehat{A'}^{r'}$;
- a proper ideal I of A with $\mathcal{K}(I) \subset g^{-1}\mathcal{K}g$, where $g := b^{-1} \circ g'^{-1} \circ b' \in \mathrm{GL}_r(\mathbb{A}_F^f)$.

Then $g' \circ b = b' \circ g^{-1}$, hence

$$g'\mathcal{K}'g'^{-1} = (b'g^{-1}\mathcal{K}gb'^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f) \supset (b'\mathcal{K}(I)b'^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f) = \mathcal{K}(IA'),$$

and by Case (i) and Theorem 3.1.1, the composition of morphisms

$$S^{r'}_{F',\mathcal{K}(IA')} \xrightarrow{\iota^{F'}_{F,b'}} S^r_{F,\mathcal{K}(I)} \xrightarrow{\pi_g} S^r_{F,\mathcal{K}}$$

is defined and finite. Because

$$(g'\mathcal{K}'g'^{-1})b'g^{-1} = b'g^{-1}(b^{-1}\mathcal{K}'b) \subset b'g^{-1}\mathcal{K},$$

this composition is invariant under the action of $g'\mathcal{K}'g'^{-1}$ on $S^{r'}_{F',\mathcal{K}(IA')}$. Hence, it induces a finite morphism $f : S^{r'}_{F',g'\mathcal{K}'g'^{-1}} \to S^r_{F,\mathcal{K}}$ such that the diagram

$$\begin{tikzcd}
S^{r'}_{F',\mathcal{K}(IA')} \arrow[r, "\iota^{F'}_{F,b'}"] \arrow[d, "\pi_1"] & S^r_{F,\mathcal{K}(I)} \arrow[d, "\pi_g"] \\
S_{F',g'\mathcal{K}'g'^{-1}} \arrow[r, "f"] & S^r_{F,\mathcal{K}}
\end{tikzcd}$$

commutes. We can now define $\iota^{F'}_{F,b} := f \circ \pi_{g'}$, where $\pi_{g'} : S^{r'}_{F',\mathcal{K}'} \to S^{r'}_{F',g'\mathcal{K}'g'^{-1}}$. For $[(\overline{\omega'}, h')] \in S^{r'}_{F',\mathcal{K}'}(\mathbb{C}_\infty)$ we indeed have

$$\iota^{F'}_{F,b}([(\overline{\omega'}, h')]) = [(\overline{\omega' \circ \varphi}, \varphi^{-1} \circ h'g'^{-1} \circ b' \circ g^{-1})] = [(\overline{\omega' \circ \varphi}, \varphi^{-1} \circ h' \circ b)],$$

independently of the choice of $\varphi : F^r \xrightarrow{\sim} F'^{r'}$ and the representative $(\overline{\omega'}, h') \in \Omega^{r'}_{F'} \times \mathrm{GL}_{r'}(\mathbb{A}^{r'}_{F'})$. This also shows that our definition of $\iota^{F'}_{F,b}$ is independent of the choice of g', b' and I.

It remains only to prove that $\iota^{F'}_{F,b}$ is injective on $\mathbb{C}_\infty$-valued points, i.e., that the map (3.2.1) is injective. For this, consider two elements $[(\overline{\omega_1'}, h_1')]$, $[(\overline{\omega_2'}, h_2')]$ of $S^{r'}_{F',\mathcal{K}'}(\mathbb{C}_\infty)$ with $\overline{\omega_1'}, \overline{\omega_2'} \in \Omega^{r'}_{F'}$ associated to $\omega_1', \omega_2' : F'^{r'}_{\infty'} \hookrightarrow \mathbb{C}_\infty$ and $h_1', h_2' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ which are mapped to the same element of $S(\mathbb{C}_\infty)$. This means that there exist $T \in \mathrm{GL}_r(F)$ and $k \in \mathcal{K}$ such that:

(i) $\overline{\omega_1' \circ \varphi \circ T^{-1}} = \overline{\omega_2' \circ \varphi}$;

(ii) $T(\varphi^{-1} \circ h_1' \circ b)k = \varphi^{-1} \circ h_2' \circ b$.

By (i), there is a $\rho \in \mathbb{C}_\infty^*$ such that the diagram

$$\begin{tikzcd}
F^r_\infty \arrow[r, "\varphi", "\sim"'] \arrow[d, "T"] & F'^{r'}_{\infty'} \arrow[r, hook, "\omega_1'"] & \mathbb{C}_\infty \arrow[d, "\rho"] \\
F^r_\infty \arrow[r, "\varphi", "\sim"'] & F'^{r'}_{\infty'} \arrow[r, hook, "\omega_2'"] & \mathbb{C}_\infty
\end{tikzcd}$$

commutes. Since the maps $\omega_1', \omega_2', \rho$ are injective and F'-linear, this implies that the F-linear automorphism $T' := \varphi \circ T \circ \varphi^{-1}$ of $F'^{r'}$ is also F'-linear and lies in $\mathrm{GL}_{r'}(F')$. Thus, we have $T' \cdot \overline{\omega_1'} = \overline{\omega_2'}$, i.e., $\overline{\omega_1'}$ and $\overline{\omega_2'}$ lie in the same $\mathrm{GL}_{r'}(F')$-orbit.

Equation (ii) implies that $T'h_1'(b \circ k \circ b^{-1}) = h_2'$ in $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. Since h_1', h_2' and T' all lie in $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$, we conclude that

$$b \circ k \circ b^{-1} \in \mathcal{K}' = (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'}),$$

i.e., $[(\overline{\omega_1'}, h_1')] = [(\overline{\omega_2'}, h_2')]$ in $S^{r'}_{F',\mathcal{K}'}(\mathbb{C}_\infty)$. □

Since the morphism $\iota^{F'}_{F,b} : S' \to S$ is injective on $\mathbb{C}_\infty$-valued points, we call it an *inclusion* of Drinfeld modular varieties (by a slight abuse of terminology). If $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}^f_F)$ is amply small (in the sense of Definition 3.1.2), we can show that it is in fact a closed immersion.

Proposition 3.2.3. *Let $\iota^{F'}_{F,b} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ be an inclusion of Drinfeld modular varieties with $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}^f_F)$ amply small. Then $\iota^{F'}_{F,b}$ is a closed immersion of varieties.*

Before giving the proof of Proposition 3.2.3, we summarize the description of the tangent spaces at the closed points of a Drinfeld modular variety $S^r_{F,\mathcal{K}}$ with $\mathcal{K} = \mathcal{K}(I)$ for a proper ideal I of A given in [Gek90].

We use for $a \in A$ the notation

$$\deg a := \log_q(|A/(a)|)$$

and denote by $\mathbb{C}_\infty\{\{\tau\}\}$ the ring of formal non-commutative power series in the variable τ with coefficients in $\mathbb{C}_\infty$ and the commutator rule $\tau\lambda = \lambda^q\tau$ for $\lambda \in \mathbb{C}_\infty$.

Definition 3.2.4. Let $\varphi : A \to \mathbb{C}_\infty\{\tau\}$ be a Drinfeld module over $\mathbb{C}_\infty$ of rank r. An $\mathbb{F}_q$-linear map $\eta : A \to \tau\mathbb{C}_\infty\{\tau\}$ is called a *derivation* with respect to φ if, for all $a, b \in A$, the derivation rule

$$\eta_{ab} = a\eta_b + \eta_a \circ \varphi_b$$

is satisfied. Such a derivation is called *reduced*, respectively *strictly reduced*, if it satisfies $\deg_\tau \eta_a \leqslant r \cdot \deg a$, respectively $\deg_\tau \eta_a < r \cdot \deg a$, for all $a \in A$. The space of reduced, respectively strictly reduced, derivations $A \to \tau\mathbb{C}_\infty\{\tau\}$ with respect to φ is denoted by $D_r(\varphi)$, respectively $D_{sr}(\varphi)$.

Theorem 3.2.5. *Let x be a $\mathbb{C}_\infty$-valued point of $S^r_{F,\mathcal{K}(I)}$ corresponding to a Drinfeld A-module φ with I-level structure α. Then there is a natural isomorphism*

$$T_x(S^r_{F,\mathcal{K}(I)}) \xrightarrow{\sim} D_{sr}(\varphi) \tag{3.2.3}$$

of vector spaces over $\mathbb{C}_\infty$.

Proof. This follows from the discussion in the in proof of Theorem 6.11 in [Gek90] and the lemmas before this proof. □

The isomorphism (3.2.3) is given as follows: a tangent vector $\xi \in T_x(S^r_{F,\mathcal{K}(I)})$ is an element of $S^r_{F,\mathcal{K}(I)}(\mathbb{C}_\infty[\varepsilon]/(\varepsilon^2))$ which projects to $x \in S^r_{F,\mathcal{K}(I)}(\mathbb{C}_\infty)$ under the canonical projection $\mathbb{C}_\infty[\varepsilon]/(\varepsilon^2) \to \mathbb{C}_\infty$. It corresponds to the isomorphism class of a Drinfeld A-module over $\mathbb{C}_\infty[\varepsilon]/(\varepsilon^2)$ with I-level structure which projects to (φ, α) under the canonical projection $\mathbb{C}_\infty[\varepsilon]/(\varepsilon^2) \to \mathbb{C}_\infty$. There is a unique Drinfeld A-module $\tilde{\varphi}$ in this isomorphism class such that, for all $a \in A$,

$$\tilde{\varphi}_a = \varphi_a + \varepsilon \cdot \eta_a$$

where $a \mapsto \eta_a$ is a strictly reduced derivation with respect to φ. The tangent vector ξ is mapped to this strictly reduced derivation under (3.2.3).

THEOREM 3.2.6. *Let φ be the Drinfeld A-module over $\mathbb{C}_\infty$ associated to an A-lattice $\Lambda \subset \mathbb{C}_\infty$. Then there is a natural isomorphism*

$$D_r(\varphi) \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, \mathbb{C}_\infty). \tag{3.2.4}$$

The $\mathbb{C}_\infty$-linear subspace $D_{sr}(\varphi) \subset D_r(\varphi)$ is mapped to a subspace of $\mathrm{Hom}_A(\Lambda, \mathbb{C}_\infty)$ which is a complement of $\mathbb{C}_\infty \cdot \mathrm{id}$, where $\mathrm{id} : \Lambda \hookrightarrow \mathbb{C}_\infty$ is the canonical inclusion.

Proof. See [Gek89, Theorems 5.14 and 6.10]. □

The isomorphism (3.2.4) is called the *de Rham isomorphism* and can be described as follows. Let η be a reduced derivation with respect to φ. Then, for all non-constant $a \in A$, there is a unique solution $F_\eta \in \mathbb{C}_\infty\{\{\tau\}\}$ satisfying the difference equation

$$F_\eta(az) - aF_\eta(z) = \eta_a(e_\Lambda(z)) \tag{3.2.5}$$

where

$$e_\Lambda(z) = z \cdot \prod_{0 \neq \lambda \in \Lambda} (1 - z/\lambda)$$

denotes the exponential function associated to the lattice Λ. This solution is independent of the choice of $a \in A$ and defines an entire function $\mathbb{C}_\infty \to \mathbb{C}_\infty$ which restricts to an A-linear map $\Lambda \to \mathbb{C}_\infty$. The reduced derivation η is mapped to $F_\eta|_\Lambda$ under (3.2.4).

Proof of Proposition 3.2.3. We use the following criterion given in [GW10, Proposition 12.94].

A proper morphism $f : X \to Y$ of varieties over an algebraically closed field K is a closed immersion if and only if the map $X(K) \to Y(K)$ induced by f is injective and, for all $x \in X(K)$, the induced map on Zariski tangent spaces $T_x(X) \to T_{f(x)}(Y)$ is injective.

Since finite morphisms are proper, by Theorem 3.2.2 we already know that $\iota^{F'}_{F,b}$ is proper and injective on $\mathbb{C}_\infty$-valued points. We therefore only have to show that, for all $x \in S^{r'}_{F',\mathcal{K}'}(\mathbb{C}_\infty)$, the induced map on Zariski tangent spaces $\iota^{F'}_{F,b_*} : T_x(S^{r'}_{F',\mathcal{K}'}) \to T_{\iota^{F'}_{F,b}(x)}(S^r_{F,\mathcal{K}})$ is injective.

Case (i). As in the proof of Theorem 3.2.2, we first consider the case where $b(\hat{A}^r) = \widehat{A'}^{r'}$ and $\mathcal{K} = \mathcal{K}(I)$ is a principal congruence subgroup modulo a proper ideal I of A. In this case, we have $\mathcal{K}' = \mathcal{K}(I')$ with $I' := IA'$. We can therefore use the description of the tangent spaces given above.

Let $x \in S^{r'}_{F',\mathcal{K}(I')}(\mathbb{C}_\infty)$ be a point corresponding to the Drinfeld A'-module φ associated to an A'-lattice $\Lambda \subset \mathbb{C}_\infty$ of rank r' with I'-level structure. Since we defined $\iota^{F'}_{F,b}$ by restricting Drinfeld A'-modules to Drinfeld A-modules, the point $\iota^{F'}_{F,b}(x) \in S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ corresponds to the Drinfeld A-module $\varphi|_A$ associated to the same $\Lambda \subset \mathbb{C}_\infty$ considered as A-lattice of rank r with some I-level structure. We can therefore consider the following diagram

$$\begin{tikzcd}
T_x(S^{r'}_{F',\mathcal{K}(I')}) \arrow[r, "(3.2.3)", "\sim"'] \arrow[d, "\iota^{F'}_{F,b_*}"] & D_{sr}(\varphi) \arrow[r, hook, "(3.2.4)"] \arrow[d] & \mathrm{Hom}_{A'}(\Lambda, \mathbb{C}_\infty) \arrow[d, hook] \\
T_{\iota^{F'}_{F,b}(x)}(S^r_{F,\mathcal{K}}) \arrow[r, "(3.2.3)", "\sim"'] & D_{sr}(\varphi|_A) \arrow[r, hook, "(3.2.4)"] & \mathrm{Hom}_A(\Lambda, \mathbb{C}_\infty)
\end{tikzcd}$$

where the vertical arrow in the middle denotes the restriction of derivations from A' to A and the one at the right the canonical inclusion. The left square of the diagram commutes by the definition of (3.2.3) because $\iota^{F'}_{F,b_*}$ has the modular interpretation of restricting Drinfeld A'-modules over $\mathbb{C}_\infty[\varepsilon]/(\varepsilon^2)$ to A. The right square also commutes because the unique solution of (3.2.5) is independent of $a \in A'$ and Λ as an A'-lattice has the same exponential function as Λ as an A-lattice.

Hence, the diagram commutes and, since the right vertical arrow is an injective map, also the other two are injective maps. In particular, the induced map $\iota^{F'}_{F,b_*}$ between tangent spaces is injective.

Case (ii). Let $b : (\mathbb{A}^f_F)^r \xrightarrow{\sim} (\mathbb{A}^f_{F'})^{r'}$ be arbitrary and $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}^f_F)$ be an arbitrary amply small subgroup. Then, by the construction in the proof of Theorem 3.2.2, there is:

- a $g' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ with $g'\mathcal{K}'g'^{-1} \subset \mathrm{GL}_{r'}(\widehat{A'})$;
- an $\mathbb{A}^f_F$-linear isomorphism $b' : (\mathbb{A}^f_F)^r \xrightarrow{\sim} (\mathbb{A}^f_{F'})^{r'}$ with $b'(\widehat{A}^r) = \widehat{A'}^{r'}$;
- a proper ideal I of A with $\mathcal{K}(I) \subset g^{-1}\mathcal{K}g$, where $g := b^{-1} \circ g'^{-1} \circ b' \in \mathrm{GL}_r(\mathbb{A}^f_F)$

such that the diagram

$$\begin{tikzcd}
S^{r'}_{F',\mathcal{K}(I')} \arrow[r, "\iota^{F'}_{F,b'}"] \arrow[d, "\pi_{g'^{-1}}"] & S^r_{F,\mathcal{K}(I)} \arrow[d, "\pi_g"] \\
S^{r'}_{F',\mathcal{K}'} \arrow[r, "\iota^{F'}_{F,b}"] & S^r_{F,\mathcal{K}}
\end{tikzcd}$$

with $I' := IA'$ commutes. By Proposition 3.1.3 and Corollary 3.1.4, the projection maps $\pi_{g'^{-1}}$ and π_g in this diagram are étale morphisms between non-singular varieties because $\mathcal{K}'$ and $\mathcal{K}$ are amply small. Hence, they induce isomorphisms on tangent spaces of closed points [Har77, Proposition 10.4]. By Case (i), the upper horizontal arrow $\iota^{F'}_{F,b'}$ induces injections on tangent spaces of closed points. Therefore, the commutativity of the diagram implies that, for all $x \in S^{r'}_{F',\mathcal{K}'}(\mathbb{C}_\infty)$, the induced map $\iota^{F'}_{F,b_*} : T_x(S^{r'}_{F',\mathcal{K}'}) \to T_{\iota^{F'}_{F,b}(x)}(S^r_{F,\mathcal{K}})$ is injective. □

Proposition 3.2.7. *Let $\iota^{F'}_{F,b_1} : S^{r'}_{F',\mathcal{K}'} \to S$ and $\iota^{F''}_{F,b_2} : S^{r''}_{F'',\mathcal{K}''} \to S$ be two inclusions of Drinfeld modular varieties with $F'' \subset F'$. Then for an $\mathbb{A}^f_{F''}$-linear isomorphism $c : (\mathbb{A}^f_{F''})^{r''} \to (\mathbb{A}^f_{F'})^{r'}$ with*

$$b_1 = c \circ b_2 \circ k \qquad (3.2.6)$$

for some $k \in \mathcal{K}$, the diagram

$$\begin{tikzcd}
S^{r'}_{F',\mathcal{K}'} \arrow[dr, "\iota^{F'}_{F,b_1}"] \arrow[dd, "\iota^{F'}_{F'',c}"'] & \\
& S \\
S^{r''}_{F'',\mathcal{K}''} \arrow[ur, "\iota^{F''}_{F,b_2}"'] &
\end{tikzcd}$$

commutes.

Proof. Note that we have

$$\mathcal{K}' = (b_1 \mathcal{K} b_1^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'}) = (c\mathcal{K}'' c^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$$

by the definition of $\mathcal{K}'$ and $\mathcal{K}''$, and (3.2.6). Therefore, there is an inclusion $\iota^{F'}_{F'',c} : S^{r'}_{F',\mathcal{K}'} \to S^{r''}_{F'',\mathcal{K}''}$.

The commutativity of the diagram follows by a direct calculation on $\mathbb{C}_\infty$-valued points using (3.2.1). $\square$

3.3 Drinfeld modular subvarieties

The image of an inclusion $\iota^{F'}_{F,b} : S' \to S$ of Drinfeld modular varieties is a subvariety of S because finite morphisms are proper.

Definition 3.3.1. A subvariety of S of the form $X = \iota^{F'}_{F,b}(S')$ for an inclusion $\iota^{F'}_{F,b}$ is called a *Drinfeld modular subvariety* of S. An irreducible component of a Drinfeld modular subvariety over $\mathbb{C}_\infty$ is called a *special subvariety* and a special subvariety of dimension 0 a *special point*.

Lemma 3.3.2. *Let $\tilde{\mathcal{K}} \subset \mathcal{K}$ be an open subgroup and $\pi_1 : S^r_{F,\tilde{\mathcal{K}}} \to S^r_{F,\mathcal{K}}$ the corresponding canonical projection. Then the following hold.*

(i) *For each Drinfeld modular subvariety $X' \subset S^r_{F,\tilde{\mathcal{K}}}$, the image $\pi_1(X')$ is a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$.*

(ii) *For each Drinfeld modular subvariety $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'}) \subset S^r_{F,\mathcal{K}}$, the preimage $\pi_1^{-1}(X)$ is a finite union of Drinfeld modular subvarieties of $S^r_{F,\tilde{\mathcal{K}}}$.*

Proof. For part (i), assume that X' is the image of the inclusion $\tilde{\iota}^{F'}_{F,b} : S^{r'}_{F',\tilde{\mathcal{K}}'} \to S^r_{F,\tilde{\mathcal{K}}}$ associated to the datum (F', b) and consider the inclusion morphism $\iota^{F'}_{F,b} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ associated to the same datum. The diagram

$$\begin{tikzcd}
S^{r'}_{F',\tilde{\mathcal{K}}'} \arrow[r, "\tilde{\iota}^{F'}_{F,b}"] \arrow[d, "\pi'_1"] & S^r_{F,\tilde{\mathcal{K}}} \arrow[d, "\pi_1"] \\
S^{r'}_{F',\mathcal{K}'} \arrow[r, "\iota^{F'}_{F,b}"] & S^r_{F,\mathcal{K}}
\end{tikzcd}$$

with π'_1 and π_1 the respective canonical projections commutes by definition of the inclusion morphisms. Hence,

$$\pi_1(X') = \iota^{F'}_{F,b}(\pi'_1(S^{r'}_{F',\tilde{\mathcal{K}}'})) = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$$

is a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$.

For part (ii), choose a set of representatives $k_1, \ldots, k_l \in \mathcal{K}$ for the left cosets $\mathcal{K}/\tilde{\mathcal{K}}$ and consider the inclusion morphisms $\iota^{F'}_{F,b\circ k_i} : S^{r'}_{F',\tilde{\mathcal{K}}'_i} \to S^r_{F,\tilde{\mathcal{K}}}$ associated to $(F', b \circ k_i)$ for $i = 1, \ldots, l$. By the definition of the inclusion morphisms we have

$$\pi_1^{-1}(X) = \bigcup_{i=1}^{l} \iota^{F'}_{F,b\circ k_i}(S^{r'}_{F',\tilde{\mathcal{K}}'_i}),$$

and hence $\pi_1^{-1}(X)$ is a finite union of Drinfeld modular subvarieties of $S^r_{F,\tilde{\mathcal{K}}}$. □

Lemma 3.3.3. *For an inclusion $\iota^{F'}_{F,b} : S' \to S$, we have*

$$\mathrm{End}(p') = \mathrm{End}(\iota^{F'}_{F,b}(p'))$$

for all $p' \in S'(\mathbb{C}_\infty)$.

Remark. This is an equality of subfields of $\mathbb{C}_\infty$ and not just an abstract isomorphism of fields.

Proof. This follows from our definitions because, for $p' = [(\overline{\omega'}, h')] \in S'(\mathbb{C}_\infty)$, we have $\mathrm{End}(p') = \{u \in \mathbb{C}_\infty : u \cdot \omega'(F'^{r'}) \subset \omega'(F'^{r'})\}$ and $\mathrm{End}(\iota^{F'}_{F,b}(p')) = \{u \in \mathbb{C}_\infty : u \cdot (\omega' \circ \varphi)(F^r) \subset (\omega' \circ \varphi)(F^r)\}$ for a chosen F-isomorphism $\varphi : F^r \xrightarrow{\sim} F'^{r'}$. □

Now we give a criterion under which two Drinfeld modular subvarieties are contained in each other.

Proposition 3.3.4. *Let $X' = \iota^{F'}_{F,b_1}(S^{r'}_{F',\mathcal{K}'})$ and $X'' = \iota^{F''}_{F,b_2}(S^{r''}_{F'',\mathcal{K}''})$ be two Drinfeld modular subvarieties of S. The following statements are equivalent.*

(i) *The inclusion $X' \subset X''$ holds.*

(ii) *There is an irreducible component of X' over $\mathbb{C}_\infty$ which is contained in X''.*

(iii) *The inclusion $F'' \subset F'$ holds and there exist $k \in \mathcal{K}$ and an $\mathbb{A}^f_{F''}$-linear isomorphism $c : (\mathbb{A}^f_{F''})^{r''} \to (\mathbb{A}^f_{F'})^{r'}$ such that $b_1 = c \circ b_2 \circ k$.*

Proof. We write $S' = S^{r'}_{F',\mathcal{K}'}$ and $S'' = S^{r''}_{F'',\mathcal{K}''}$.

The implication (i) ⇒ (ii) is trivial and (iii) ⇒ (i) follows from Proposition 3.2.7.

For (ii) ⇒ (iii) assume that $\iota^{F'}_{F,b_1}(Y') \subset \iota^{F''}_{F,b_2}(S'')$ for an irreducible component Y' of S' over $\mathbb{C}_\infty$. By Lemma 2.1.6 there is a $p' = [(\overline{\omega'}, h')] \in Y'(\mathbb{C}_\infty)$ with $\mathrm{End}(p') = F'$. Now let $\iota^{F'}_{F,b_1}(p') = \iota^{F''}_{F,b_2}(p'')$ for a suitable $p'' = [(\overline{\omega''}, h'')] \in S''(\mathbb{C}_\infty)$. Lemmas 2.1.5 and 3.3.3 yield

$$F' = \mathrm{End}(p') = \mathrm{End}(\iota^{F'}_{F,b_1}(p')) = \mathrm{End}(\iota^{F''}_{F,b_2}(p'')) = \mathrm{End}(p'') \supset F''.$$

Because $\iota^{F'}_{F,b_1}(p') = \iota^{F''}_{F,b_2}(p'')$, we have

$$[(\overline{\omega' \circ \varphi_1}, \varphi_1^{-1} \circ h' \circ b_1)] = [(\overline{\omega'' \circ \varphi_2}, \varphi_2^{-1} \circ h'' \circ b_2)]$$

for F-linear isomorphisms $\varphi_1 : F^r \xrightarrow{\sim} F'^{r'}$ and $\varphi_2 : F^r \xrightarrow{\sim} F''^{r''}$. Hence, there are $T \in \mathrm{GL}_r(F)$ and $k \in \mathcal{K}$ such that:

(1) $\overline{\omega' \circ \varphi_1} = \overline{\omega'' \circ \varphi_2 \circ T^{-1}}$;

(2) $\varphi_1^{-1} \circ h' \circ b_1 = T(\varphi_2^{-1} \circ h'' \circ b_2)k$.

Because of (1) and $F'' \subset F'$, one concludes as in the proof of Theorem 3.2.2 that the F-linear isomorphism $\psi := \varphi_1 \circ T \circ \varphi_2^{-1} : F''^{r''} \to F'^{r'}$ is F''-linear.

We set $c := b_1 \circ k^{-1} \circ b_2^{-1} : (\mathbb{A}^f_{F''})^{r''} \to (\mathbb{A}^f_{F'})^{r'}$. By (2) this is equal to

$$c = h'^{-1} \circ \varphi_1 \circ T \circ \varphi_2^{-1} \circ h'' = h'^{-1} \circ \psi \circ h''.$$

Since ψ is F''-linear and $F'' \subset F'$ we conclude that c is an $\mathbb{A}^f_{F''}$-linear isomorphism. Furthermore, we have $b_1 = c \circ b_2 \circ k$ by the definition of c, which shows part (iii) of the proposition. □

Corollary 3.3.5. *Let $X' = \iota^{F''}_{F,b'}(S^{r''}_{F'',\mathcal{K}''})$ be a fixed Drinfeld modular subvariety of S. Then the assignment*

$$X \longmapsto \iota^{F''}_{F,b'}(X)$$

is a bijection from the set of Drinfeld modular subvarieties of $S^{r''}_{F'',\mathcal{K}''}$ to the set of Drinfeld modular subvarieties of S contained in X'.

Proof. Since $\iota^{F''}_{F,b'}$ is injective on $\mathbb{C}_\infty$-valued points, it is enough to show the following.

(i) The variety $\iota^{F''}_{F,b'}(X)$ is a Drinfeld modular subvariety of S for each Drinfeld modular subvariety X of $S^{r''}_{F'',\mathcal{K}''}$.

(ii) The variety $(\iota^{F''}_{F,b'})^{-1}(X)$ is a Drinfeld modular subvariety of $S^{r''}_{F'',\mathcal{K}''}$ for every Drinfeld modular subvariety $X \subset X'$ of S.

For (i), let $X = \iota^{F'}_{F'',c}(S^{r'}_{F',\mathcal{K}'})$ be a Drinfeld modular subvariety of $S^{r''}_{F'',\mathcal{K}''}$. The map

$$b := c \circ b' : (\mathbb{A}^f_F)^r \to (\mathbb{A}^f_{F'})^{r'}$$

is an $\mathbb{A}^f_F$-linear isomorphism, hence we can apply Proposition 3.2.7 to conclude that

$$\iota^{F''}_{F,b'}(X) = \iota^{F''}_{F,b'}(\iota^{F'}_{F'',c}(S^{r'}_{F',\mathcal{K}'})) = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$$

is a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$.

For (ii), let $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ be a Drinfeld modular subvariety of S which is contained in X'. By Proposition 3.3.4, we have $F \subset F'' \subset F'$ and there are an $\mathbb{A}^f_{F''}$-linear isomorphism $c : (\mathbb{A}^f_{F''})^{r''} \xrightarrow{\sim} (\mathbb{A}^f_{F'})^{r'}$ and a $k \in \mathcal{K}$ such that

$$b = c \circ b' \circ k.$$

By Proposition 3.2.7, we have

$$X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'}) = \iota^{F''}_{F,b'}(\iota^{F'}_{F'',c}(S^{r'}_{F',\mathcal{K}'})).$$

Since $\iota^{F''}_{F,b'}$ is injective on $\mathbb{C}_\infty$-valued points, this implies that $(\iota^{F''}_{F,b'})^{-1}(X) = \iota^{F'}_{F'',c}(S^{r'}_{F',\mathcal{K}'})$ is a Drinfeld modular subvariety of $S^{r''}_{F'',\mathcal{K}''}$. □

From Proposition 3.3.4, the following criterion for equality of Drinfeld modular subvarieties follows.

Corollary 3.3.6. *Let $X' = \iota^{F'}_{F,b_1}(S^{r'}_{F',\mathcal{K}'})$ and $X'' = \iota^{F''}_{F,b_2}(S^{r''}_{F'',\mathcal{K}''})$ be two Drinfeld modular subvarieties of S. The following statements are equivalent.*

(i) *The equality $X' = X''$ holds.*

(ii) *The varieties X' and X'' have a common irreducible component over $\mathbb{C}_\infty$.*

(iii) *The equality $F' = F''$ holds (hence $r' = r''$) and there exist $s \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ and $k \in \mathcal{K}$ such that $b_1 = s \circ b_2 \circ k$.*

In particular, each special subvariety of S is an irreducible component over $\mathbb{C}_\infty$ of a unique Drinfeld modular subvariety of S.

COROLLARY 3.3.7. *For a Drinfeld modular subvariety* $X' \subset S$ *there is a unique extension* $F' \subset \mathbb{C}_\infty$ *of* F *and a unique conjugacy class* C *of compact open subgroups of* $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ *with* $r' = r/[F'/F]$ *such that* $F'' = F'$ *and* $\mathcal{K}'' \in C$ *for all inclusions* $\iota^{F''}_{F,c} : S^{r''}_{F'',\mathcal{K}''} \to S$ *with image* X'.

Proof. By definition, X' is the image of some inclusion $\iota^{F'}_{F,b} : S^{r'}_{F',\mathcal{K}'} \to S$. For any other inclusion $\iota^{F''}_{F,c} : S^{r''}_{F'',\mathcal{K}''} \to S$ with image X', Corollary 3.3.6 implies that $F'' = F'$ and $b = s \circ c \circ k$ for suitable $s \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ and $k \in \mathcal{K}$. The latter implies $\mathcal{K}' = s\mathcal{K}''s^{-1}$, i.e., $\mathcal{K}''$ lies in the conjugacy class of $\mathcal{K}'$ in $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. □

The preceding corollary allows us to make the following definition.

DEFINITION 3.3.8. For a Drinfeld modular subvariety $X' = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ of S, the extension $F' \subset \mathbb{C}_\infty$ of F is called the *reflex field* of X', and the index of $\mathcal{K}'$ in a maximal compact subgroup of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ is called the *index* of X' and is denoted by $i(X')$. Furthermore, the product

$$D(X') := |\mathrm{Cl}(F')| \cdot i(X'),$$

where $\mathrm{Cl}(F')$ denotes the class group of $A' \subset F'$, is called the *predegree* of X'.

By Corollary 3.3.6, each special subvariety of S is an irreducible component of a unique Drinfeld modular subvariety of S. This allows us to define the reflex field of a special subvariety.

DEFINITION 3.3.9. For a special subvariety V of S which is an irreducible component of a Drinfeld modular subvariety X' of S, the reflex field of V is defined to be the reflex field of X'.

If $\mathcal{K} = \mathrm{GL}_r(\hat{A})$, Corollary 3.3.6 immediately implies the following characterization of the set of Drinfeld modular subvarieties of S with a given reflex field F'.

COROLLARY 3.3.10. *Assume that* $S = S^r_{F,\mathcal{K}}$ *with* $\mathcal{K} = \mathrm{GL}_r(\hat{A})$ *and let* $F' \subset \mathbb{C}_\infty$ *be an extension of* F *of degree* r/r' *for some integer* $r' \geqslant 1$ *with only one place* ∞' *lying over* ∞. *Then the set of Drinfeld modular subvarieties of* S *with reflex field* F' *is in bijective correspondence with the set of orbits of the action of* $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ *on the set of free* $\hat{A}$*-submodules of rank* r *of* $(\mathbb{A}^f_{F'})^{r'}$ *via the assignment*

$$\iota^{F'}_{F,b}(S') \longmapsto \mathrm{GL}_{r'}(\mathbb{A}^f_{F'}) \cdot b(\hat{A}^r).$$

PROPOSITION 3.3.11. *The natural action of the absolute Galois group* $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ *on the set of subvarieties of* $S = S^r_{F,\mathcal{K}}$ *which are defined over* $\overline{F}$ *restricts to an action on the set of Drinfeld modular subvarieties of* S. *For* $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$ *and a Drinfeld modular subvariety* $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$, *the Galois conjugate* $\sigma(X)$ *is given by* $\iota^{\sigma(F')}_{F,\sigma\circ b}(S^{r'}_{\sigma(F'),\sigma\circ\mathcal{K}'\circ\sigma^{-1}})$.

Remark. In the above formula for the Galois conjugate $\sigma(X)$, the $\mathbb{A}^f_F$-linear isomorphism $(\mathbb{A}^f_{F'})^{r'} \xrightarrow{\sim} (\mathbb{A}^f_{\sigma(F')})^{r'}$ obtained by tensoring $\sigma : F' \xrightarrow{\sim} \sigma(F')$ with $(\mathbb{A}^f_F)^{r'}$ over F is also denoted by σ.

Proof. As explained in § 1.1, we identify $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ with $\mathrm{Aut}_F(\overline{F})$ via the unique extension of the elements of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ to $\overline{F}$.

Case (i). We first consider the case where $S = S^r_{F,\mathcal{K}(I)}$ for a proper ideal I of A and

$$X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$$

for an inclusion morphism $\iota^{F'}_{F,b}$ associated to a datum (F',b) satisfying $b(\widehat{A}^r)=\widehat{A'}^{r'}$ with A' the integral closure of A in F'. As explained in the proof of Theorem 3.2.2, in this case we have $\mathcal{K}'=\mathcal{K}(I')$ with $I'=IA'$ and $\iota^{F'}_{F,b}$ is defined by the morphism (3.2.2) of functors from $\mathcal{F}^{r'}_{F',I'}$ to $\mathcal{F}^{r}_{F,I}$ (restricted to the subcategory of F'-schemes) using the modular interpretation of $S^{r'}_{F',\mathcal{K}(I')}$ and $S^{r}_{F,\mathcal{K}(I)}$.

Note that, for any Drinfeld A'-module $\varphi : A' \to \overline{F}\{\tau\}$ over $\overline{F}$,

$$\varphi^\sigma : \begin{array}{ccc} \sigma(A') & \longrightarrow & \overline{F}\{\tau\} \\ \sigma(a') & \longmapsto & (\varphi_{a'})^\sigma \end{array}$$

where $(\varphi_{a'})^\sigma$ is obtained from $\varphi_{a'}$ by applying σ to its coefficients, is a Drinfeld $\sigma(A')$-module over $\overline{F}$. Furthermore, for any I'-level structure $\alpha : (I'^{-1}/A')^{r'} \xrightarrow{\sim} \varphi_{I'} \subset \overline{F}$ on φ, the composition

$$(\sigma(I')^{-1}/\sigma(A'))^{r'} \xrightarrow{\sigma^{-1}} (I'^{-1}/A')^{r'} \xrightarrow{\alpha} \varphi_{I'} \xrightarrow{\sigma} (\varphi^\sigma)_{\sigma(I')}$$

is a $\sigma(I')$-level structure on φ^σ. Using the modular interpretation of $S^{r'}_{F',\mathcal{K}(I')}$ and $S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))}$, the assignment

$$(\varphi,\alpha) \longmapsto (\varphi^\sigma, \sigma\circ\alpha\circ\sigma^{-1})$$

defines a map $g_\sigma : S^{r'}_{F',\mathcal{K}(I')}(\overline{F}) \to S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))}(\overline{F})$. By construction, the map g_σ is bijective with inverse $g_{\sigma^{-1}}$.

Note that we have $(\sigma\circ b)(\hat{A}^r)=\widehat{\sigma(A')}^{r'}$. Hence the datum $(\sigma(F'),\sigma\circ b)$ defines an inclusion map

$$\iota^{\sigma(F')}_{F,\sigma\circ b} : S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))} \longrightarrow S^{r}_{F,\mathcal{K}(I)},$$

which is defined by a morphism of functors from $\mathcal{F}^{r'}_{\sigma(F'),\sigma(I')}$ to $\mathcal{F}^{r}_{F,I}$ (restricted to the subcategory of $\sigma(F')$-schemes). A straightforward verification shows that the diagram

$$\begin{tikzcd}
S^{r'}_{F',\mathcal{K}(I')}(\overline{F}) \arrow[r, "\iota^{F'}_{F,b}"] \arrow[d, "g_\sigma"] & S^{r}_{F,\mathcal{K}(I)}(\overline{F}) \arrow[d, "\sigma"] \\
S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))}(\overline{F}) \arrow[r, "\iota^{\sigma(F')}_{F,\sigma\circ b}"] & S^{r}_{F,\mathcal{K}(I)}(\overline{F})
\end{tikzcd}$$

commutes, where the right vertical map is given by the natural action of σ on the closed points of $S^{r}_{F,\mathcal{K}(I)}$ defined over $\overline{F}$.

Since, for any subvariety $Y \subset S$ defined over $\overline{F}$, the set $Y(\overline{F})$ of $\overline{F}$-valued points (viewed as a subset of the closed points of $Y \subset S$) is Zariski dense in Y (see, e.g., [Bor91, Corollary AG. 13.3]), the commutativity of the above diagram implies that

$$\sigma(X)=\iota^{\sigma(F')}_{F,\sigma\circ b}(g_\sigma(S^{r'}_{F',\mathcal{K}(I')}))=\iota^{\sigma(F')}_{F,\sigma\circ b}(S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))})$$

for $X=\iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}(I')})$. Hence, $\sigma(X)$ is a Drinfeld modular subvariety of S and it is of the desired form because $\sigma\circ\mathcal{K}'\circ\sigma^{-1}=\mathcal{K}(\sigma(I'))$.

Case (ii). For a general $X=\iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'}) \subset S^{r}_{F,\mathcal{K}}$, by the construction in the proof of Theorem 3.2.2, there is:

- a $g' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ with $g'\mathcal{K}'g'^{-1} \subset \mathrm{GL}_{r'}(\widehat{A'})$;
- an $\mathbb{A}^f_F$-linear isomorphism $b' : (\mathbb{A}^f_F)^r \xrightarrow{\sim} (\mathbb{A}^f_{F'})^{r'}$ with $b'(\widehat{A}^r)=\widehat{A'}^{r'}$;

- a proper ideal I of A with $\mathcal{K}(I) \subset g^{-1}\mathcal{K}g$, where $g := b^{-1} \circ g'^{-1} \circ b' \in \mathrm{GL}_r(\mathbb{A}^f_F)$

such that the diagram

$$\begin{tikzcd}
S^{r'}_{F',\mathcal{K}(I')} \arrow[r, "\iota^{F'}_{F,b'}"] \arrow[d, "\pi_{g'^{-1}}"] & S^{r}_{F,\mathcal{K}(I)} \arrow[d, "\pi_g"] \\
S^{r'}_{F',\mathcal{K}'} \arrow[r, "\iota^{F'}_{F,b}"] & S^{r}_{F,\mathcal{K}}
\end{tikzcd}$$

with $I' := IA'$ commutes where π_g and $\pi_{g'^{-1}}$ are surjective and defined over F. This implies, together with Case (i),

$$\begin{aligned}
\sigma(X) &= \sigma(\iota^{F'}_{F,b}(\pi_{g'^{-1}}(S^{r'}_{F',\mathcal{K}(I')}))) = \sigma(\pi_g(\iota^{F'}_{F,b'}(S^{r'}_{F',\mathcal{K}(I')}))) \\
&= \pi_g(\sigma(\iota^{F'}_{F,b'}(S^{r'}_{F',\mathcal{K}(I')}))) = \pi_g(\iota^{\sigma(F')}_{F,\sigma\circ b'}(S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))})).
\end{aligned}$$

By a similar commutative diagram, this is equal to

$$\iota^{\sigma(F')}_{F,\sigma\circ b}(S^{r'}_{\sigma(F'),\sigma\circ\mathcal{K}'\circ\sigma^{-1}}),$$

hence a Drinfeld modular subvariety of S of the desired form. □

3.4 Determinant map and irreducible components

For a general Drinfeld modular variety $S^r_{F,\mathcal{K}}$, we denote by $\det \mathcal{K} \subset (\mathbb{A}^f_F)^*$ the image of $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}^f_F)$ under the determinant map. Since the determinant map is a group homomorphism and maps principal congruence subgroups of $\mathrm{GL}_r(\mathbb{A}^f_F)$ to principal congruence subgroups of $(\mathbb{A}^f_F)^*$, the subgroup $\det \mathcal{K} \subset (\mathbb{A}^f_F)^*$ is open and compact.

Definition 3.4.1. The map $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty) \to S^1_{F,\det\mathcal{K}}(\mathbb{C}_\infty)$ given by

$$\begin{aligned}
\mathrm{GL}_r(F)\backslash(\Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}) &\longrightarrow F^*\backslash(\mathbb{A}^f_F)^*/\det\mathcal{K} \\
[(\overline{\omega}, h)] &\longmapsto [\det h]
\end{aligned}$$

is called *determinant map* and is denoted by det.

Remark. The determinant map can be described in terms of the modular interpretation, using the construction of exterior powers of Drinfeld modules in [Hei04, Theorem 3.3]. We refrain from doing so because we do not need that.

Proposition 3.4.2. *The determinant map is surjective and its fibers are exactly the irreducible components of* $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$.

Proof. The surjectivity is immediate because $\det : \mathrm{GL}_r(\mathbb{A}^f_F) \to (\mathbb{A}^f_F)^*$ is surjective.

We know by Proposition 2.1.3 that the irreducible components of $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ are in bijective correspondence with the double coset space $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}$. A point $[(\overline{\omega}, h)] \in S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ lies in the irreducible component corresponding to a double coset $[g] \in \mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}$ if and only if $h \in [g]$.

We show that, for every $g \in \mathrm{GL}_r(\mathbb{A}^f_F)$, the fiber of $[\det g] \in S^1_{F,\det\mathcal{K}}(\mathbb{C}_\infty)$ is equal to the irreducible component corresponding to $[g] \in \mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}$. By the above remarks, this is equivalent to

$$h \in \mathrm{GL}_r(F)\cdot g\cdot\mathcal{K} \Longleftrightarrow \det h \in F^*\cdot(\det g)\cdot(\det\mathcal{K})$$

for all $h \in \mathrm{GL}_r(\mathbb{A}^f_F)$.

If $h \in \mathrm{GL}_r(F) \cdot g \cdot \mathcal{K}$, then we have $\det h \in F^* \cdot (\det g) \cdot (\det \mathcal{K})$ by the multiplicativity of the determinant. Conversely, assume that $\det h \in F^* \cdot (\det g) \cdot (\det \mathcal{K})$. Then there are $T \in \mathrm{GL}_r(F)$ and $k \in \mathcal{K}$ such that

$$\det h = \det(T \cdot g \cdot k),$$

and hence $Tgkh^{-1} \in \mathrm{SL}_r(\mathbb{A}_F^f)$. By the strong approximation theorem [Pra77] for semi-simple simply connected groups over function fields, $\mathrm{SL}_r(F)$ is dense in $\mathrm{SL}_r(\mathbb{A}_F^f)$. Since $h\mathcal{K}h^{-1}$ is an open subgroup of $\mathrm{GL}_r(\mathbb{A}_F^f)$, we therefore have

$$\mathrm{SL}_r(\mathbb{A}_F^f) = \mathrm{SL}_r(F) \cdot ((h\mathcal{K}h^{-1}) \cap \mathrm{SL}_r(\mathbb{A}_F^f)).$$

So there are $T' \in \mathrm{SL}_r(F)$ and $k' \in \mathcal{K} \cap \mathrm{SL}_r(\mathbb{A}_F^f)$ such that $Tgkh^{-1} = T'hk'h^{-1}$. This implies

$$h = T'^{-1}Tgkk'^{-1} \in \mathrm{GL}_r(F) \cdot g \cdot \mathcal{K}. \qquad \square$$

By Proposition 3.4.2, the determinant map induces a bijection

$$\det_* : \pi_0(S^r_{F,\mathcal{K}}) \xrightarrow{\sim} S^1_{F,\det \mathcal{K}}$$

between the set $\pi_0(S^r_{F,\mathcal{K}})$ of irreducible components of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$ and the set $S^1_{F,\det \mathcal{K}}$ (we identify the latter set with $S^1_{F,\det \mathcal{K}}(\mathbb{C}_\infty)$ as explained in § 2.2). We now consider the natural action of the absolute Galois group $G_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$ on these two sets.

PROPOSITION 3.4.3. *The bijection* $\det_*$ *is* G_F*-equivariant.*

Proof. We consider separable extensions $F' \subset \mathbb{C}_\infty$ of F of degree r with only one place above ∞. The intersection F'' of all these extensions is equal to F. This follows by induction over r.

Assume by contradiction that $F'' \supsetneq F$ with $[F''/F] = r' > 1$. By Eisenstein's criterion ([Sti93, Proposition III.1.14]) we find a second extension $F''_2 \neq F''$ of F of degree r' with only one place ∞''_2 above ∞. By the induction hypothesis, the intersection of all separable extensions of F''_2 of degree r/r' with only one place above ∞''_2 is equal to F''_2. These extensions of F''_2 are all separable extensions of F of degree r with only one place above ∞, and hence its intersection F''_2 contains F''. This is not possible, because $F''_2 \neq F''$ and $[F''_2/F] = [F''/F] = r'$.

The equality $F'' = F$ implies that the subgroups $\mathrm{Gal}(F^{\mathrm{sep}}/F') \subset G_F$ where F' runs over all separable extensions of F of degree r with only one place above ∞ generate the whole absolute Galois group G_F. Therefore it is enough to show that $\det_*$ is $\mathrm{Gal}(F^{\mathrm{sep}}/F')$-equivariant for all these extensions F'.

From now on, let F'/F be a fixed extension of the above form, Y an irreducible component of $S^r_{F,\mathcal{K}}$ and $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F')$. We have to show that $\det_*(\sigma(Y)) = \sigma(\det_*(Y))$. We assume that Y corresponds to the class of $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ in $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$ via the bijective correspondence from Proposition 2.1.3. We choose an F-linear isomorphism

$$\varphi : F^r \xrightarrow{\sim} F',$$

and define

$$b := \varphi \circ g : (\mathbb{A}_F^f)^r \xrightarrow{\sim} \mathbb{A}_{F'}^f.$$

The datum (F', b) defines an inclusion morphism $\iota^{F'}_{F,b} : S^1_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$. By its definition, the point $p' := [1] \in S^1_{F',\mathcal{K}'} = F'^*\backslash(\mathbb{A}_{F'}^f)^*/\mathcal{K}'$ is mapped to the closed point

$$p := \iota^{F'}_{F,b}([1]) = [(\overline{i \circ \varphi}, \varphi^{-1} \circ 1 \circ b)] = [(\overline{i \circ \varphi}, g)]$$

of $S^r_{F,\mathcal{K}}$, where i denotes the canonical inclusion $F'_{\infty'} \hookrightarrow \mathbb{C}_\infty$. This point lies in the irreducible component Y, which corresponds to the class of g in $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}$.

By Proposition 1.2.2, the point $p' \in S^1_{F',\mathcal{K}'}$ is defined over $F'^{\mathrm{sep}} = F^{\mathrm{sep}}$. Since $\iota^{F'}_{F,b}$ is defined over F', the closed point $p = \iota^{F'}_{F,b}(p') \in S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ is also defined over F^{sep} and we have

$$\iota^{F'}_{F,b}(\sigma(p')) = \sigma(p) \in \sigma(Y),$$

i.e., $\sigma(Y)$ is the unique irreducible component of $S^r_{F,\mathcal{K}}$ containing $\iota^{F'}_{F,b}(\sigma(p'))$. The equality $\det_*(\sigma(Y)) = \sigma(\det_*(Y))$ is therefore equivalent to

$$\det(\iota^{F'}_{F,b}(\sigma(p'))) = \sigma(\det p). \tag{3.4.1}$$

We use the description of the Galois action on $S^1_{F',\mathcal{K}'}$ and $S^1_{F,\det\mathcal{K}}$ given by Theorem 2.2.1 to calculate both sides of (3.4.1). For this, let H/F (respectively H'/F') be the finite abelian extensions corresponding to the closed finite index subgroups $F^* \cdot \det\mathcal{K} \subset (\mathbb{A}^f_F)^*$ (respectively $F'^* \cdot \mathcal{K}' \subset (\mathbb{A}^f_{F'})^*$) in class field theory, and let E be the compositum of H and H'. Then the diagram of Artin maps

$$\begin{tikzcd}
(\mathbb{A}^f_F)^* \arrow[r, "\psi_{H/F}"] & \mathrm{Gal}(H/F) \\
(\mathbb{A}^f_{F'})^* \arrow[u, "N_{F'/F}"'] \arrow[r, "\psi_{E/F'}"] \arrow[dr, "\psi_{H'/F'}"] & \mathrm{Gal}(E/F') \arrow[u, "r_{E/H}"'] \arrow[d, "r_{E/H'}"] \\
 & \mathrm{Gal}(H'/F')
\end{tikzcd}$$

commutes with $N_{F'/F}$, the norm map, and $r_{E/H}$, $r_{E/H'}$, the restriction maps. Therefore, if $h' \in (\mathbb{A}^f_{F'})^*$ is chosen such that $\psi_{E/F'}(h') = \sigma|_E$, then we have

$$\psi_{H'/F'}(h') = \sigma|_{H'},$$
$$\psi_{H/F}(N_{F'/F}(h')) = \sigma|_H.$$

With Theorem 2.2.1 this implies

$$\begin{aligned}
\det(\iota^{F'}_{F,b}(\sigma(p'))) &= \det(\iota^{F'}_{F,b}([h'^{-1}])) = \det(\varphi^{-1} \circ h'^{-1} \circ b) \\
&= \det(\varphi^{-1} \circ h'^{-1} \circ \varphi) \cdot \det g = [N_{F'/F}(h')^{-1} \cdot \det g] \\
&= \sigma([\det g]) = \sigma(\det p).
\end{aligned}$$

So we have shown (3.4.1), which is equivalent to $\det_*(\sigma(Y)) = \sigma(\det_*(Y))$. □

Corollary 3.4.4. *The determinant map is induced by a unique morphism $S^r_{F,\mathcal{K}} \to S^1_{F,\det\mathcal{K}}$ defined over F.*

Proof. By Proposition 3.4.2, the determinant map is constant on the irreducible components of $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$. Since these irreducible components and all closed points of $S^1_{F,\det\mathcal{K}}$ are defined over F^{sep}, the determinant map is therefore induced by a unique morphism defined over F^{sep}.

By Proposition 3.4.3, this morphism over F^{sep} is G_F-equivariant. Hence, by [Bor91, AG 14.3] it is defined over F. □

Corollary 3.4.5. *The Drinfeld modular variety $S^r_{F,\mathcal{K}}$ is F-irreducible and has exactly*

$$|S^1_{F,\det\mathcal{K}}| = |F^*\backslash(\mathbb{A}^f_F)^*/\det\mathcal{K}| = |\mathrm{Cl}(F)|\cdot|\hat{A}^*/(\mathbb{F}^*_q\cdot\det\mathcal{K})|$$

irreducible components over $\mathbb{C}_\infty$.

Proof. By Corollary 2.2.2 and Proposition 3.4.3, it follows that the absolute Galois group G_F acts transitively on the set of irreducible components of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$. Hence, $S^r_{F,\mathcal{K}}$ is F-irreducible by Proposition 1.2.2.

It only remains to show the second equality. Note that

$$(\mathbb{A}^f_F)^*/(F^*\cdot\hat{A}^*) \cong \mathrm{Cl}(F)$$

by the direct adaptation of [Neu07, Proposition VI.1.3] to the function field case. Therefore we have

$$|F^*\backslash(\mathbb{A}^f_F)^*/\det\mathcal{K}| = |\mathrm{Cl}(F)|\cdot|(F^*\cdot\hat{A}^*)/(F^*\cdot\det\mathcal{K})|.$$

The claim now follows from

$$(F^*\cdot\hat{A}^*)/(F^*\cdot\det\mathcal{K}) \cong \hat{A}^*/((F^*\cdot\det\mathcal{K})\cap\hat{A}^*)$$

and

$$(F^*\cdot\det\mathcal{K})\cap\hat{A}^* = (F^*\cap\hat{A}^*)\cdot\det\mathcal{K} = \mathbb{F}^*_q\cdot\det\mathcal{K}.$$

□

Corollary 3.4.6. *Each Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$ with reflex field F' is F'-irreducible.*

Proof. A Drinfeld modular subvariety X of $S^r_{F,\mathcal{K}}$ with reflex field F' is the image of an inclusion morphism $\iota^{F'}_{F,b} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$. Since $\iota^{F'}_{F,b}$ is defined over F' by Theorem 3.2.2, Corollary 3.4.5 immediately implies the F'-irreducibility of X. □

4. Degree of subvarieties

4.1 Compactification of Drinfeld modular varieties

In [Pin12] Pink constructs the *Satake compactification* $\overline{S}^r_{F,\mathcal{K}}$ of a Drinfeld modular variety $S^r_{F,\mathcal{K}}$ with $\mathcal{K}\subset\mathrm{GL}_r(\hat{A})$. It is a normal projective variety which contains $S^r_{F,\mathcal{K}}$ as an open dense subvariety and it is characterized up to unique isomorphism by a certain universal property.

If $\mathcal{K}$ is amply small, $\overline{S}^r_{F,\mathcal{K}}$ is endowed with a natural ample invertible sheaf $\mathcal{L}^r_{F,\mathcal{K}}$. In [Pin12], the space of global sections of its kth power is defined to be the space of algebraic modular forms of weight k on $S^r_{F,\mathcal{K}}$.

If $\mathcal{K}\subset\mathrm{GL}_r(\mathbb{A}^f_F)$ is arbitrary (not necessarily contained in $\mathrm{GL}_r(\hat{A})$) and $g\in\mathrm{GL}_r(\mathbb{A}^f_F)$ is chosen such that $g\mathcal{K}g^{-1}\subset\mathrm{GL}_r(\hat{A})$, we define

$$\overline{S}^r_{F,\mathcal{K}} := \overline{S}^r_{F,g\mathcal{K}g^{-1}} \tag{4.1.1}$$

and, if $\mathcal{K}$ is amply small,

$$\mathcal{L}^r_{F,\mathcal{K}} := \mathcal{L}^r_{F,g\mathcal{K}g^{-1}}. \tag{4.1.2}$$

As in Step (v) of the proof of Theorem 2.1.2, one can show, using part (i) of the following proposition for $\mathcal{K}\subset\mathrm{GL}_r(\hat{A})$, that this defines $\overline{S}^r_{F,\mathcal{K}}$ and $\mathcal{L}^r_{F,\mathcal{K}}$ up to isomorphism.

PROPOSITION 4.1.1. (i) *For* $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ *and a compact open subgroup* $\mathcal{K}' \subset g^{-1}\mathcal{K}g$ *the morphism* $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ *defined in* § 3.1 *extends uniquely to a finite morphism* $\overline{\pi_g} : \overline{S}^r_{F,\mathcal{K}'} \to \overline{S}^r_{F,\mathcal{K}}$ *defined over* F *with* $\deg \overline{\pi_g} = \deg \pi_g$. *If* $\mathcal{K}$ *is amply small, then there is a canonical isomorphism*

$$\mathcal{L}^r_{F,\mathcal{K}'} \cong \overline{\pi_g}^* \mathcal{L}^r_{F,\mathcal{K}}.$$

(ii) *Any inclusion* $\iota^{F'}_{F,b} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ *of Drinfeld modular varieties extends uniquely to a finite morphism* $\overline{\iota^{F'}_{F,b}} : \overline{S}^{r'}_{F',\mathcal{K}'} \to \overline{S}^r_{F,\mathcal{K}}$ *defined over* F' *with* $\deg \overline{\iota^{F'}_{F,b}} = \deg \iota^{F'}_{F,b}$. *If* $\mathcal{K}$ *is amply small, then there is a canonical isomorphism*

$$\mathcal{L}^{r'}_{F',\mathcal{K}'} \cong \overline{\iota^{F'}_{F,b}}^* \mathcal{L}^r_{F,\mathcal{K}}.$$

Proof. This follows from [Pin12, Propositions 4.11 and 4.12 and Lemma 5.1]. Note that these statements automatically hold for arbitrary levels $\mathcal{K}$ and $\mathcal{K}'$ (not necessarily contained in $\mathrm{GL}_r(\hat{A})$, respectively $\mathrm{GL}_{r'}(\widehat{A'})$) because (4.1.1) and (4.1.2) define the Satake compactification of a general Drinfeld modular variety as the Satake compactification of a Drinfeld modular variety with level contained in $\mathrm{GL}_r(\hat{A})$ respectively $\mathrm{GL}_{r'}(\widehat{A'})$. The equalities $\deg \overline{\pi_g} = \deg \pi_g$ and $\deg \overline{\iota^{F'}_{F,b}} = \deg \iota^{F'}_{F,b}$ hold because each Drinfeld modular variety is dense in its Satake compactification. □

4.2 Degree of subvarieties

In this subsection, $S^r_{F,\mathcal{K}}$ always denotes a Drinfeld modular variety with $\mathcal{K}$ amply small.

DEFINITION 4.2.1. The *degree* of an irreducible subvariety $X \subset S^r_{F,\mathcal{K}}$ is defined to be the degree of its Zariski closure $\overline{X}$ in $\overline{S}^r_{F,\mathcal{K}}$ with respect to $\mathcal{L}^r_{F,\mathcal{K}}$, i.e., the integer

$$\deg X := \deg_{\mathcal{L}^r_{F,\mathcal{K}}} \overline{X} = \int_{\overline{S}^r_{F,\mathcal{K}}} c_1(\mathcal{L}^r_{F,\mathcal{K}})^{\dim X} \cap [\overline{X}], \tag{4.2.1}$$

where $c_1(\mathcal{L}^r_{F,\mathcal{K}}) \in A^1\overline{S}^r_{F,\mathcal{K}}$ denotes the first Chern class of $\mathcal{L}^r_{F,\mathcal{K}}$, the cycle class of $\overline{X}$ in $A_{\dim X}\overline{S}^r_{F,\mathcal{K}}$ is denoted by $[\overline{X}]$ and $\cap$ is the cap-product between $A^{\dim X}\overline{S}^r_{F,\mathcal{K}}$ and $A_{\dim X}\overline{S}^r_{F,\mathcal{K}}$.

The degree of a reducible subvariety $X \subset S^r_{F,\mathcal{K}}$ is the sum of the degrees of all irreducible components of X.

Remarks.

- Note that our definition of degree for reducible subvarieties differs from the one used in many textbooks where only the sum over the irreducible components of maximal dimension is taken.
- The formula (4.2.1) also holds for reducible subvarieties $X \subset S^r_{F,\mathcal{K}}$ whose irreducible components all have the same dimension.

LEMMA 4.2.2. *The degree of a subvariety* $X \subset S^r_{F,\mathcal{K}}$ *is at least the number of irreducible components of* X.

Proof. This follows by our definition of degree because $\mathcal{L}^r_{F,\mathcal{K}}$ is ample and the degree of an irreducible subvariety of a projective variety with respect to an ample invertible sheaf is a positive integer (see, e.g., [Ful98, Lemma 12.1]). □

PROPOSITION 4.2.3. (i) *Let* $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ *be the morphism defined in* § 3.1 *for* $g \in \mathrm{GL}_r(\mathbb{A}^f_F)$ *and* $\mathcal{K}' \subset g^{-1}\mathcal{K}g$. *Then*

$$\deg \pi_g^{-1}(X) = [g^{-1}\mathcal{K}g : \mathcal{K}'] \cdot \deg X \tag{4.2.2}$$

for subvarieties $X \subset S^r_{F,\mathcal{K}}$ *and*

$$\deg \pi_g(X') \leqslant \deg X' \tag{4.2.3}$$

for subvarieties $X' \subset S^r_{F,\mathcal{K}'}$. *In particular, we have*

$$\deg T_g(X) \leqslant [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g] \cdot \deg X \tag{4.2.4}$$

for subvarieties $X \subset S^r_{F,\mathcal{K}}$.

(ii) *For any inclusion* $\iota^{F'}_{F,b} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ *of Drinfeld modular varieties and for any subvariety* $X \subset S^{r'}_{F',\mathcal{K}'}$, *we have*

$$\deg X = \deg \iota^{F'}_{F,b}(X). \tag{4.2.5}$$

Proof. We use the *projection formula* for Chern classes (see, e.g., [Ful98, Proposition 2.5(c)]).

If $f : X \to Y$ *is a proper morphism of varieties and* $\mathcal{L}$ *is an invertible sheaf on* Y, *then, for all* k*-cycles* $\alpha \in A_k(X)$, *we have the equality*

$$f_*(c_1(f^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap f_*(\alpha) \tag{4.2.6}$$

of $(k-1)$*-cycles in* $A_{k-1}(Y)$.

For the proof of (4.2.2) and (4.2.3), we first assume that $X \subset S^r_{F,\mathcal{K}}$ and $X' \subset S^r_{F,\mathcal{K}'}$ are irreducible. For this, note that $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is finite of degree $[g^{-1}\mathcal{K}g : \mathcal{K}']$ and étale by Proposition 3.1.3 because $\mathcal{K}$ is amply small. The latter implies that the restriction of π_g to the subvariety $\pi_g^{-1}(X)$ is also finite of degree $[g^{-1}\mathcal{K}g : \mathcal{K}']$ and, because $\deg \overline{\pi_g} = \deg \pi_g$, we have the equality

$$\overline{\pi_g}_*[\overline{\pi_g^{-1}(X)}] = [g^{-1}\mathcal{K}g : \mathcal{K}'] \cdot [\overline{X}]$$

of cycles on $\overline{S}^r_{F,\mathcal{K}}$. For $d := \dim X$, with Proposition 4.1.1(i) and the above projection formula we get

$$\begin{aligned}
\deg \pi_g^{-1}(X) &= \deg_{\overline{\pi_g}^*\mathcal{L}^r_{F,\mathcal{K}}} \overline{\pi_g^{-1}(X)} = \int_{\overline{S}^r_{F,\mathcal{K}'}} c_1(\overline{\pi_g}^*\mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{\pi_g^{-1}(X)}] \\
&= \int_{\overline{S}^r_{F,\mathcal{K}}} \overline{\pi_g}_*(c_1(\overline{\pi_g}^*\mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{\pi_g^{-1}(X)}]) \\
&= \int_{\overline{S}^r_{F,\mathcal{K}}} c_1(\mathcal{L}^r_{F,\mathcal{K}})^d \cap \overline{\pi_g}_*[\overline{\pi_g^{-1}(X)}] \\
&= [g^{-1}\mathcal{K}g : \mathcal{K}'] \cdot \int_{\overline{S}^r_{F,\mathcal{K}}} c_1(\mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{X}] = [g^{-1}\mathcal{K}g : \mathcal{K}'] \cdot \deg X.
\end{aligned}$$

For the proof of (4.2.3), we note that

$$\overline{\pi_g}_*[\overline{X'}] = \deg(\pi_g|_{X'}) \cdot [\overline{\pi_g(X')}]$$

as cycles on $\overline{S}^r_{F,\mathcal{K}}$. The same calculation as above gives

$$\deg X' = \deg(\pi_g|_{X'}) \cdot \deg \pi_g(X') \geqslant \deg \pi_g(X').$$

If $X \subset S^r_{F,\mathcal{K}}$ is reducible with irreducible components $X_1, \ldots, X_n$, we have

$$\deg \pi_g^{-1}(X) = \sum_{i=1}^{n} \deg \pi_g^{-1}(X_i)$$

because the set of irreducible components of $\pi_g^{-1}(X)$ is the disjoint union of the sets of irreducible components of the $\pi_g^{-1}(X_i)$. Therefore, (4.2.2) follows from the irreducible case.

If $X' \subset S^r_{F,\mathcal{K}'}$ is reducible with irreducible components $X'_1, \ldots, X'_k$, then the set of irreducible components of $\pi_g(X')$ is a subset of $\{\pi_g(X'_1), \ldots, \pi_g(X'_k)\}$, hence we have

$$\deg \pi_g(X') \leqslant \sum_{i=1}^{k} \deg \pi_g(X'_i),$$

and the inequality (4.2.3) follows from the irreducible case.

The inequality (4.2.4) immediately follows from (4.2.2) and (4.2.3) because

$$T_g(X) = \pi_g(\pi_1^{-1}(X))$$

where π_1 and π_g are projection morphisms $S^r_{F,\mathcal{K}_g} \to S^r_{F,\mathcal{K}}$ with $\mathcal{K}_g := \mathcal{K} \cap g^{-1}\mathcal{K}g$ and

$$\deg \pi_1 = [\mathcal{K} : \mathcal{K}_g] = [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g].$$

Finally, for the proof of (4.2.5) we use that $\iota^{F'}_{F,b} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is a closed immersion by Proposition 3.2.3 because $\mathcal{K}$ is amply small. We therefore have $\deg \overline{\iota^{F'}_{F,b}} = \deg \iota^{F'}_{F,b} = 1$ and, for an irreducible subvariety $X \subset S^{r'}_{F',\mathcal{K}'}$, the equality

$$\overline{\iota^{F'}_{F,b}}_*[\overline{X}] = [\overline{\iota^{F'}_{F,b}(X)}]$$

of cycles on $\overline{S}^r_{F,\mathcal{K}}$ holds. The same calculation as in the proof of (4.2.2) therefore gives

$$\deg \iota^{F'}_{F,b}(X) = \deg X$$

because $\overline{\iota^{F'}_{F,b}}^* \mathcal{L}^r_{F,\mathcal{K}} \cong \mathcal{L}^{r'}_{F',\mathcal{K}'}$ by Proposition 4.1.1(ii).

If $X \subset S^{r'}_{F',\mathcal{K}'}$ is reducible with irreducible components $X_1, \ldots, X_l$, then $\iota^{F'}_{F,b}(X)$ has exactly the irreducible components $\iota^{F'}_{F,b}(X_1), \ldots, \iota^{F'}_{F,b}(X_l)$ because $\iota^{F'}_{F,b}$ is a closed immersion. Therefore, the formula (4.2.5) for X reducible follows from the irreducible case. □

We will use the following two consequences of Bézout's theorem to get an upper bound for the degree of the intersection of two subvarieties of $S^r_{F,\mathcal{K}}$.

LEMMA 4.2.4. *For subvarieties V, W of a projective variety U and an ample invertible sheaf $\mathcal{L}$ on U, we have*

$$\deg V \cap W \leqslant \deg V \cdot \deg W,$$

where deg *denotes the degree with respect to $\mathcal{L}$.*

Proof. See [Ful98, Example 8.4.6] in the case where V and W are irreducible.

If $V = V_1 \cup \cdots \cup V_k$ and $W = W_1 \cup \cdots \cup W_l$ are decompositions into irreducible components, then

$$V \cap W = \bigcup_{i,j} V_i \cap W_j.$$

Therefore, each irreducible component of $V \cap W$ is an irreducible component of some $V_i \cap W_j$. By our definition of degree for reducible varieties this implies

$$\deg V \cap W \leqslant \sum_{i,j} \deg(V_i \cap W_j).$$

Hence, by the case that V and W are irreducible, we get

$$\deg V \cap W \leqslant \sum_{i,j} \deg V_i \cdot \deg W_j = \left(\sum_i \deg V_i\right) \cdot \left(\sum_j \deg W_j\right) = \deg V \cdot \deg W. \qquad \square$$

LEMMA 4.2.5. *For subvarieties V, W of $S^r_{F,\mathcal{K}}$ we have*

$$\deg V \cap W \leqslant \deg V \cdot \deg W.$$

Proof. In view of the previous lemma, it is enough to show the following inequality of degrees of Zariski closures in $\overline{S}^r_{F,\mathcal{K}}$ with respect to $\mathcal{L}^r_{F,\mathcal{K}}$:

$$\deg \overline{V \cap W} \leqslant \deg \overline{V} \cap \overline{W}.$$

For this, it suffices to show that each irreducible component of $\overline{V \cap W}$ is an irreducible component of $\overline{V} \cap \overline{W}$. Note that $\overline{V \cap W} \subset \overline{V} \cap \overline{W}$ and

$$\overline{V \cap W} \cap S^r_{F,\mathcal{K}} = V \cap W = (\overline{V} \cap S^r_{F,\mathcal{K}}) \cap (\overline{W} \cap S^r_{F,\mathcal{K}}) = (\overline{V} \cap \overline{W}) \cap S^r_{F,\mathcal{K}}$$

because $S^r_{F,\mathcal{K}}$ is Zariski open in $\overline{S}^r_{F,\mathcal{K}}$. Therefore

$$\overline{V} \cap \overline{W} = \overline{V \cap W} \ \cup \ (Y \cap (\overline{V} \cap \overline{W})) \tag{4.2.7}$$

where $Y := \overline{S}^r_{F,\mathcal{K}} \backslash S^r_{F,\mathcal{K}}$ denotes the boundary of the compactification. Since the irreducible components of $\overline{V \cap W}$ are the Zariski closures of the irreducible components of $V \cap W$, they are all not contained in Y and therefore, by (4.2.7), irreducible components of $\overline{V} \cap \overline{W}$. $\square$

4.3 Degree of Drinfeld modular subvarieties

We let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety.

PROPOSITION 4.3.1. *If $\mathcal{K}$ is amply small, there is a constant $C > 0$ only depending on F, $\mathcal{K}$ and r such that*

$$\deg(X) \geqslant C \cdot D(X)$$

for all Drinfeld modular subvarieties $X \subset S^r_{F,\mathcal{K}}$ with $D(X)$ the predegree of X from Definition 3.3.8.

Remark. We expect that one could also prove an upper bound for $\deg(X)$ of the form $\deg(X) \leqslant C' \cdot D(X)$ with a constant C' depending on F, $\mathcal{K}$ and r. Because of this expectation, we call $D(X)$ the predegree of X. We refrain from proving an upper bound because we only need a lower bound in the following.

Proof. Since $\mathcal{K}$ is amply small, there is a proper ideal I of A and a $g \in \mathrm{GL}_r(\mathbb{A}^f_F)$ such that $g\mathcal{K}g^{-1} \subset \mathcal{K}(I)$. As explained in the beginning of the proof of Proposition 3.2.3, for each Drinfeld modular subvariety $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ of S, there is a $g' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ such that $\mathcal{K}' \subset g'\mathcal{K}(I')g'^{-1}$ where $I' := IA'$. Therefore, by Proposition 4.2.3, we have

$$\deg(X) = \deg(S^{r'}_{F',\mathcal{K}'}) = [g'\mathcal{K}(I')g'^{-1} : \mathcal{K}'] \cdot \deg(S^{r'}_{F',\mathcal{K}(I')}).$$

Since $S^{r'}_{F',\mathcal{K}(I')}$ has at least $|\mathrm{Cl}(F')|$ irreducible components over $\mathbb{C}_\infty$ by Corollary 3.4.5, we have $\deg(S^{r'}_{F',\mathcal{K}(I')}) \geqslant |\mathrm{Cl}(F')|$. Using $i(X) = [g'\mathrm{GL}_{r'}(\widehat{A'})g'^{-1} : \mathcal{K}']$ we therefore get

$$\deg(X) \geqslant \frac{i(X)}{[\mathrm{GL}_{r'}(\widehat{A'}) : \mathcal{K}(I')]} \cdot |\mathrm{Cl}(F')| = \frac{1}{[\mathrm{GL}_{r'}(\widehat{A'}) : \mathcal{K}(I')]} \cdot D(X).$$

Because $[\mathrm{GL}_{r'}(\widehat{A'}) : \mathcal{K}(I')] \leqslant [\mathrm{GL}_r(A) : \mathcal{K}(I)]$, we conclude that $\deg(X) \geqslant C \cdot D(X)$ for $C := 1/[\mathrm{GL}_r(A) : \mathcal{K}(I)]$ only depending on $\mathcal{K}$, F and r. □

Theorem 4.3.2. *For each sequence (X_n) of pairwise distinct Drinfeld modular subvarieties of S, the sequence of predegrees $(D(X_n))$ is unbounded. In particular, if $\mathcal{K}$ is amply small, the degrees $\deg(X_n)$ are unbounded.*

Proof. By Proposition 4.3.1, it is enough to show that the sequence

$$D(X_n) = i(X_n) \cdot |\mathrm{Cl}(F_n)|$$

where F_n is the reflex field of X_n is unbounded.

The following two propositions imply that there are only finitely many extensions F' of F of degree dividing r and bounded class number.

Proposition 4.3.3. *There are only finitely many finite extensions $F' \subset \mathbb{C}_\infty$ of F of fixed genus g' and bounded degree.*

Proof. By the Hurwitz genus formula (see e.g. [Sti93, Theorem III.4.12]) the degree of the different divisor of F'/F and therefore also the degree of the discriminant divisor of F'/F is bounded as F' runs over all finite separable extensions of F of fixed genus and bounded degree. Hence, [Gos98, Theorem 8.23.5] implies that there are only finitely many separable extensions of F with fixed genus and bounded degree. Since each finite extension of F can be decomposed into a separable and a totally inseparable extension and each global function field has at most one totally inseparable extension of a given degree, the proposition follows. □

Proposition 4.3.4. *Let F' be a function field of genus g' with field of constants $\mathbb{F}_{q'}$. Then*

$$|\mathrm{Cl}(F')| \geqslant \frac{(q'-1)(q'^{2g'} - 2g'q'^{g'} + 1)}{2g'(q'^{g'+1} - 1)}.$$

Proof. See [Bre05, Proposition 3.1]. □

Therefore, the sequence $D(X_n)$ is unbounded if the set of reflex fields F_n is infinite. So it suffices to show unboundedness of the predegree $D(X_n)$ in a sequence of pairwise distinct Drinfeld modular subvarieties of S with fixed reflex field. This follows from the next theorem. Thus we have reduced the proof of Theorem 4.3.2 to Theorem 4.3.5. □

Theorem 4.3.5. *For each sequence (X_n) of pairwise distinct Drinfeld modular subvarieties of S with fixed reflex field F', the indices $i(X_n)$ are unbounded.*

Proof. We first note that we can assume without loss of generality that the given compact subgroup $\mathcal{K}$ equals $\mathrm{GL}_r(\hat{A})$. Indeed, if $\mathcal{K}$ is replaced by a compact open subgroup $\mathcal{L} \supset \mathcal{K}$ and the X_n by their images under the canonical projection $\pi_1 : S^r_{F,\mathcal{K}} \to S^r_{F,\mathcal{L}}$, the indices $i(X_n)$ decrease by Definition 3.3.8. Hence, we can assume that $\mathcal{K}$ is a maximal compact open subgroup and therefore some conjugate $h\mathrm{GL}_r(\hat{A})h^{-1}$ of $\mathrm{GL}_r(\hat{A})$. If we further replace the X_n by their images under the isomorphism $\pi_{h^{-1}} : S^r_{F,h\mathrm{GL}_r(\hat{A})h^{-1}} \to S^r_{F,\mathrm{GL}_r(\hat{A})}$, then the $i(X_n)$ obviously do not change because the X_n are the image of an inclusion from the same $S^{r'}_{F',\mathcal{K}'}(\mathbb{C}_\infty)$. Therefore, we can without loss of generality assume $\mathcal{K} = \mathrm{GL}_r(\hat{A})$.

For the following considerations, we assume that $X_n = \iota^{F'}_{F,b_n}(S^{r'}_{F',\mathcal{K}'_n}(\mathbb{C}_\infty))$ with $\mathbb{A}^f_F$-linear isomorphisms $b_n : (\mathbb{A}^f_F)^r \to (\mathbb{A}^f_{F'})^{r'}$. We denote by Λ_n the $\hat{A}$-lattices $b_n(\hat{A}^r)$ in $(\mathbb{A}^f_{F'})^{r'}$. By Corollary 3.3.10, they are determined up to and only up to the action of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$, and their orbits under the action of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ are pairwise distinct.

We have the product decomposition $\Lambda_n = \prod_{\mathfrak{p}\neq\infty} \Lambda_{n,\mathfrak{p}} := \prod_{\mathfrak{p}\neq\infty} b_{n,\mathfrak{p}}(A^r_{\mathfrak{p}})$, where $\Lambda_{n,\mathfrak{p}} \subset {F'}_{\mathfrak{p}}^{r'}$ are free $A_{\mathfrak{p}}$-submodules of rank r. The $A'_{\mathfrak{p}}$-modules $A'_{\mathfrak{p}} \cdot \Lambda_{n,\mathfrak{p}}$ are finitely generated submodules of ${F'}_{\mathfrak{p}}^{r'}$ with $F'_{\mathfrak{p}} \cdot \Lambda_{n,\mathfrak{p}} = {F'}_{\mathfrak{p}}^{r'}$, and hence free of rank r' because $A'_{\mathfrak{p}}$ is a direct product of principal ideal domains. This implies that $\hat{A}' \cdot \Lambda_n$ is a free $\hat{A}'$-submodule of $(\mathbb{A}^f_{F'})^{r'}$ of rank r'. Since the Λ_n are determined up to and only up to the action of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$, we may therefore assume without loss of generality that $\hat{A}' \cdot \Lambda_n = {\hat{A}'}^{r'}$ for all n.

Note that we have

$$\mathcal{K}'_n = (b_n \mathrm{GL}_r(\hat{A}) b_n^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'}) = \mathrm{Stab}_{\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})} \Lambda_n.$$

Since $\hat{A}' \cdot \Lambda_n = {\hat{A}'}^{r'}$, these compact open subgroups of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ are all contained in the maximal compact subgroup $\mathrm{GL}_{r'}(\hat{A}') = \mathrm{Stab}_{\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})} {\hat{A}'}^{r'}$. Hence, we can write the indices $i(X_n)$ as

$$i(X_n) = [\mathrm{GL}_{r'}(\hat{A}') : \mathrm{Stab}_{\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})} \Lambda_n]$$

and, using the above product decompositions, as $i(X_n) = \prod_{\mathfrak{p}\neq\infty} i_{n,\mathfrak{p}}$, where

$$i_{n,\mathfrak{p}} = [\mathrm{GL}_{r'}(A'_{\mathfrak{p}}) : \mathrm{Stab}_{\mathrm{GL}_{r'}(F'_{\mathfrak{p}})} \Lambda_{n,\mathfrak{p}}].$$

For each n, almost all factors of this product are 1 because $\Lambda_{n,\mathfrak{p}} = {A'}_{\mathfrak{p}}^{r'}$ for almost all $\mathfrak{p}$.

Since we assumed that $A'_{\mathfrak{p}} \cdot \Lambda_{n,\mathfrak{p}} = {A'}_{\mathfrak{p}}^{r'}$, by the Proposition 4.3.6 below, we get the estimates $i_{n,\mathfrak{p}} \geqslant C \cdot [{A'}_{\mathfrak{p}}^{r'} : \Lambda_{n,\mathfrak{p}}]^{1/r}$, where the constant C is independent of n and $\mathfrak{p}$.

We now finish the proof by assuming (for contradiction) that the sequence $(i(X_n))$ is bounded. This implies by the above product decomposition of $i(X_n)$ and estimates of $i_{n,\mathfrak{p}}$ that $[{A'}_{\mathfrak{p}}^{r'} : \Lambda_{n,\mathfrak{p}}] \leqslant D$ for all n and $\mathfrak{p}$ for some uniform constant D.

However, note that as a finite $A_{\mathfrak{p}}$-module ${A'}_{\mathfrak{p}}^{r'}/\Lambda_{n,\mathfrak{p}}$ is isomorphic to some product

$$A_{\mathfrak{p}}/\mathfrak{p}^{m_1} A_{\mathfrak{p}} \times \cdots \times A_{\mathfrak{p}}/\mathfrak{p}^{m_l} A_{\mathfrak{p}}.$$

If $\Lambda_{n,\mathfrak{p}} \not\supseteq \mathfrak{p}^N \cdot {A'}_{\mathfrak{p}}^{r'}$, we have $m_i \geqslant N+1$ for some i and therefore

$$|k(\mathfrak{p})|^{N+1} \leqslant [{A'}_{\mathfrak{p}}^{r'} : \Lambda_{n,\mathfrak{p}}] \leqslant D.$$

In particular, we have $|k(\mathfrak{p})| \leqslant D$ whenever $\Lambda_{n,\mathfrak{p}} \neq {A'}_{\mathfrak{p}}^{r'}$. Since there are only finitely many primes $\mathfrak{p}$ with $|k(\mathfrak{p})| \leqslant D$, we conclude the following.

- There are finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ such that $\Lambda_{n,\mathfrak{p}} = {A'}_{\mathfrak{p}}^{r'}$ for all n and $\mathfrak{p} \neq \mathfrak{p}_1, \ldots, \mathfrak{p}_k$.
- There is an $N \in \mathbb{N}$ such that, for all $\mathfrak{p}$ and n, the $A_{\mathfrak{p}}$-lattice $\Lambda_{n,\mathfrak{p}}$ contains $\mathfrak{p}^N {A'}_{\mathfrak{p}}^{r'}$.

Because the quotients ${A'}_{\mathfrak{p}}^{r'}/\mathfrak{p}^N {A'}_{\mathfrak{p}}^{r'}$ are finite, the second statement implies that for all $1 \leqslant i \leqslant k$ there are only finitely many possibilities for $\Lambda_{n,\mathfrak{p}_i}$. Since for $\mathfrak{p} \neq \mathfrak{p}_1, \ldots, \mathfrak{p}_k$ the lattices $\Lambda_{n,\mathfrak{p}}$ are independent of n, this implies that only finitely many $\hat{A}$-lattices $\Lambda_n \subset (\mathbb{A}^f_{F'})^{r'}$ occur, which is a contradiction of our assumptions. □

Proposition 4.3.6. *Let K be a complete field with respect to a discrete valuation v with finite residue field containing $\mathbb{F}_q$ and let R be the corresponding discrete valuation ring with maximal ideal $\mathfrak{m}$. Let $K' := L_1 \times \cdots \times L_m$ with L_i finite field extensions of K and $R' := S_1 \times \cdots \times S_m$ with $S_i \subset L_i$ the discrete valuation ring associated to the unique extension of v to L_i. Suppose that $r' \geqslant 1$, and set $r := r' \cdot \sum_{i=1}^m [L_i : K]$.*

There is a constant $C > 0$ only depending on q and r such that, for any free R-submodule $\Lambda \subset K'^{r'}$ of rank r with $R' \cdot \Lambda = R'^{r'}$, we have

$$[\mathrm{GL}_{r'}(R') : \mathrm{Stab}_{\mathrm{GL}_{r'}(K')}(\Lambda)] \geqslant C \cdot [R'^{r'} : \Lambda]^{1/r}.$$

Proof. We introduce the notation

$$H := \{T \in \mathrm{Mat}_{r'}(R') : T \cdot \Lambda \subseteq \Lambda\}.$$

This set of matrices is an R-subalgebra of $\mathrm{Mat}_{r'}(R')$ with $H^* = \mathrm{Stab}_{\mathrm{GL}_{r'}(R')}(\Lambda)$.

Note that, if $g_1, \ldots, g_r$ is an R-basis of Λ, then $\Lambda = \xi(R^r) \subset K'^{r'}$ for

$$\xi : \begin{array}{ccc} K^r & \longrightarrow & K'^{r'} \\ (x_1, \ldots, x_r) & \longmapsto & x_1 g_1 + \cdots + x_r g_r. \end{array}$$

Since K is complete, ξ is a homeomorphism (cf. [Neu07, Proposition 4.9]). This implies that $\Lambda \subset R'^{r'}$ is open.

Hence, there is a $k \in \mathbb{N}$ such that $\mathfrak{m}^k R'^{r'} \subseteq \Lambda$. Therefore $\mathrm{Mat}_{r'}(\mathfrak{m}^k R') \subseteq H$ and

$$H/\mathrm{Mat}_{r'}(\mathfrak{m}^k R') = \{\overline{T} \in \mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R') : \overline{T} \cdot (\Lambda/\mathfrak{m}^k R'^{r'}) \subseteq \Lambda/\mathfrak{m}^k R'^{r'}\}$$

if we identify $\mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R')$ with $\mathrm{Mat}_{r'}(R')/\mathrm{Mat}_{r'}(\mathfrak{m}^k R')$. For the stabilizer of $\Lambda/\mathfrak{m}^k R'^{r'}$ under the action of $\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')$, this means that

$$(H/\mathrm{Mat}_{r'}(\mathfrak{m}^k R'))^* = \mathrm{Stab}_{\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')}(\Lambda/\mathfrak{m}^k R'^{r'}).$$

The orbit of Λ under $\mathrm{GL}_{r'}(R')$ is in bijective correspondence with the orbit of $\Lambda/\mathfrak{m}^k R'^{r'}$ under $\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')$ via

$$T \cdot \Lambda \longmapsto (T \cdot \Lambda)/\mathfrak{m}^k R'^{r'}.$$

Therefore the above formulas for the corresponding stabilizers give us the following estimate:

$$[\mathrm{GL}_{r'}(R') : H^*] = [\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R') : (H/\mathrm{Mat}_{r'}(\mathfrak{m}^k R'))^*] \geqslant \frac{|\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')|}{[H : \mathrm{Mat}_{r'}(\mathfrak{m}^k R')]}.$$

Lemma 4.3.7. *There is a constant C only depending on q and r (namely $C = (1 - 1/q)^r$) such that*

$$|\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')| \geqslant C \cdot |\mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R')|.$$

Proof. By the definition of R' the quotient $R'/\mathfrak{m}^k R'$ is isomorphic to $S_1/\mathfrak{m}^k S_1 \times \cdots \times S_m/\mathfrak{m}^k S_m$, and hence

$$\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R') \cong \mathrm{GL}_{r'}(S_1/\mathfrak{m}^k S_1) \times \cdots \times \mathrm{GL}_{r'}(S_m/\mathfrak{m}^k S_m),$$
$$\mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R') \cong \mathrm{Mat}_{r'}(S_1/\mathfrak{m}^k S_1) \times \cdots \times \mathrm{Mat}_{r'}(S_m/\mathfrak{m}^k S_m).$$

Now note that, for any $l \geqslant 1$ and any discrete valuation ring U with maximal ideal $\mathfrak{n}$ and residue field $\mathbb{F}_{q'}$ containing $\mathbb{F}_q$, a matrix $T \in \mathrm{Mat}_{r'}(U)$ is invertible if and only if its reduction modulo $\mathfrak{n}^l$

is invertible in $\mathrm{Mat}_{r'}(U/\mathfrak{n}^l)$. In particular, $\mathrm{GL}_{r'}(U/\mathfrak{n}^l)$ exactly consists of the matrices with reduction modulo $\mathfrak{n}$ lying in $\mathrm{GL}_{r'}(U/\mathfrak{n})$. As the fibers of the projection $\mathrm{Mat}_{r'}(U/\mathfrak{n}^l) \to \mathrm{Mat}_{r'}(U/\mathfrak{n})$ have all cardinality $|\mathfrak{n}/\mathfrak{n}^l|^{r'^2} = q'^{(l-1)r'^2}$, we get

$$\begin{aligned} |\mathrm{GL}_{r'}(U/\mathfrak{n}^l)| &= q'^{(l-1)r'^2} |\mathrm{GL}_{r'}(\mathbb{F}_{q'})| \\ &= q'^{(l-1)r'^2} (q'^{r'} - 1)(q'^{r'} - q') \cdots (q'^{r'} - q'^{r'-1}) \\ &\geqslant q'^{lr'^2} \left(1 - \frac{1}{q}\right)^{r'} \\ &= \left(1 - \frac{1}{q}\right)^{r'} |\mathrm{Mat}_{r'}(U/\mathfrak{n}^l)|. \end{aligned}$$

Since $m \leqslant r/r'$, altogether we have

$$|\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')| \geqslant \left(1 - \frac{1}{q}\right)^{mr'} |\mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R')| \geqslant C \cdot |\mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R')|$$

with $C = (1 - 1/q)^r$. □

Proof of Proposition 4.3.6 (continued). By Lemma 4.3.7 and the preceding estimate, we have

$$[\mathrm{GL}_{r'}(R') : H^*] \geqslant C \cdot \frac{[\mathrm{Mat}_{r'}(R') : \mathrm{Mat}_{r'}(\mathfrak{m}^k R')]}{[H : \mathrm{Mat}_{r'}(\mathfrak{m}^k R')]} = C \cdot [\mathrm{Mat}_{r'}(R') : H].$$

To finish the proof of Proposition 4.3.6, we consider an R-basis $g_1, \ldots, g_r$ of Λ and the R-module homomorphism

$$\begin{array}{ccc} \mathrm{Mat}_{r'}(R')^r & \longrightarrow & R'^{r'}/\Lambda \\ (T_1, \ldots, T_r) & \longmapsto & (T_1 \cdot g_1 + \cdots + T_r \cdot g_r) \mod \Lambda. \end{array}$$

It is surjective and its kernel contains H^r. Therefore, we have

$$[\mathrm{Mat}_{r'}(R') : H]^r = [\mathrm{Mat}_{r'}(R')^r : H^r] \geqslant [R'^{r'} : \Lambda]$$

and in total

$$[\mathrm{GL}_{r'}(R') : \mathrm{Stab}_{\mathrm{GL}_{r'}(K')}(\Lambda)] \geqslant C \cdot [\mathrm{Mat}_{r'}(R') : H] \geqslant C \cdot [R'^{r'} : \Lambda]^{1/r}.$$

This completes the proof of Proposition 4.3.6. □

5. Zariski density of Hecke orbits

In the whole of this section, $S = S^r_{F,\mathcal{K}}$ denotes a Drinfeld modular variety and C a set of representatives in $\mathrm{GL}_r(\mathbb{A}^f_F)$ for $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}$. We use the description of the irreducible components of S over $\mathbb{C}_\infty$ given in Proposition 2.1.3. We let Y_h be the irreducible component of S over $\mathbb{C}_\infty$ corresponding to $h \in C$ and identify its $\mathbb{C}_\infty$-valued points $Y_h(\mathbb{C}_\infty) \subset \mathrm{GL}_r(F)\backslash(\Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K})$ with $\Gamma_h\backslash\Omega^r_F$ where $\Gamma_h := h\mathcal{K}h^{-1} \cap \mathrm{GL}_r(F)$ via the isomorphism from Proposition 2.1.3.

5.1 Definition and explicit description of $(T_g + T_{g^{-1}})$-orbits

For $g \in \mathrm{GL}_r(\mathbb{A}^f_F)$ and closed subvarieties $Z \subset S$ we define

$$(T_g + T_{g^{-1}})(Z) := T_g(Z) \cup T_{g^{-1}}(Z),$$

and recursively define

$$\begin{aligned}(T_g+T_{g^{-1}})^0(Z) &:= Z\\ (T_g+T_{g^{-1}})^n(Z) &:= (T_g+T_{g^{-1}})\big((T_g+T_{g^{-1}})^{n-1}(Z)\big), \quad n\geqslant 1.\end{aligned}$$

Definition 5.1.1. For a geometric point $x\in S(\mathbb{C}_\infty)$ and $g\in \mathrm{GL}_r(\mathbb{A}_F^f)$, the union

$$T_g^\infty(x) := \bigcup_{n\geqslant 0}(T_g+T_{g^{-1}})^n(x)\subset S(\mathbb{C}_\infty)$$

is called the $(T_g+T_{g^{-1}})$-*orbit* of x.

Note that $T_g^\infty(x)$ is the smallest subset of $S(\mathbb{C}_\infty)$ containing x which is mapped into itself under T_g and $T_{g^{-1}}$.

We now give an explicit description of the intersection of $T_g^\infty(x)$ with the irreducible components of S over $\mathbb{C}_\infty$ for $x\in S(\mathbb{C}_\infty)$ and $g\in\mathrm{GL}_r(\mathbb{A}_F^f)$.

Proposition 5.1.2. *Let $h_1,h_2\in C$ and assume that $x\in Y_{h_1}(\mathbb{C}_\infty)$ with $x=[\omega]\in\Gamma_{h_1}\backslash\Omega_F^r$. Then the intersection of $T_g^\infty(x)$ with $Y_{h_2}(\mathbb{C}_\infty)$ is given by*

$$T_g^\infty(x)\cap Y_{h_2}(\mathbb{C}_\infty)=\{[T\omega]\in\Gamma_{h_2}\backslash\Omega_F^r : T\in h_2\langle\mathcal{K}g\mathcal{K}\rangle h_1^{-1}\cap\mathrm{GL}_r(F)\},$$

where $\langle\mathcal{K}g\mathcal{K}\rangle$ denotes the subgroup of $\mathrm{GL}_r(\mathbb{A}_F^f)$ generated by the double coset $\mathcal{K}g\mathcal{K}$.

Proof. By assumption, we have $x=[(\omega,h_1)]\in\mathrm{GL}_r(F)\backslash\big(\Omega_F^r\times\mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}\big)$. Hence, by Definition 3.1.5 and the recursive definition of $(T_g+T_{g^{-1}})^n(x)$, the elements of $T_g^\infty(x)$ are exactly those of the form $[(\omega,h_1k_1g_1k_2g_2\cdots k_ng_n)]$ with $n\geqslant 0$, $k_i\in\mathcal{K}$ and $g_i\in\{g,g^{-1}\}$. Hence, an element $y\in T_g^\infty(x)\cap Y_{h_2}(\mathbb{C}_\infty)$ can be written as $y=[(\omega,h_1s)]$ with $s\in\langle\mathcal{K}g\mathcal{K}\rangle$. Since y lies in Y_{h_2}, there exist $T\in\mathrm{GL}_r(F)$ and $k\in\mathcal{K}$ with $Th_1sk=h_2$. Therefore

$$y=[(\omega,h_1s)]=[(T\omega,Th_1sk)]=[(T\omega,h_2)]$$

is equal to $[T\omega]\in\Gamma_{h_2}\backslash\Omega_F^r$, where $T\in h_2\langle\mathcal{K}g\mathcal{K}\rangle h_1^{-1}\cap\mathrm{GL}_r(F)$.

Conversely, an element $[T\omega]\in\Gamma_{h_2}\backslash\Omega_F^r$ with $T=h_2sh_1^{-1}\in h_2\langle\mathcal{K}g\mathcal{K}\rangle h_1^{-1}\cap\mathrm{GL}_r(F)$ is equal to

$$[(T\omega,h_2)]=[(\omega,T^{-1}h_2)]=[(\omega,h_1s^{-1}h_2^{-1}h_2)]=[(\omega,h_1s^{-1})]$$

with $s^{-1}\in\langle\mathcal{K}g\mathcal{K}\rangle$, and hence lies in $T_g^\infty(x)\cap Y_{h_2}(\mathbb{C}_\infty)$. □

5.2 Zariski density

We give a sufficient condition for a subset $M\subset S(\mathbb{C}_\infty)$ to be Zariski dense in one irreducible component Y_h of S over $\mathbb{C}_\infty$. Recall that, for a place $\mathfrak{p}\neq\infty$ of F, by $\mathbb{A}_F^{f,\mathfrak{p}}$ we denote the adeles outside ∞ and $\mathfrak{p}$.

Proposition 5.2.1. *Let M be a subset of $S(\mathbb{C}_\infty)$ contained in an irreducible component Y_h of S over $\mathbb{C}_\infty$ for $h\in C$ and suppose that M contains an element $x=[\omega]\in Y_h(\mathbb{C}_\infty)=\Gamma_h\backslash\Omega_F^r$ such that there exists a place $\mathfrak{p}\neq\infty$ of F and an open subgroup $\mathcal{K}'\subset\mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$ with*

$$M':=\{[T\omega]\in\Gamma_h\backslash\Omega_F^r : T\in(\mathrm{SL}_r(F_\mathfrak{p})\times\mathcal{K}')\cap\mathrm{GL}_r(F)\}\subset M.$$

Then M is Zariski dense in Y_h.

Proof. We denote the Zariski closure of M' by Y. It is enough to show that $Y(\mathbb{C}_\infty)=Y_h(\mathbb{C}_\infty)$. As the non-singular locus Y^{ns} of Y over $\mathbb{C}_\infty$ is Zariski open and dense in Y [Har77, Theorem I.5.3], the intersection $Y^{\mathrm{ns}}(\mathbb{C}_\infty) \cap M'$ is non-empty. Since $(\mathrm{SL}_r(F_\mathfrak{p}) \times \mathcal{K}') \cap \mathrm{GL}_r(F)$ is a subgroup of $\mathrm{GL}_r(F)$, we can therefore assume that $x = [\omega]$ lies in $Y^{\mathrm{ns}}(\mathbb{C}_\infty)$. Hence it is enough to show that the tangent space $T_x Y$ of Y at x is of dimension $r - 1 = \dim S$.

Since $\mathcal{K}'$ is open in $\mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$, there is an $N \in A$ with $N \notin \mathfrak{p}$ such that $K'(N) \subset \mathcal{K}'$, where $K'(N)$ denotes the principal congruence subgroup modulo N of $\mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$. Now let $l \geqslant 1$ such that $\mathfrak{p}^l = (\pi)$ is a principal ideal of A and consider for $1 \leqslant i \leqslant r - 1$ and $k \geqslant 1$ the matrices

$$A_{ik} := \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & \frac{N}{\pi^k} & & 1 \end{pmatrix} \in \mathrm{SL}_r(F),$$

with the entry N/π^k in the ith column. As elements of $\mathrm{GL}_r(\mathbb{A}_F^f)$ (diagonally embedded) they lie in $\mathrm{SL}_r(F_\mathfrak{p}) \times K'(N) \subset \mathrm{SL}_r(F_\mathfrak{p}) \times \mathcal{K}'$. Hence, for all $1 \leqslant i \leqslant r - 1$ and $k \geqslant 1$, $[A_{ik}\omega]$ lies in $M' \subset Y(\mathbb{C}_\infty)$.

We now view Ω_F^r as a subset of $\mathbb{A}^{r-1}(\mathbb{C}_\infty)$ by identifying $[\omega_1 : \cdots : \omega_{r-1} : 1]$ with $(\omega_1, \ldots, \omega_{r-1})$ (note that the rth projective coordinate ω_r of an arbitrary element of Ω_F^r can be assumed to be 1 because the F_∞-rational hyperplane $\omega_r = 0$ does not belong to Ω_F^r). Assume that we have $\omega = (\omega_1, \ldots, \omega_{r-1})$ in this identification. Then, using (2.1.2), we see that

$$A_{ik}\omega = \left(\omega_1, \ldots, \omega_i - \frac{N}{\pi^k}, \ldots, \omega_{r-1}\right)$$

for all $1 \leqslant i \leqslant r - 1$ and $k \geqslant 1$. Note that $\omega_i - N/\pi^k$ converges to ω_i in $\mathbb{C}_\infty$ for $k \to \infty$ and that $\{[A_{ik}\omega]\}_{k \geqslant 1} \subset Y(\mathbb{C}_\infty)$ for all $1 \leqslant i \leqslant r - 1$. Since $Y(\mathbb{C}_\infty) \subset Y_h(\mathbb{C}_\infty) = \Gamma_h \backslash \Omega_F^r$ is closed in the rigid-analytic topology, it follows that there is an $\varepsilon > 0$ such that for all $1 \leqslant i \leqslant r - 1$ and $c \in \mathbb{C}_\infty$ with $|c|_\infty < \varepsilon$

$$[(\omega_1, \ldots, \omega_i + c, \ldots, \omega_{r-1})] \in Y(\mathbb{C}_\infty).$$

This implies $\dim T_x Y = r - 1$ and $Y(\mathbb{C}_\infty) = Y_h(\mathbb{C}_\infty)$. □

Now let $\mathfrak{p} \neq \infty$ be a place of F and $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ trivial outside $\mathfrak{p}$, i.e., $g := (1, \ldots, g_\mathfrak{p}, \ldots, 1)$ for some $g_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$. Using Proposition 5.2.1, we prove a sufficient condition for the $(T_g + T_{g^{-1}})$-orbit $T_g^\infty(x)$ to be Zariski dense in the irreducible component of S over $\mathbb{C}_\infty$ containing x. This result is a generalization of Theorem 4.11 in [Bre12].

Theorem 5.2.2. *Assume that the image of the cyclic subgroup $\langle g_\mathfrak{p} \rangle \subset \mathrm{GL}_r(F_\mathfrak{p})$ in $\mathrm{PGL}_r(F_\mathfrak{p})$ is unbounded and, for $x \in S(\mathbb{C}_\infty)$, let Y_x be the irreducible component of S over $\mathbb{C}_\infty$ containing x. Then, for all $x \in S(\mathbb{C}_\infty)$ and $g := (1, \ldots, g_\mathfrak{p}, \ldots, 1)$, the intersection of the $(T_g + T_{g^{-1}})$-orbit $T_g^\infty(x)$ with $Y_x(\mathbb{C}_\infty)$ is Zariski dense in Y_x.*

Proof. We assume that $Y_x = Y_h$ for some $h \in C$. Then, by Proposition 5.1.2, we have

$$T_g^\infty(x) \cap Y_x(\mathbb{C}_\infty) = \{[T\omega] \in \Gamma_h \backslash \Omega_F^r : T \in h\langle \mathcal{K} g \mathcal{K} \rangle h^{-1} \cap \mathrm{GL}_r(F)\}.$$

Since $h\mathcal{K}h^{-1}$ is an open subgroup of $\mathrm{GL}_r(\mathbb{A}_F^f)$, we can find compact open subgroups $\mathcal{K}_\mathfrak{p} \subset \mathrm{GL}_r(F_\mathfrak{p})$ and $\mathcal{K}' \subset \mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$ such that $\mathcal{K}_\mathfrak{p} \times \mathcal{K}' \subset h\mathcal{K}h^{-1}$ and hence

$$\langle \mathcal{K}_\mathfrak{p} h_\mathfrak{p} g_\mathfrak{p} h_\mathfrak{p}^{-1} \mathcal{K}_\mathfrak{p} \rangle \times \mathcal{K}' \subset h\langle \mathcal{K} g \mathcal{K} \rangle h^{-1}.$$

We now consider the open subgroup $U_{\mathfrak{p}} := \langle K_{\mathfrak{p}}\, h_{\mathfrak{p}}\, g_{\mathfrak{p}}\, h_{\mathfrak{p}}^{-1} K_{\mathfrak{p}} \rangle \cap \mathrm{SL}_r(F_{\mathfrak{p}})$ of $\mathrm{SL}_r(F_{\mathfrak{p}})$. It is normalized by the image of $\langle g_{\mathfrak{p}} \rangle$ in $\mathrm{PGL}_r(F_{\mathfrak{p}})$, which is unbounded by assumption. Since PGL_r is a connected adjoint absolutely simple linear algebraic group over the local field $F_{\mathfrak{p}}$ and $\mathrm{SL}_r \hookrightarrow \mathrm{GL}_r \to \mathrm{PGL}_r$ is its universal covering, we conclude by [Pin00, Theorem 2.2] that $U_{\mathfrak{p}}$ is equal to $\mathrm{SL}_r(F_{\mathfrak{p}})$.

Hence, $\mathrm{SL}_r(F_{\mathfrak{p}})$ is contained in $\langle K_{\mathfrak{p}}\, h_{\mathfrak{p}}\, g_{\mathfrak{p}}\, h_{\mathfrak{p}}^{-1} K_{\mathfrak{p}} \rangle$ and we have

$$\{[T\omega] \in \Gamma_h \backslash \Omega_F^r : T \in (\mathrm{SL}_r(F_{\mathfrak{p}}) \times \mathcal{K}') \cap \mathrm{GL}_r(F)\} \subset T_g^{\infty}(x) \cap Y_x(\mathbb{C}_\infty).$$

Therefore, we can apply Proposition 5.2.1 to the subset $T_g^{\infty}(x) \cap Y_x(\mathbb{C}_\infty)$ of $S(\mathbb{C}_\infty)$ and conclude that $T_g^{\infty}(x) \cap Y_x(\mathbb{C}_\infty)$ is Zariski dense in Y_x. □

6. Geometric criterion for being a Drinfeld modular subvariety

Proposition 6.1.1. *Let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety and $Z \subset S$ an irreducible subvariety over $\mathbb{C}_\infty$ such that $Z = T_g Z = T_{g^{-1}} Z$ for some $g = (1, \dots, g_{\mathfrak{p}}, \dots, 1)$ with $g_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$. If the cyclic subgroup of $\mathrm{PGL}_r(F_{\mathfrak{p}})$ generated by the image of $g_{\mathfrak{p}}$ is unbounded, then Z is an irreducible component of S over $\mathbb{C}_\infty$.*

Proof. Let $x \in Z(\mathbb{C}_\infty)$ be a geometric point of Z. By assumption we have $T_g(x) \subset T_g Z = Z$ and $T_{g^{-1}}(x) \subset T_{g^{-1}} Z = Z$, and hence

$$(T_g + T_{g^{-1}})(x) \subset Z.$$

Iterating we get for all $n \geqslant 1$

$$(T_g + T_{g^{-1}})^n(x) \subset Z,$$

so the $(T_g + T_{g^{-1}})$-orbit $T_g^{\infty}(x)$ of x is contained in Z. Since Z is irreducible over $\mathbb{C}_\infty$, the orbit $T_g^{\infty}(x)$ is contained in one irreducible component Y of S over $\mathbb{C}_\infty$. So $T_g^{\infty}(x)$ is Zariski dense in Y by Theorem 5.2.2. Since Z is Zariski closed in S, it follows that $Z = Y$ is an irreducible component of S over $\mathbb{C}_\infty$. □

Definition 6.1.2. A subvariety X defined over $\overline{F}$ of a Drinfeld modular subvariety $S^r_{F,\mathcal{K}}$ is called *Hodge-generic* if none of its irreducible components over $\mathbb{C}_\infty$ is contained in a proper Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$.

Theorem 6.1.3. *Let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety with $\mathcal{K} = \mathcal{K}_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}$ amply small where $\mathcal{K}_{\mathfrak{p}} \subset \mathrm{GL}_r(F_{\mathfrak{p}})$ and $\mathcal{K}^{(\mathfrak{p})} \subset \mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$. Suppose that $Z \subset S$ is an F-irreducible Hodge-generic subvariety with $\dim Z \geqslant 1$ such that $Z \subset T_g Z$ for some $g = (1, \dots, g_{\mathfrak{p}}, \dots, 1)$ with $g_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$. If, for all $k_1, k_2 \in \mathcal{K}_{\mathfrak{p}}$, the cyclic subgroup of $\mathrm{PGL}_r(F_{\mathfrak{p}})$ generated by the image of $k_1 \cdot g_{\mathfrak{p}} \cdot k_2$ is unbounded, then $Z = S$.*

Remark. Note that the unboundedness condition in this theorem is stronger than the one in Proposition 6.1.1. For example, for $r = 2$, $\mathcal{K}_{\mathfrak{p}} = \mathrm{GL}_2(A_{\mathfrak{p}})$ and a uniformizer $\pi_{\mathfrak{p}} \in F_{\mathfrak{p}}$, the image of $g_{\mathfrak{p}} = \left(\begin{smallmatrix} \pi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ generates an unbounded subgroup of $\mathrm{PGL}_2(F_{\mathfrak{p}})$, but for $k_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{K}_{\mathfrak{p}}$, the image of $k_1 g_{\mathfrak{p}}$ generates a bounded subgroup of $\mathrm{PGL}_2(F_{\mathfrak{p}})$ because $(k_1 g_{\mathfrak{p}})^2$ is a scalar matrix.

Proof. In this proof, for simplicity of notation, we identify $\mathrm{GL}_r(F_{\mathfrak{p}})$ as a subgroup of $\mathrm{GL}_r(\mathbb{A}_F^f)$ via the inclusion

$$h_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}}) \longmapsto (1, \dots, h_{\mathfrak{p}}, \dots, 1) \in \mathrm{GL}_r(\mathbb{A}_F^f).$$

Let $Z = Z_1 \cup \dots \cup Z_s$ be a decomposition of Z into irreducible components over $\mathbb{C}_\infty$. Since Z is defined over F, the irreducible component Z_1 is defined over some finite, separable extension E of F. By the F-irreducibility of S and Z, it is enough to show that Z_1 is an irreducible component of S over $\mathbb{C}_\infty$. We divide the proof into two steps.

Step (i) We show that there is an open subgroup $\mathcal{K}' \subset \mathcal{K}$ with associated canonical projection $\pi : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ and an E-irreducible component Z_1' of $\pi^{-1}(Z_1)$ which is also irreducible over $\mathbb{C}_\infty$ such that $T_{h_\mathfrak{p}} Z_1'$ is E-irreducible for all $h_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$.

Step (ii) Using Proposition 6.1.1, we prove that Z_1' is an irreducible component of $S^r_{F,\mathcal{K}'}$ over $\mathbb{C}_\infty$.

Steps (i) and (ii) imply that $Z_1 = \pi(Z_1')$ is an irreducible component of $S = S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$.

Step (i). Note that, by Proposition 3.1.3, the canonical projections

$$\pi_{U_\mathfrak{p}} : S^r_{F,U_\mathfrak{p}\times\mathcal{K}^{(\mathfrak{p})}} \longrightarrow S$$

where $U_\mathfrak{p}$ runs over all open normal subgroups of $\mathcal{K}_\mathfrak{p}$ form a projective system of finite étale Galois covers defined over F with Galois groups $\mathcal{K}_\mathfrak{p}/U_\mathfrak{p}$. Hence, by Proposition 3.1.3

$$\pi_\mathfrak{p} : S^{(\mathfrak{p})} := \varprojlim_{U_\mathfrak{p}} S^r_{F,U_\mathfrak{p}\times\mathcal{K}^{(\mathfrak{p})}} \longrightarrow S$$

is a pro-étale Galois cover with group $\varprojlim_{U_\mathfrak{p}} \mathcal{K}_\mathfrak{p}/U_\mathfrak{p}$. Since $\mathcal{K}_\mathfrak{p}$ is a profinite group, this group is isomorphic to $\mathcal{K}_\mathfrak{p}$ and we have the following isomorphisms of rigid-analytic spaces:

$$\begin{aligned} S^{(\mathfrak{p})}(\mathbb{C}_\infty) &\cong \varprojlim_{U_\mathfrak{p}} \mathrm{GL}_r(F)\backslash(\Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)/(U_\mathfrak{p}\times\mathcal{K}^{(\mathfrak{p})})) \\ &\cong \mathrm{GL}_r(F)\backslash(\Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}^{(\mathfrak{p})}). \end{aligned}$$

By Proposition 3.1.3 and these identifications, the automorphism of the $\mathcal{K}_\mathfrak{p}$-cover $\pi_\mathfrak{p}$ corresponding to a $k_\mathfrak{p} \in \mathcal{K}_\mathfrak{p}$ is given by

$$\varprojlim_{U_\mathfrak{p}} \pi_{k_\mathfrak{p}} : [(\overline{\omega}, h)] \mapsto [(\overline{\omega}, hk_\mathfrak{p}^{-1})]$$

on $\mathbb{C}_\infty$-valued points of $S^{(\mathfrak{p})}$.

We now denote by Y the non-singular locus of the variety Z_1 over $\mathbb{C}_\infty$. By [Har77, Theorem I.5.3.], Y is a non-empty open subset of Z_1 and Y is also defined over E.

Let $y \in Y(\mathbb{C}_\infty) \subset S(\mathbb{C}_\infty)$ be a geometric point of Y. We denote by $\pi_1^{\mathrm{arithm}}(Y, y)$ the arithmetic fundamental group of the variety Y over E, i.e., $\pi_1^{\mathrm{arithm}}(Y, y) := \pi_1(Y_0, y)$ if $Y = (Y_0)_{\mathbb{C}_\infty}$ for a scheme Y_0 over E. Furthermore we fix a geometric point $x = [(\overline{\omega}, h)] \in S^{(\mathfrak{p})}(\mathbb{C}_\infty)$ with $\pi_\mathfrak{p}(x) = y$ and consider the monodromy representation

$$\rho : \pi_1^{\mathrm{arithm}}(Y, y) \longrightarrow \mathcal{K}_\mathfrak{p}$$

associated to $x \in S^{(\mathfrak{p})}(\mathbb{C}_\infty)$ and the $\mathcal{K}_\mathfrak{p}$-cover $\pi_\mathfrak{p}$.

By [BP05, Theorem 4] the image of ρ is open in $\mathrm{GL}_r(F_\mathfrak{p})$ under the assumptions:

- $\mathcal{K}$ is amply small;
- Y is a smooth irreducible locally closed subvariety of S with $\dim Y \geqslant 1$;
- the Zariski closure of Y in S is Hodge-generic.

These assumptions are satisfied in our case, and hence $\mathcal{K}'_\mathfrak{p} := \rho(\pi_1^{\mathrm{arithm}}(Y, y))$ is open in $\mathcal{K}_\mathfrak{p}$.

Now we set $\mathcal{K}' := \mathcal{K}'_\mathfrak{p} \times \mathcal{K}^{(\mathfrak{p})}$ and consider the canonical projection

$$\pi : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}.$$

The orbit of the point $x' := [(\overline{\varpi}, h)] \in S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty)$ lying between our base points $x \in S^{(\mathfrak{p})}(\mathbb{C}_\infty)$ and $y \in S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ under the action of $\pi_1^{\mathrm{arithm}}(Y, y)$ on the fiber $\pi^{-1}(y)$ equals

$$\{[(\overline{\varpi}, hk'^{-1}_{\mathfrak{p}})] \in S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty) : k'_{\mathfrak{p}} \in \rho(\pi_1^{\mathrm{arithm}}(Y, y)) = \mathcal{K}'_{\mathfrak{p}}\}$$

and is therefore of cardinality 1. Hence, the E-irreducible component Y' of $\pi^{-1}(Y)$ containing x' is mapped isomorphically onto Y by π. Since Y is irreducible over $\mathbb{C}_\infty$, it follows that Y' is also irreducible over $\mathbb{C}_\infty$.

Note, furthermore, for any open subgroup $\tilde{\mathcal{K}}'_{\mathfrak{p}} \subset \mathcal{K}'_{\mathfrak{p}}$ and $\tilde{\mathcal{K}}' := \tilde{\mathcal{K}}'_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}$ with canonical projection $\pi' : S^r_{F,\tilde{\mathcal{K}}'} \to S^r_{F,\mathcal{K}'}$ that

$$\pi'^{-1}(x') = \{[(\overline{\varpi}, hk'_{\mathfrak{p}})] \in S^r_{F,\tilde{\mathcal{K}}'}(\mathbb{C}_\infty) : k'_{\mathfrak{p}} \in \mathcal{K}'_{\mathfrak{p}}\}$$

is exactly one orbit under the action of $\pi_1^{\mathrm{arithm}}(Y, y)$ on $\pi'^{-1}(\pi^{-1}(y))$. Therefore, $\pi'^{-1}(Y')$ is E-irreducible. Since this holds for every open subgroup $\tilde{\mathcal{K}}'_{\mathfrak{p}} \subset \mathcal{K}'_{\mathfrak{p}}$, this implies that $T_{h_{\mathfrak{p}}}Y'$ is E-irreducible for all $h_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$.

We now define Z'_1 to be the Zariski closure of Y' in $S^r_{F,\mathcal{K}'}$. Since Y' is irreducible over $\mathbb{C}_\infty$, its Zariski closure Z'_1 is also irreducible over $\mathbb{C}_\infty$, and, moreover, by dimension reasons, an irreducible component of $\pi^{-1}(Z_1)$ over $\mathbb{C}_\infty$. Since Y' is also E-irreducible, we similarly conclude that Z'_1 is an E-irreducible component of $\pi^{-1}(Z_1)$.

Note that, for all $h_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$, the projections π_1 and $\pi_{h_{\mathfrak{p}}}$ in the definition of the Hecke correspondence $T_{h_{\mathfrak{p}}}$ on $S^r_{F,\mathcal{K}'}$ are open and closed because they are finite and étale. By the E-irreducibility of $T_{h_{\mathfrak{p}}}Y'$ this implies that

$$T_{h_{\mathfrak{p}}}Z'_1 = \pi_{h_{\mathfrak{p}}}(\pi_1^{-1}(\overline{Y'})) = \overline{\pi_{h_{\mathfrak{p}}}(\pi_1^{-1}(Y'))} = \overline{T_{h_{\mathfrak{p}}}Y'}$$

is E-irreducible and concludes Step (i).

Step (ii). By the assumption $Z \subset T_g Z$, the irreducible component Z_1 of Z is contained in $T_g Z_i$ for some i. Since Z is F-irreducible, there is an element $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$ with $Z_i = \sigma(Z_1)$. This gives for $Z'_1 \subset S^r_{F,\mathcal{K}'}$

$$Z'_1 \subset \pi^{-1}(Z_1) \subset \pi^{-1}(T_g\sigma(Z_1)) = \sigma(\pi^{-1}(T_g Z_1)), \tag{6.1.1}$$

where the last equality holds because all our projection morphisms are defined over F.

A direct computation shows that

$$\pi^{-1}(T_g Z_1) = \bigcup_{i,j=1}^{l} T_{k_i^{-1} g_{\mathfrak{p}} k_j} Z'_1 \tag{6.1.2}$$

where $\{k_1, \ldots, k_l\}$ is a set of representatives for the left cosets in $\mathcal{K}_{\mathfrak{p}}/\mathcal{K}'_{\mathfrak{p}}$. By Step (i), all $T_{k_i^{-1} g_{\mathfrak{p}} k_j} Z'_1$ are E-irreducible.

Since Z'_1 is E-irreducible, the relations (6.1.1) and (6.1.2) imply the existence of indices i and j such that for $h_{\mathfrak{p}} := k_i^{-1} g_{\mathfrak{p}} k_j$

$$Z'_1 = \sigma(T_{h_{\mathfrak{p}}} Z'_1).$$

Iterating this gives the inclusion

$$Z'_1 = \sigma(T_{h_{\mathfrak{p}}}\sigma(T_{h_{\mathfrak{p}}} Z'_1)) = \sigma^2(T_{h_{\mathfrak{p}}}(T_{h_{\mathfrak{p}}} Z'_1)) \supset \sigma^2(T_{h_{\mathfrak{p}}^2} Z'_1),$$

which must be an equality because both sides are of the same dimension and Z_1' is E-irreducible. Repeating the same argument gives

$$Z_1' = \sigma^i(T_{h_{\mathfrak{p}}^i} Z_1')$$

for all $i \geqslant 1$. There is an $n \geqslant 1$ with $\sigma^n \in \mathrm{Gal}(F^{\mathrm{sep}}/E)$. Since $T_{h_{\mathfrak{p}}^n} Z_1'$ is defined over E, we conclude the relations

$$\begin{aligned} Z_1' &= \sigma^n(T_{h_{\mathfrak{p}}^n} Z_1') = T_{h_{\mathfrak{p}}^n} Z_1', \\ T_{h_{\mathfrak{p}}^{-n}} Z_1' &= T_{h_{\mathfrak{p}}^{-n}}(T_{h_{\mathfrak{p}}^n} Z_1') \supset Z_1'. \end{aligned}$$

Again, the latter relation must be an equality because $T_{h_{\mathfrak{p}}^{-n}} Z_1'$ is E-irreducible and of the same dimension as Z_1'. Note that the cyclic subgroup of $\mathrm{PGL}_r(F_{\mathfrak{p}})$ generated by the image of $h_{\mathfrak{p}}^n = (k_i^{-1} g_{\mathfrak{p}} k_j)^n$ is unbounded by our assumption. So we can apply Proposition 6.1.1 and conclude that Z_1' is an irreducible component of $S^r_{F,\mathcal{K}'}$ over $\mathbb{C}_\infty$. □

7. Existence of good primes and suitable Hecke operators

7.1 Good primes

In this subsection, $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ denotes a Drinfeld modular subvariety of a Drinfeld modular variety $S^r_{F,\mathcal{K}}$ associated to the datum (F', b).

Definition 7.1.1. For a prime $\mathfrak{p}$ of F, a free $A_{\mathfrak{p}}$-submodule $\Lambda_{\mathfrak{p}} \subset F_{\mathfrak{p}}^r$ of rank r is called an *$A_{\mathfrak{p}}$-lattice.*

Definition 7.1.2. A prime $\mathfrak{p}$ is called *good* for $X \subset S^r_{F,\mathcal{K}}$ if there exists an $A_{\mathfrak{p}}$-lattice $\Lambda_{\mathfrak{p}} \subset F_{\mathfrak{p}}^r$ such that the following hold.

(i) We have $\mathcal{K} = \mathcal{K}_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}$ with $\mathcal{K}_{\mathfrak{p}}$ the kernel of the natural map

$$\mathrm{Stab}_{\mathrm{GL}_r(F_{\mathfrak{p}})}(\Lambda_{\mathfrak{p}}) \to \mathrm{Aut}_{k(\mathfrak{p})}(\Lambda_{\mathfrak{p}}/\mathfrak{p}\cdot\Lambda_{\mathfrak{p}})$$

for a $\mathcal{K}^{(\mathfrak{p})} \subset \mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$.

(ii) There is a prime $\mathfrak{p}'$ of F' above $\mathfrak{p}$ with local degree $[F'_{\mathfrak{p}'}/F_{\mathfrak{p}}] = 1$.

(iii) The $A_{\mathfrak{p}}$-module $b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})$ is an $A'_{\mathfrak{p}}$-submodule of $F'^{r'}_{\mathfrak{p}}$.

Remarks.

- The definition is independent of the datum (F', b) describing X because F' is uniquely determined by X and $b'_{\mathfrak{p}} = s_{\mathfrak{p}} \circ b_{\mathfrak{p}} \circ k_{\mathfrak{p}}$ with $s_{\mathfrak{p}} \in \mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ and $k_{\mathfrak{p}} \in \mathcal{K}_{\mathfrak{p}} \subset \mathrm{Stab}_{\mathrm{GL}_r(F_{\mathfrak{p}})}(\Lambda_{\mathfrak{p}})$ for a second datum (F', b') describing X by Corollary 3.3.6.
- The existence of a good prime $\mathfrak{p}$ for X implies that the reflex field F' of X is separable over F because there exists a prime $\mathfrak{p}'$ of F' which is unramified over F.
- If $\Lambda_{\mathfrak{p}} = s_{\mathfrak{p}} A_{\mathfrak{p}}^r$ for an $s_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$, then condition (i) is equivalent to

  $$\mathcal{K} = s_{\mathfrak{p}} \mathcal{K}(\mathfrak{p}) s_{\mathfrak{p}}^{-1} \times \mathcal{K}^{(\mathfrak{p})},$$

  where $\mathcal{K}(\mathfrak{p}) \subset \mathrm{GL}_r(A_{\mathfrak{p}})$ is the principal congruence subgroup modulo $\mathfrak{p}$.
- Condition (i) implies that $\mathcal{K}' = (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'}) = \mathcal{K}'_{\mathfrak{p}} \times \mathcal{K}'^{(\mathfrak{p})}$ with $\mathcal{K}'_{\mathfrak{p}}$ the kernel of the natural map

  $$\mathrm{Stab}_{\mathrm{GL}_{r'}(F'_{\mathfrak{p}})}(b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})) \to \mathrm{Aut}_{k(\mathfrak{p})}(b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})/\mathfrak{p}\cdot b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})).$$

Since $b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})$ is an $A'_{\mathfrak{p}}$-submodule of ${F'_{\mathfrak{p}'}}^{r'}$ by condition (iii), this means that $\mathcal{K}'_{\mathfrak{p}}$ is conjugate to the principal congruence subgroup modulo $\mathfrak{p}$ of $\mathrm{GL}_{r'}(A'_{\mathfrak{p}})$.

Proposition 7.1.3. *Let $\mathfrak{p}$ be a good prime for X. Suppose that X is contained in a Drinfeld modular subvariety $X' = \iota^{F''}_{F,b'}(S^{r''}_{F'',\mathcal{K}''}) \subset S^{r}_{F,\mathcal{K}}$.*

Then $X'' := (\iota^{F''}_{F,b'})^{-1}(X)$ is a Drinfeld modular subvariety of $S^{r''}_{F'',\mathcal{K}''}$ and there is a prime $\mathfrak{p}''$ of F'' above $\mathfrak{p}$ with $k(\mathfrak{p}) = k(\mathfrak{p}'')$ such that $\mathfrak{p}''$ is good for $X'' \subset S^{r''}_{F'',\mathcal{K}''}$.

Proof. By Corollary 3.3.5, $X'' = (\iota^{F''}_{F,b'})^{-1}(X)$ is a Drinfeld modular subvariety of $S^{r''}_{F'',\mathcal{K}''}$. In the proof of Corollary 3.3.5 we saw that $F \subset F'' \subset F'$ and there are an $\mathbb{A}^{f}_{F''}$-linear isomorphism $c : (\mathbb{A}^{f}_{F''})^{r''} \xrightarrow{\sim} (\mathbb{A}^{f}_{F'})^{r'}$ and a $k \in \mathcal{K}$ such that

$$b = c \circ b' \circ k \tag{7.1.1}$$

and $X'' = \iota^{F'}_{F'',c}(S^{r'}_{F',\mathcal{K}'})$. The situation is summarized in the following commutative diagram where all arrows are bijections on $\mathbb{C}_\infty$-valued points.

$$\begin{tikzcd}
 & X \arrow[r, phantom, "\subset"] & X' \arrow[r, phantom, "\subset"] & S^{r}_{F,\mathcal{K}} \\
S^{r'}_{F',\mathcal{K}'} \arrow[ur, "\iota^{F'}_{F,b}"] \arrow[dr, "\iota^{F'}_{F'',c}"'] & & & \\
 & X'' \arrow[uu, "\iota^{F''}_{F,b'}|_{X''}"'] \arrow[r, phantom, "\subset"] & S^{r''}_{F'',\mathcal{K}''} \arrow[uu, "\iota^{F''}_{F,b'}"'] &
\end{tikzcd}$$

Let $\Lambda_{\mathfrak{p}}$ be an $A_{\mathfrak{p}}$-lattice and $\mathfrak{p}'$ a prime of F' above $\mathfrak{p}$ for which the conditions (i)–(iii) of Definition 7.1.2 are satisfied. We define $\mathfrak{p}''$ to be the prime of F'' lying between $\mathfrak{p}$ and $\mathfrak{p}'$. Since $\mathfrak{p}'$ is of local degree 1 over F, we have $k(\mathfrak{p}) = k(\mathfrak{p}') = k(\mathfrak{p}'')$. We now show that $\mathfrak{p}''$ is a good prime for $X'' = \iota^{F'}_{F'',c}(S^{r'}_{F',\mathcal{K}'}) \subset S^{r''}_{F'',\mathcal{K}''}$.

By construction, $\mathfrak{p}'$ is also of local degree 1 over F'', i.e., condition (ii) in Definition 7.1.2 is satisfied for $\mathfrak{p}''$.

By condition (iii), $b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})$ is an $A'_{\mathfrak{p}}$-submodule of ${F'_{\mathfrak{p}}}^{r'}$. Hence, we can write

$$b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}) = \Lambda'_{\mathfrak{p}''} \times \Lambda'^{(\mathfrak{p}'')}$$

with $\Lambda'_{\mathfrak{p}''} \subset {F'_{\mathfrak{p}''}}^{r'}$ an $A'_{\mathfrak{p}''}$-submodule (recall that $A'_{\mathfrak{p}''} = A' \otimes_{A''} A''_{\mathfrak{p}''}$ by our conventions). Since c is $\mathbb{A}^{f}_{F''}$-linear and $A'' \subset A'$, it follows that $\Lambda''_{\mathfrak{p}''} := c^{-1}_{\mathfrak{p}''}(\Lambda'_{\mathfrak{p}''})$ is an $A''_{\mathfrak{p}''}$-lattice in ${F''_{\mathfrak{p}''}}^{r''}$. By construction, condition (iii) in Definition 7.1.2 holds for $\mathfrak{p}''$ and $\Lambda''_{\mathfrak{p}''}$.

We note that condition (i) implies $\mathcal{K}'' = (b'\mathcal{K}b'^{-1}) \cap \mathrm{GL}_{r''}(\mathbb{A}^{f}_{F''}) = \mathcal{K}''_{\mathfrak{p}} \times \mathcal{K}''^{(\mathfrak{p})}$ with $\mathcal{K}''_{\mathfrak{p}}$ the kernel of the natural map $\mathrm{Stab}_{\mathrm{GL}_{r''}(F''_{\mathfrak{p}})}(b'_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})) \to \mathrm{Aut}_{k(\mathfrak{p})}(b'_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})/\mathfrak{p} \cdot b'_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}))$. Note that

$$b'_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}) = b'_{\mathfrak{p}}(k_{\mathfrak{p}}\Lambda_{\mathfrak{p}}) = c^{-1}_{\mathfrak{p}}(b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})) = c^{-1}_{\mathfrak{p}}(\Lambda'_{\mathfrak{p}''} \times \Lambda'^{(\mathfrak{p}'')}) = \Lambda''_{\mathfrak{p}''} \times \Lambda''^{(\mathfrak{p}'')}.$$

Since $k(\mathfrak{p}) = k(\mathfrak{p}'')$ and $\mathfrak{p}A''_{\mathfrak{p}''} = \mathfrak{p}''A''_{\mathfrak{p}''}$, we therefore see that condition (i) is also satisfied for $\mathfrak{p}''$ and $\Lambda''_{\mathfrak{p}''}$. □

7.2 Suitable Hecke correspondences

PROPOSITION 7.2.1. *Let* $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'}) \subset S^r_{F,\mathcal{K}}$ *be a Drinfeld modular subvariety and* $g' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. *Then, we have*

$$X \subset T_g X$$

for $g := b^{-1} \circ g' \circ b \in \mathrm{GL}_r(\mathbb{A}^f_F)$.

Proof. Let $p = \iota^{F'}_{F,b}([(\overline{\omega'}, h')]) \in X(\mathbb{C}_\infty)$ for some $\overline{\omega'} \in \Omega^{r'}_{F'}$ and $h' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. Then we have

$$p = [(\overline{\omega' \circ \varphi}, \varphi^{-1} \circ h' \circ b)] = [(\overline{\omega' \circ \varphi}, \varphi^{-1} \circ h'g' \circ b \circ g^{-1})]$$

for an F-linear isomorphism $\varphi : F^r \xrightarrow{\sim} F'^{r'}$, and therefore p lies in $T_g(\iota^{F'}_{F,b}([(\overline{\omega'}, h'g')]))$ and therefore in $T_g X(\mathbb{C}_\infty)$. Since $p \in X(\mathbb{C}_\infty)$ was arbitrary, we conclude $X \subset T_g X$. □

THEOREM 7.2.2. *Let* $\mathfrak{p}$ *be a good prime for a Drinfeld modular subvariety* $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'}) \subset S^r_{F,\mathcal{K}}$ *and let* $\mathfrak{p}'$ *be a prime of* F' *above* $\mathfrak{p}$ *with local degree* 1 *over* F. *Then there is a*

$$g' = (1, \ldots, g'_{\mathfrak{p}'}, \ldots, 1) \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$$

with $g'_{\mathfrak{p}'} \in \mathrm{GL}_{r'}(F'_{\mathfrak{p}'})$ *such that the following hold for* $g := b^{-1} \circ g' \circ b \in \mathrm{GL}_r(\mathbb{A}^f_F)$:

(i) $X \subset T_g X$;

(ii) $\deg T_g = [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g] = |k(\mathfrak{p})|^{r-1}$;

(iii) *for all* $k_1, k_2 \in \mathcal{K}_\mathfrak{p}$, *the cyclic subgroup of* $\mathrm{PGL}_r(F_\mathfrak{p})$ *generated by the image of* $k_1 \cdot g_\mathfrak{p} \cdot k_2$ *is unbounded.*

Proof. Suppose that the conditions (i)–(iii) in Definition 7.1.2 are satisfied for the $A_\mathfrak{p}$-lattice $\Lambda_\mathfrak{p} \subset F^r_\mathfrak{p}$.

By condition (iii) in Definition 7.1.2, $b_\mathfrak{p}(\Lambda_\mathfrak{p})$ is an $A'_\mathfrak{p}$-submodule of $F'^{r'}_\mathfrak{p}$. Hence we can write

$$b_\mathfrak{p}(\Lambda_\mathfrak{p}) = \Lambda'_{\mathfrak{p}'} \times \Lambda'^{(\mathfrak{p}')}_\mathfrak{p}$$

with $\Lambda'_{\mathfrak{p}'} \subset F'^{r'}_{\mathfrak{p}'}$ a free $A'_{\mathfrak{p}'}$-submodule of rank r'. Let $g'_{\mathfrak{p}'} : F'^{r'}_{\mathfrak{p}'} \to F'^{r'}_{\mathfrak{p}'}$ be given by

$$\mathrm{diag}(\pi_{\mathfrak{p}'}, 1, \ldots, 1)$$

for a uniformizer $\pi_{\mathfrak{p}'} \in A'_{\mathfrak{p}'}$ with respect to an $A'_{\mathfrak{p}'}$-basis of $\Lambda'_{\mathfrak{p}'}$.

We now check the conditions (i)–(iii) for $g := b^{-1} \circ g' \circ b$ where $g' = (1, \ldots, g'_{\mathfrak{p}'}, \ldots, 1) \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. Statement (i) follows by Proposition 7.2.1.

For conditions (ii) and (iii), note that each $A'_{\mathfrak{p}'}$-basis of $\Lambda'_{\mathfrak{p}'}$ is also an $A_\mathfrak{p}$-basis of $\Lambda'_{\mathfrak{p}'}$ and can be extended to an $A_\mathfrak{p}$-basis of $b_\mathfrak{p}(\Lambda_\mathfrak{p})$ because the local degree $[F'_{\mathfrak{p}'}/F_\mathfrak{p}]$ is equal to 1. In particular, the $\mathfrak{p}$-component $g'_\mathfrak{p} \in \mathrm{GL}_{r'}(F'_\mathfrak{p}) = \prod_{\mathfrak{q}'|\mathfrak{p}} \mathrm{GL}_{r'}(F'_{\mathfrak{q}'})$ of $g' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ viewed as an $F_\mathfrak{p}$-linear map $F'^{r'}_\mathfrak{p} \to F'^{r'}_\mathfrak{p}$ is given by the diagonal matrix

$$D_\mathfrak{p} := \mathrm{diag}(\pi_\mathfrak{p}, 1, \ldots, 1) \in \mathrm{GL}_r(F_\mathfrak{p})$$

with respect to some $A_\mathfrak{p}$-basis B' of $b_\mathfrak{p}(\Lambda_\mathfrak{p})$ for a uniformizer $\pi_\mathfrak{p} \in A_\mathfrak{p}$. It follows that the $\mathfrak{p}$-component $g_\mathfrak{p} : F^r_\mathfrak{p} \to F^r_\mathfrak{p}$ of $g = b^{-1} \circ g' \circ b \in \mathrm{GL}_r(\mathbb{A}^f_F)$ is also given by $D_\mathfrak{p}$ with respect to the $A_\mathfrak{p}$-basis $b^{-1}_\mathfrak{p}(B')$ of $\Lambda_\mathfrak{p}$. Hence, there is an $s_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$ such that

$$g_\mathfrak{p} = s_\mathfrak{p} D_\mathfrak{p} s^{-1}_\mathfrak{p},$$
$$\Lambda_\mathfrak{p} = s_\mathfrak{p} A^r_\mathfrak{p}.$$

By the remark after Definition 7.1.2, we therefore have

$$\mathcal{K}_{\mathfrak{p}} = s_{\mathfrak{p}} \mathcal{K}(\mathfrak{p}) s_{\mathfrak{p}}^{-1}$$

with $\mathcal{K}(\mathfrak{p})$ the principal congruence subgroup of $\mathrm{GL}_r(A_{\mathfrak{p}})$ modulo $\mathfrak{p}$.

Hence, we can and do assume $\mathcal{K}_{\mathfrak{p}} = \mathcal{K}(\mathfrak{p})$ and $g_{\mathfrak{p}} = D_{\mathfrak{p}}$ because conditions (ii) and (iii) are invariant under conjugation.

For the proof of condition (ii), consider the map

$$\alpha : \begin{array}{ccl} \mathcal{K}(\mathfrak{p}) & \longrightarrow & (A_{\mathfrak{p}}/(\pi_{\mathfrak{p}}))^{r-1} \\ h & \longmapsto & ([\pi_{\mathfrak{p}}^{-1} \cdot h_{21}], \ldots, [\pi_{\mathfrak{p}}^{-1} \cdot h_{r1}]). \end{array}$$

For $h, h' \in \mathcal{K}(\mathfrak{p})$, we have for $2 \leqslant i \leqslant r$

$$\begin{aligned} \pi_{\mathfrak{p}}^{-1} \cdot (hh')_{i1} &= (\pi_{\mathfrak{p}}^{-1} h_{i1}) h'_{11} + h_{ii} (\pi_{\mathfrak{p}}^{-1} h'_{i1}) + \sum_{j \neq i,1} (\pi_{\mathfrak{p}}^{-1} h_{ij}) h'_{j1} \\ &\equiv \pi_{\mathfrak{p}}^{-1} h_{i1} + \pi_{\mathfrak{p}}^{-1} h'_{i1} + 0 \pmod{\mathfrak{p}}, \end{aligned}$$

and therefore α is a homomorphism of groups. It is, furthermore, surjective, and its kernel is exactly equal to $\mathcal{K}(\mathfrak{p}) \cap D_{\mathfrak{p}} \mathcal{K}(\mathfrak{p}) D_{\mathfrak{p}}^{-1}$. Hence, we have

$$[\mathcal{K} : \mathcal{K} \cap g^{-1} \mathcal{K} g] = [\mathcal{K}_{\mathfrak{p}} : \mathcal{K}_{\mathfrak{p}} \cap g_{\mathfrak{p}}^{-1} \mathcal{K}_{\mathfrak{p}} g_{\mathfrak{p}}] = |k(\mathfrak{p})|^{r-1}.$$

For condition (iii), let $k_1, k_2 \in \mathcal{K}_{\mathfrak{p}} = \mathcal{K}(\mathfrak{p})$ be arbitrary. We prove that the eigenvalues of $(k_1 g_{\mathfrak{p}} k_2)^{-1} = k_2^{-1} D_{\mathfrak{p}}^{-1} k_1^{-1}$ do not all have the same $\mathfrak{p}$-valuation by showing that the Newton polygon of the characteristic polynomial

$$\chi(\lambda) = \lambda^r + a_{r-1} \lambda^{r-1} + \cdots + a_1 \lambda + a_0$$

of $k_2^{-1} D_{\mathfrak{p}}^{-1} k_1^{-1}$ consists at least of two line segments. This implies that the cyclic subgroup of $\mathrm{PGL}_r(F_{\mathfrak{p}})$ generated by the image of $k_1 g_{\mathfrak{p}} k_2$ is unbounded.

Since k_1, k_2 are elements of $\mathrm{GL}_r(A_{\mathfrak{p}})$, we have $\det(k_1), \det(k_2) \in A_{\mathfrak{p}}^*$ and hence

$$v_{\mathfrak{p}}(a_0) = v_{\mathfrak{p}}(\det(k_2^{-1} D_{\mathfrak{p}}^{-1} k_1^{-1})) = 0 - v_{\mathfrak{p}}(\det(D_{\mathfrak{p}})) + 0 = -1.$$

The coefficient a_{r-1} can be expressed as

$$a_{r-1} = -\mathrm{tr}(k_2^{-1} D_{\mathfrak{p}}^{-1} k_1^{-1}) = -\sum_i (k_2^{-1})_{i1} \pi_{\mathfrak{p}}^{-1} (k_1^{-1})_{1i} - \sum_i \sum_{j \neq 1} (k_2^{-1})_{ij} (k_1^{-1})_{ji}.$$

Because of $k_1, k_2 \in \mathcal{K}(\mathfrak{p})$, we have $v_{\mathfrak{p}}((k_1^{-1})_{ij}), v_{\mathfrak{p}}((k_2^{-1})_{ij}) \geqslant 0$ with equality exactly for $i = j$. Therefore, in the above expression for a_{r-1}, the summand for $i = 1$ in the first sum has $\mathfrak{p}$-valuation -1 and all the other summands have $\mathfrak{p}$-valuation at least 0. We conclude

$$v_{\mathfrak{p}}(a_{r-1}) = -1.$$

Hence, the point $(r-1, v_{\mathfrak{p}}(a_{r-1}))$ lies below the line through $(0, v_{\mathfrak{p}}(a_0))$ and $(r, 0)$. This implies that the Newton polygon of χ consists at least of two line segments. □

7.3 Existence of good primes

Proposition 7.3.1. *Let $X = \iota_{F,b}^{F'}(S_{F',\mathcal{K}'}^{r'}) \subset S_{F,\mathcal{K}}^r$ be a Drinfeld modular subvariety and $\mathfrak{p}$ a prime of F such that the following hold.*

(i) *There is a prime* $\mathfrak{p}'$ *of* F' *above* $\mathfrak{p}$ *with local degree* $[F'_{\mathfrak{p}'}/F_{\mathfrak{p}}]=1$.

(ii) *We have* $\mathcal{K}=\mathcal{K}_{\mathfrak{p}}\times\mathcal{K}^{(\mathfrak{p})}$ *with* $\mathcal{K}_{\mathfrak{p}}\subset\mathrm{GL}_r(F_{\mathfrak{p}})$ *a maximal compact subgroup and* $\mathcal{K}^{(\mathfrak{p})}\subset\mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$.

(iii) *The subgroup* $\mathcal{K}'_{\mathfrak{p}}:=(b_{\mathfrak{p}}\mathcal{K}_{\mathfrak{p}}b_{\mathfrak{p}}^{-1})\cap\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ *of* $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ *is maximal compact.*

Then there is a subgroup $\tilde{\mathcal{K}}\subset\mathcal{K}$ *and a Drinfeld modular subvariety* $\tilde{X}\subset S^r_{F,\tilde{\mathcal{K}}}$ *such that:*

(a) $\pi_1(\tilde{X})=X$ *for the canonical projection* $\pi_1:S^r_{F,\tilde{\mathcal{K}}}\to S^r_{F,\mathcal{K}}$;

(b) $\mathfrak{p}$ *is good for* $\tilde{X}\subset S^r_{F,\tilde{\mathcal{K}}}$;

(c) $[\mathcal{K}:\tilde{\mathcal{K}}]<|k(\mathfrak{p})|^{r^2}$.

Proof. As $\mathcal{K}_{\mathfrak{p}}$ is a maximal compact subgroup of $\mathrm{GL}_r(F_{\mathfrak{p}})$, there is an $s_{\mathfrak{p}}\in\mathrm{GL}_r(F_{\mathfrak{p}})$ with $\mathcal{K}_{\mathfrak{p}}=s_{\mathfrak{p}}\mathrm{GL}_r(A_{\mathfrak{p}})s_{\mathfrak{p}}^{-1}$. We define $\Lambda_{\mathfrak{p}}$ to be the lattice $s_{\mathfrak{p}}\cdot A^r_{\mathfrak{p}}$, for which we have

$$\mathcal{K}_{\mathfrak{p}}=\mathrm{Stab}_{\mathrm{GL}_r(F_{\mathfrak{p}})}(\Lambda_{\mathfrak{p}}).$$

Now, we let $\tilde{\mathcal{K}}_{\mathfrak{p}}$ be the kernel of the natural map

$$\mathrm{Stab}_{\mathrm{GL}_r(F_{\mathfrak{p}})}(\Lambda_{\mathfrak{p}})\to\mathrm{Aut}_{k(\mathfrak{p})}(\Lambda_{\mathfrak{p}}/\mathfrak{p}\cdot\Lambda_{\mathfrak{p}})$$

and define $\tilde{\mathcal{K}}:=\tilde{\mathcal{K}}_{\mathfrak{p}}\times\mathcal{K}^{(\mathfrak{p})}$.

By construction, we get the upper bound (c) for the index of $\tilde{\mathcal{K}}$ in $\mathcal{K}$:

$$[\mathcal{K}:\tilde{\mathcal{K}}]=[\mathcal{K}_{\mathfrak{p}}:\tilde{\mathcal{K}}_{\mathfrak{p}}]=|\mathrm{Aut}_{k(\mathfrak{p})}(\Lambda_{\mathfrak{p}}/\mathfrak{p}\cdot\Lambda_{\mathfrak{p}})|=|\mathrm{GL}_r(k(\mathfrak{p}))|<|k(\mathfrak{p})|^{r^2}.$$

We denote by $\tilde{\iota}^{F'}_{F,b}$ the inclusion $S^{r'}_{F',\tilde{\mathcal{K}}'}\to S^r_{F,\tilde{\mathcal{K}}}$ associated to the same datum (F',b) as $\iota^{F'}_{F,b}$ and set $\tilde{X}:=\iota^{F'}_{F,b}(S^{r'}_{F',\tilde{\mathcal{K}}'})$. The proof of Lemma 3.3.2(i) shows that $\tilde{X}$ is a Drinfeld modular subvariety of $S^r_{F,\tilde{\mathcal{K}}}$ with $\pi_1(\tilde{X})=X$.

It remains to show that $\mathfrak{p}$ is good for $\tilde{X}\subset S^r_{F,\tilde{\mathcal{K}}}$. Condition (i) in Definition 7.1.2 is satisfied by construction of $\tilde{K}$ and condition (ii) by assumption. So we only have to check that $\Lambda'_{\mathfrak{p}}:=b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})$ is an $A'_{\mathfrak{p}}$-submodule of $F'^{r'}_{\mathfrak{p}}$. Since $\mathcal{K}_{\mathfrak{p}}$ is the stabilizer of $\Lambda_{\mathfrak{p}}$ in $\mathrm{GL}_r(F_{\mathfrak{p}})$, the stabilizer of $\Lambda'_{\mathfrak{p}}$ in $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ is exactly

$$\mathcal{K}'_{\mathfrak{p}}:=(b_{\mathfrak{p}}\mathcal{K}_{\mathfrak{p}}b_{\mathfrak{p}}^{-1})\cap\mathrm{GL}_{r'}(F'_{\mathfrak{p}}),$$

which is a maximal compact subgroup of $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ by assumption. Since $A'^*_{\mathfrak{p}}$ is the unique maximal compact subgroup of $F'^*_{\mathfrak{p}}$, we therefore have

$$\mathrm{Stab}_{F'^*_{\mathfrak{p}}}(\Lambda'_{\mathfrak{p}})=\mathcal{K}'_{\mathfrak{p}}\cap F'^*_{\mathfrak{p}}=A'^*_{\mathfrak{p}},$$

where $F'^*_{\mathfrak{p}}$ is embedded in $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ as scalars. Since $A'^*_{\mathfrak{p}}$ generates $A'_{\mathfrak{p}}$ as a ring, we conclude that $\Lambda'_{\mathfrak{p}}$ is an $A'_{\mathfrak{p}}$-submodule of $F'^{r'}_{\mathfrak{p}}$. □

THEOREM 7.3.2. *Let* $S=S^r_{F,\mathcal{K}}$ *be a Drinfeld modular variety and* $N>0$. *For every prime* $\mathfrak{q}$ *of* F, *denote by* $\mathcal{K}_{\mathfrak{q}}$ *the projection of* $\mathcal{K}$ *to* $\mathrm{GL}_r(F_{\mathfrak{q}})$. *Then, for almost all Drinfeld modular subvarieties* $X=\iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ *with separable reflex field* F' *over* F, *there is a prime* $\mathfrak{p}$ *with the following properties.*

(i) *There is a prime* $\mathfrak{p}'$ *of* F' *above* $\mathfrak{p}$ *with local degree* $[F'_{\mathfrak{p}'}/F_{\mathfrak{p}}]=1$.

(ii) *The subgroup* $\mathcal{K}_{\mathfrak{p}}$ *of* $\mathrm{GL}_r(F_{\mathfrak{p}})$ *is maximal compact and* $\mathcal{K}=\mathcal{K}_{\mathfrak{p}}\times\mathcal{K}^{(\mathfrak{p})}$ *with* $\mathcal{K}^{(\mathfrak{p})}\subset\mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$.

(iii) *The subgroup* $\mathcal{K}'_{\mathfrak{p}} := (b_{\mathfrak{p}}\mathcal{K}_{\mathfrak{p}}b_{\mathfrak{p}}^{-1}) \cap \mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ *of* $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ *is maximal compact.*

(iv) *We have* $|k(\mathfrak{p})|^N < D(X)$ *where* $D(X)$ *denotes the predegree of* X *from Definition 3.3.8.*

Before giving the proof of this theorem, we show two lemmas.

Lemma 7.3.3. *There are absolute constants* $C_1, C_2 > 0$ *such that for all global function fields* F' *with field of constants containing* $\mathbb{F}_q$

$$g(F') \leqslant C_1 + C_2 \cdot \log_q(|\mathrm{Cl}(F')|)$$

where $g(F')$ *denotes the genus of* F' *and* $|\mathrm{Cl}(F')|$ *the class number of* F'.

Proof. Let F' be a global function field with field of constants $\mathbb{F}_{q'} \supset \mathbb{F}_q$. Then, with Proposition 4.3.4 we get the estimate

$$|\mathrm{Cl}(F')| \geqslant \frac{(q'-1)(q'^{2g(F')} - 2g(F')q'^{g(F')} + 1)}{2g(F')(q'^{g(F')+1} - 1)} \geqslant (q-1) \cdot \left(\frac{q^{g(F')-1}}{2g(F')} - \frac{1}{q}\right),$$

which implies

$$\frac{q^{g(F')-1}}{g(F')} \leqslant \frac{2|\mathrm{Cl}(F')|}{q-1} + \frac{2}{q} \leqslant 4|\mathrm{Cl}(F')|,$$

and, because $x/2 \geqslant \log_q x - 1$,

$$\frac{g(F')}{2} - 2 \leqslant g(F') - 1 - \log_q g(F') \leqslant \log_q(4|\mathrm{Cl}(F')|).$$

So the desired estimate holds for the absolute constants $C_1 := 8$ and $C_2 := 2$. □

Lemma 7.3.4. *There are constants* $C_3, C_4 > 0$ *only depending on* r *such that for all finite separable extensions* F'/F *of global function fields with* $[F'/F] \leqslant r$

$$g(E') \leqslant C_3 + C_4 \cdot g(F')$$

where E' *denotes the normal closure of the extension* F'/F.

Proof. Let F'/F be a finite separable extension of global function fields of degree $r' \leqslant r$. Its normal closure E' is the compositum of all Galois conjugates $F'_1, \ldots, F'_{r'}$ of F' over F. We use Castelnuovo's inequality [Sti93, Theorem III.10.3] to bound its genus.

If a global function field K *is the compositum of two subfields* K_1 *and* K_2 *with* $n_i := [K/K_i] < \infty$ *for* $i = 1, 2$, *then*

$$g(K) \leqslant n_1 \cdot g(K_1) + n_2 \cdot g(K_2) + (n_1 - 1)(n_2 - 1).$$

For $K_1 = F'_1$ and $K_2 = F'_2$ this gives

$$g(F'_1F'_2) \leqslant r' \cdot g(F') + r' \cdot g(F') + (r'-1)^2 \leqslant 2r' \cdot g(F') + r'^2$$

because all Galois conjugates of F' over F have the same genus, and $[F'_1F'_2/F'_1] \leqslant [F'_2/F] = r'$ and $[F'_1F'_2/F'_2] \leqslant [F'_1/F] = r'$. With induction over k we get

$$g(F'_1 \cdots F'_k) \leqslant kr'^{k-1} \cdot g(F') + (k-1)r'^k,$$

and with $k = r'$ we get

$$g(E') \leqslant r'^{r'} \cdot g(F') + (r'-1) \cdot r'^{r'} \leqslant (r-1)r^r + r^r \cdot g(F').$$

□

Proof of Theorem 7.3.2. For a Drinfeld modular subvariety $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ with separable reflex field over F, we denote by $n(X)$ the number of primes of F for which properties (ii) and (iii) in

Theorem 7.1.3 do not both hold, and by $m(X, N)$ the number of primes of F with properties (i) and (iv). We show the following statements for Drinfeld modular subvarieties X of S with separable reflex field.

(a) We have $n(X) \leqslant C_5 + C_6 \cdot \log_q(i(X))$ for constants C_5, C_6 independent of X where $i(X)$ denotes the index of X as defined in Definition 3.3.8.

(b) There is an $M > 0$ such that $m(X, N) > n(X)$ for all X with $D(X) > M$.

Statement (b) implies the theorem because $D(X) > M$ for almost all Drinfeld modular subvarieties X of S by Theorem 4.3.2.

Proof of (a). For a Drinfeld modular subvariety $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ of S we have

$$\mathcal{K}' = (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$$

and the index $i(X)$ is the index of $\mathcal{K}'$ in a maximal compact subgroup of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$.

For a prime $\mathfrak{p}$ for which property (ii) holds, we can write $\mathcal{K}_\mathfrak{p} = \mathrm{Stab}_{\mathrm{GL}_r(F_\mathfrak{p})}(\Lambda_\mathfrak{p})$ for some $A_\mathfrak{p}$-lattice $\Lambda_\mathfrak{p} \subset F^r_\mathfrak{p}$ and

$$\mathcal{K}'_\mathfrak{p} = (b_\mathfrak{p}\mathcal{K}_\mathfrak{p}b_\mathfrak{p}^{-1}) \cap \mathrm{GL}_{r'}(F'_\mathfrak{p}) = \mathrm{Stab}_{\mathrm{GL}_{r'}(F'_\mathfrak{p})}(\Lambda'_\mathfrak{p})$$

with $\Lambda'_\mathfrak{p} := b_\mathfrak{p}(\Lambda_\mathfrak{p})$. Note that $A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p}$ is a free $A'_\mathfrak{p}$-submodule of rank r' because $A'_\mathfrak{p}$ is a direct product of principal ideal domains. Therefore with Proposition 4.3.6 we get the estimate

$$[\mathrm{Stab}_{\mathrm{GL}_{r'}(F'_\mathfrak{p})}(A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p}) : \mathcal{K}'_\mathfrak{p}] \geqslant C \cdot [A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p} : \Lambda'_\mathfrak{p}]^{1/r}$$

for some constant $C > 0$ only depending on q and r. If $\mathcal{K}'_\mathfrak{p}$ is not a maximal compact subgroup of $\mathrm{GL}_{r'}(F'_\mathfrak{p})$ (i.e., property (iii) does not hold for $\mathfrak{p}$), then $\Lambda'_\mathfrak{p}$ cannot be an $A'_\mathfrak{p}$-submodule of $F'^{r'}_\mathfrak{p}$, i.e., we have $\Lambda'_\mathfrak{p} \subsetneq A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p}$ and

$$[\mathrm{Stab}_{\mathrm{GL}_{r'}(F'_\mathfrak{p})}(A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p}) : \mathcal{K}'_\mathfrak{p}] \geqslant C \cdot |k(\mathfrak{p})|^{1/r}$$

because each finite non-trivial $A_\mathfrak{p}$-module has at least $|k(\mathfrak{p})|$ elements.

Since, for each prime $\mathfrak{p}$ satisfying property (ii), we have $\mathcal{K}' = \mathcal{K}'_\mathfrak{p} \times \mathcal{K}'^{(\mathfrak{p})}$ for some subgroup $\mathcal{K}'^{(\mathfrak{p})} \subset \mathrm{GL}_{r'}(F' \otimes \mathbb{A}^{f,\mathfrak{p}}_F)$, we conclude that

$$i(X) \geqslant C \cdot |k(\mathfrak{p})|^{n_3(X)/r} \geqslant C \cdot q^{n_3(X)/r},$$

where $n_3(X)$ is the number of primes of F for which property (ii) holds, but property (iii) does not hold. If n_2 is the number of primes of F, for which property (ii) does not hold, then we conclude

$$n(X) = n_2 + n_3(X) \leqslant n_2 - r \cdot \log_q(C) + r \cdot \log_q(i(X)).$$

This finishes the proof of (a), because n_2 is independent of X.

Proof of (b). Let X be a Drinfeld modular subvariety of S with separable reflex field F' over F. We denote the normal closure of the extension F'/F by E'. To give a lower bound for $m(X, N)$ we note that all primes $\mathfrak{p}$ of F which completely split in E' satisfy property (i). We bound the number of such primes with fixed degree using an effective version of Čebotarev's theorem.

For the application of Čebotarev's theorem we fix some notations. We denote the constant extension degree of E'/F by n and its geometric extension degree by k. Since we assumed F to have field of constants $\mathbb{F}_q$, the field of constants of E' is $\mathbb{F}_{q^n}$ and $k = [E'/\mathbb{F}_{q^n} \cdot F]$. We furthermore fix a separating transcendence element θ of $F/\mathbb{F}_q$ (i.e., an element θ of F such that $F/\mathbb{F}_q(\theta)$ is finite and separable) and set $d := [F/\mathbb{F}_q(\theta)]$.

The effective version of Čebotarev's theorem in [FJ05, Proposition 6.4.8] says that for all $i \geqslant 1$ with $n|i$

$$\left| |C_i(E'/F)| - \frac{q^i}{ik} \right| < \frac{2}{ik}((k + g(E'))q^{i/2} + k(2g(F) + 1)q^{i/4} + g(E') + dk)$$

where

$$\begin{aligned} C_i(E'/F) := \{\mathfrak{p} \text{ place of } F \mid k(\mathfrak{p}) = \mathbb{F}_{q^i},\ \mathfrak{p} \text{ completely splits in } E' \text{ and} \\ \mathfrak{p} \text{ is unramified over } \mathbb{F}_q(\theta)\}. \end{aligned}$$

We apply this for all X with predegree $D(X) \geqslant q^{4Nr!}$. Because $n \leqslant [E'/F] \leqslant r!$, for these X we have $q^n \leqslant D(X)^{1/4N}$. Therefore there are $j \geqslant 1$ with $n|j$ and $q^j < D(X)^{1/N}$ and we can define

$$i := \max\{j \geqslant 1 : n \mid j, q^j < D(X)^{1/N}\}.$$

Our choice of i ensures that

$$m(X, N) \geqslant |C_i(E'/F)|.$$

By our choice of i and X we have $q^i < D(X)^{1/N}$, $q^{n+i} \geqslant D(X)^{1/N}$ and $q^n \leqslant D(X)^{1/4N}$. Hence we have the bounds

$$q^i < D(X)^{1/N}, \quad q^i = \frac{q^{n+i}}{q^n} \geqslant D(X)^{3/4N}.$$

Furthermore, Lemmas 7.3.3 and 7.3.4 imply

$$\begin{aligned} g(F') &\leqslant C_1 + C_2 \cdot \log_q(D(X)), \\ g(E') &\leqslant C_3 + C_4 \cdot g(F'). \end{aligned}$$

Since d is independent of X and $1 \leqslant n, k \leqslant r!$ for all X, the above conclusion of Čebotarev's theorem and these bounds imply

$$m(X, N) \geqslant \frac{C_1' \cdot D(X)^{3/4N}}{\log_q(D(X))} - \frac{C_2' + C_3' \log_q(D(X))}{\log_q(D(X))}(D(X)^{1/2N} + D(X)^{1/4N} + 1)$$

with $C_1', C_2', C_3' > 0$ independent of X. On the other hand, our statement (a) gives the bound

$$n(X) \leqslant C_5 + C_6 \cdot \log_q(D(X))$$

with C_5, C_6 independent of X. Since $x^{1/2N}(\log_q(x))^2 = o(x^{3/4N})$ for $x \to \infty$, these bounds imply the existence of an $M > 0$ such that $m(X, N) > n(X)$ for all X with $D(X) > M$. □

8. The André–Oort conjecture for Drinfeld modular varieties

8.1 Statement and first reduction

Conjecture 8.1.1 (André–Oort Conjecture for Drinfeld modular varieties). Let S be a Drinfeld modular variety and Σ a set of special points of S. Then each irreducible component over $\mathbb{C}_\infty$ of the Zariski closure of Σ is a special subvariety of S.

Our main result is the following theorem.

Theorem 8.1.2. *Conjecture 8.1.1 is true if the reflex fields of all special points in Σ are separable over F.*

Since the reflex field of a special point in $S^r_{F,\mathcal{K}}$ is of degree r over F, special points with inseparable reflex field over F can only occur if r is divisible by $p = \mathrm{char}(F)$. Hence, Theorem 8.1.2 implies the following corollary.

COROLLARY 8.1.3. *Conjecture 8.1.1 is true if r is not a multiple of $p = \mathrm{char}(F)$.*

Theorem 8.1.2 follows from the following crucial statement, whose proof we give in the next subsection.

THEOREM 8.1.4. *Let S be a Drinfeld modular variety and $Z \subset S$ an F-irreducible subvariety. Suppose that Σ is a set of Drinfeld modular subvarieties of S, all of the same dimension $d < \dim Z$ and with separable reflex field over F, whose union is Zariski dense in Z. Then, for almost all $X \in \Sigma$, there is a Drinfeld modular subvariety X' of S with $X \subsetneq X' \subset Z$.*

Remark. By Proposition 3.3.4, the proper inclusion $X \subsetneq X'$ implies that $\dim X < \dim X'$ because the reflex field of X' is properly contained in the reflex field of X.

PROPOSITION 8.1.5. *Theorem 8.1.4 implies Theorem 8.1.2.*

Proof of Proposition 8.1.5. We can assume without loss of generality that the Zariski closure Y of Σ is irreducible over $\mathbb{C}_\infty$. Since each special point in Σ is defined over F^{sep}, the Zariski closure Y of Σ is also defined over F^{sep}. Hence, we can consider the subvariety $Z := \mathrm{Gal}(F^{\mathrm{sep}}/F) \cdot Y$, which is F-irreducible by Proposition 1.2.2. The union Σ' of all $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-conjugates of the elements of Σ is Zariski dense in Z. Proposition 3.3.11 implies that Σ' is a union of Drinfeld modular subvarieties of dimension 0 with separable reflex field over F.

Hence, we can apply Theorem 8.1.4 with $d = 0$ and find a finite subset $\tilde{\Sigma} \subset \Sigma$ such that for all $X \in \Sigma \backslash \tilde{\Sigma}$, there is a Drinfeld modular subvariety X' with $X \subsetneq X' \subset Z$. We denote the set of these Drinfeld modular subvarieties X' by Σ'. Since $\tilde{\Sigma}$ is finite, the union of all subvarieties in Σ' is Zariski dense in Z.

Note that Proposition 3.3.4 implies that all elements X' of Σ' are of positive dimension. Therefore there is a $d' > 0$ with $d' \leqslant \dim Z$ such that the Zariski closure of the union of all subvarieties of dimension d' in Σ' is of codimension 0 in Z. We let Σ'' be the set of all $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-conjugates of the subvarieties of dimension d' in Σ'. Since Z is F-irreducible, this is a set of Drinfeld modular subvarieties of S, all of the same dimension $d' > 0$, whose union is Zariski dense in Z.

If $d' = \dim Z$, then Y is an irreducible component over $\mathbb{C}_\infty$ of an element in Σ'' and therefore special. If $d' < \dim Z$, we apply Theorem 8.1.4 with $d = d' > 0$ one more time to get a set of Drinfeld modular subvarieties of dimension $d'' > d'$ whose union is Zariski dense in Z. We iterate this process until we eventually get such a set with $d'' = \dim Z$, which implies that Y is special. ☐

8.2 Inductive proof in the separable case

The proof of Theorem 8.1.4 requires the results from § 7.3 about the existence of good primes and the following theorem. We first give an inductive proof of the latter theorem using our results about existence of suitable Hecke correspondences from § 7.2 and our geometric criterion in Theorem 6.1.3.

THEOREM 8.2.1. *Let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety and $X \subset S$ a Drinfeld modular subvariety over F which is contained in an F-irreducible subvariety $Z \subset S$ with $\dim Z > \dim X$.*

Suppose that $\mathfrak{p}$ *is a good prime for* $X \subset S$ *and*

$$\deg(X) > |k(\mathfrak{p})|^{(r-1)\cdot(2^s-1)} \cdot \deg(Z)^{2^s}$$

for $s := \dim Z - \dim X$. *Then there is a Drinfeld modular subvariety* X' *of* S *with* $X \subsetneq X' \subset Z$.

Remark. The degree $\deg(X)$ makes sense here because $\mathcal{K}$ is amply small by condition (i) in Definition 7.1.2.

Proof. In this proof, by 'irreducible component' we always mean an irreducible component over $\mathbb{C}_\infty$. We assume that $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$. Note that F' is separable over F by the remark after Definition 7.1.2.

We prove the following statements for all $n \geqslant 1$.

(i) If the theorem is true for $s = n$ and Z Hodge-generic (i.e., no irreducible component of Z lies in a proper Drinfeld modular subvariety of S, see Definition 6.1.2), then it is true for $s = n$ and general Z.

(ii) If the theorem is true for all s with $1 \leqslant s < n$ and general Z, then it is true for $s = n$ and Z Hodge-generic.

These two statements imply the theorem by induction over s.

Proof of (i). We assume that the theorem is true for $s = n$ and Z Hodge-generic and have to show that it is true for $s = n$ if Z is not Hodge-generic. In this case, there is an irreducible component of Z which is contained in a proper Drinfeld modular subvariety of S. Since $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts transitively on the irreducible components of Z (Proposition 1.2.2) and $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on the set of Drinfeld modular subvarieties of S (Proposition 3.3.11), also the other irreducible components of Z are contained in a proper Drinfeld modular subvariety of S. In particular, this is the case for some chosen irreducible component Z' of Z which contains an irreducible component V of X.

We now consider a minimal Drinfeld modular subvariety $Y = \iota^{F''}_{F,b'}(S^{r''}_{F'',\mathcal{K}''})$ of S with $Z' \subset Y \subsetneq S$. By Proposition 3.3.4, the reflex field F'' of Y is contained in F' and is therefore also separable over F. Since Y is defined over F'', the F''-irreducible component $Z'' := \mathrm{Gal}(F^{\mathrm{sep}}/F'') \cdot Z'$ of Z is contained in Y. Furthermore, the F'-irreducibility of X (see Corollary 3.4.6) implies

$$X = \mathrm{Gal}(F^{\mathrm{sep}}/F') \cdot V \subset \mathrm{Gal}(F^{\mathrm{sep}}/F'') \cdot V \subset \mathrm{Gal}(F^{\mathrm{sep}}/F'') \cdot Z' = Z'' \subset Y.$$

We now set $\tilde{X} := (\iota^{F''}_{F,b'})^{-1}(X)$ and $\tilde{Z} := (\iota^{F''}_{F,b'})^{-1}(Z'')$. These are subvarieties of $S^{r''}_{F'',\mathcal{K}''}$ with

$$\tilde{X} \subset \tilde{Z} \subset S^{r''}_{F'',\mathcal{K}''}$$

and

$$\dim \tilde{Z} - \dim \tilde{X} = \dim Z - \dim X = n.$$

The subvariety $\tilde{Z} = (\iota^{F''}_{F,b'})^{-1}(Z'')$ is F''-irreducible because $Z'' \subset \iota^{F''}_{F,b'}(S^{r''}_{F'',\mathcal{K}''})$ is F''-irreducible and $\iota^{F''}_{F,b'}$ is a closed immersion defined over F'' by Proposition 3.2.3.

By Corollary 3.3.5 and minimality of Y, the subvariety $\tilde{Z} \subset S^{r''}_{F'',\mathcal{K}''}$ is Hodge-generic and $\tilde{X}$ is a Drinfeld modular subvariety of $S^{r''}_{F'',\mathcal{K}''}$ with separable reflex field F' over F''. Furthermore, by Proposition 7.1.3, there is a prime $\mathfrak{p}''$ of F'' above $\mathfrak{p}$ with $k(\mathfrak{p}) = k(\mathfrak{p}'')$ such that $\mathfrak{p}''$ is good for $\tilde{X} \subset S^{r''}_{F'',\mathcal{K}''}$.

Proposition 4.2.3 (ii) implies

$$\begin{aligned}\deg \tilde{X} &= \deg X,\\ \deg \tilde{Z} &= \deg Z'' \leqslant \deg Z.\end{aligned}$$

Because $k(\mathfrak{p}) = k(\mathfrak{p}'')$ and $r'' < r$, the assumption

$$\deg(\tilde{X}) > |k(\mathfrak{p}'')|^{(r''-1)\cdot(2^n-1)} \cdot \deg(\tilde{Z})^{2^n}$$

is satisfied. So if Theorem 8.2.1 is true for Z Hodge-generic and $s = n$ then there is a Drinfeld modular subvariety $\tilde{X}'$ of $S^{r''}_{F'',\mathcal{K}''}$ with $\tilde{X} \subsetneq \tilde{X}' \subset \tilde{Z}$ and $X' := \iota^{F''}_{F,b'}(\tilde{X}')$ is the desired Drinfeld modular subvariety of S with $X \subsetneq X' \subset Z$. This concludes the proof of (i).

Proof of (ii). We assume that the theorem is true for all s with $1 \leqslant s < n$ and have to show that it is true for Z Hodge-generic and $\dim Z - \dim X = n$. Since $\mathfrak{p}$ is a good prime for X, we can apply Theorem 7.2.2 and find a $g \in \mathrm{GL}_r(\mathbb{A}^f_F)$ with the following properties:

(a) $X \subset T_g X$;

(b) $\deg T_g = [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g] = |k(\mathfrak{p})|^{r-1}$;

(c) for all $k_1, k_2 \in \mathcal{K}_{\mathfrak{p}}$, the cyclic subgroup of $\mathrm{PGL}_r(F_{\mathfrak{p}})$ generated by the image of $k_1 \cdot g_{\mathfrak{p}} \cdot k_2$ is unbounded.

Because of property (a) and $X \subset Z$, we have

$$X \subset Z \cap T_g Z.$$

Lemma 4.2.5 together with Proposition 4.2.3 and property (b) of our $g \in \mathrm{GL}_r(\mathbb{A}^f_F)$ give us the upper bound

$$\deg(Z \cap T_g Z) \leqslant \deg Z \cdot \deg T_g Z \leqslant (\deg Z)^2 \cdot \deg T_g = (\deg Z)^2 \cdot |k(\mathfrak{p})|^{r-1}.$$

With the assumption on $\deg X$ and $n = \dim Z - \dim X \geqslant 1$ we conclude

$$\deg X > |k(\mathfrak{p})|^{(r-1)\cdot(2^n-1)} \cdot \deg(Z)^{2^n} \geqslant \deg(Z \cap T_g Z).$$

Therefore X cannot be a union of irreducible components of $Z \cap T_g Z$. Note that $Z \cap T_g Z$ is defined over F, hence also over the reflex field F' of X. Since X is F'-irreducible, there is an F'-irreducible component Y' of $Z \cap T_g Z$ with $X \subset Y'$. We have $X \subsetneq Y'$ because X is not a union of irreducible components (over $\mathbb{C}_\infty$) of $Z \cap T_g Z$.

Now we set $Y := \mathrm{Gal}(F^{\mathrm{sep}}/F) \cdot Y'$. This is an F-irreducible component of $Z \cap T_g Z$ which contains X with $\dim X < \dim Y$. We distinguish two cases.

Case 1. $Y = Z$. Because $Y \subset Z \cap T_g Z$, this is only possible if $Z \subset T_g Z$. Since Z is F-irreducible and Hodge-generic, property (c) from the above list holds and $\mathcal{K}$ is amply small, we can apply our geometric criterion (Theorem 6.1.3) and conclude that $Z = S$. So $X' := Z = S$ satisfies the conclusion of the theorem.

Case 2. $Y \subsetneq Z$. Set $s' := \dim Y - \dim X$. Since Y and Z are F-irreducible, we have $1 \leqslant s' < n = \dim Z - \dim X$. Hence, by our assumption, we can apply the theorem to $X \subset Y \subset S$ and the prime $\mathfrak{p}$ provided that the inequality of degrees

$$\deg X > |k(\mathfrak{p})|^{(r-1)\cdot(2^{s'}-1)} \cdot \deg(Y)^{2^{s'}}$$

holds.

To check the latter, note that Y is a union of irreducible components (over $\mathbb{C}_\infty$) of $Z \cap T_g Z$, because it is an F-irreducible component of $Z \cap T_g Z$, whence

$$\deg Y \leqslant \deg(Z \cap T_g Z) \leqslant |k(\mathfrak{p})|^{r-1} \cdot (\deg Z)^2.$$

Therefore we indeed have

$$\begin{aligned} |k(\mathfrak{p})|^{(r-1)\cdot(2^{s'}-1)} \cdot \deg(Y)^{2^{s'}} &\leqslant |k(\mathfrak{p})|^{(r-1)\cdot(2^{n-1}-1)} \cdot \deg(Y)^{2^{n-1}} \\ &\leqslant |k(\mathfrak{p})|^{(r-1)\cdot(2^{n-1}-1)} \cdot |k(\mathfrak{p})|^{(r-1)\cdot 2^{n-1}} \cdot (\deg Z)^{2^n} \\ &= |k(\mathfrak{p})|^{(r-1)\cdot(2^{n}-1)} \cdot (\deg Z)^{2^n} < \deg X. \end{aligned}$$

So we find a Drinfeld modular subvariety X' of S with $X \subsetneq X' \subset Y \subset Z$ as desired. □

Proof of Theorem 8.1.4. We first reduce ourselves to the case $S = S^r_{F,\mathcal{K}}$ with $\mathcal{K}$ amply small. If $\mathcal{K}$ is not amply small, there is an amply small open subgroup $\mathcal{L} \subset \mathcal{K}$ with corresponding canonical projection $\pi_1 : S^r_{F,\mathcal{L}} \to S^r_{F,\mathcal{K}}$. We choose an F-irreducible component $\tilde{Z}$ of $\pi_1^{-1}(Z)$ with $\dim Z = \dim \tilde{Z}$ and set

$$\tilde{\Sigma} := \{\tilde{X} \subset \tilde{Z} \ F'\text{-irreducible component of } \pi_1^{-1}(X) \mid X \in \Sigma \text{ with reflex field } F'\}.$$

Since Drinfeld modular subvarieties with reflex field F' are F'-irreducible by Corollary 3.4.6, all $\tilde{X} \in \tilde{\Sigma}$ are Drinfeld modular subvarieties of $S^r_{F,\mathcal{L}}$ by Lemma 3.3.2. They are all contained in $\tilde{Z}$ and their union is Zariski dense in $\tilde{Z}$ by our assumption on Σ. If Theorem 8.1.4 is true for $\mathcal{K}$ amply small, we conclude that, for almost all $\tilde{X} \in \tilde{\Sigma}$, there is a Drinfeld modular subvariety $\tilde{X}'$ of $S^r_{F,\mathcal{L}}$ with $\tilde{X} \subsetneq \tilde{X}' \subset \tilde{Z}$. For such an $\tilde{X}'$, again by Lemma 3.3.2, $X' := \pi_1(\tilde{X}')$ is a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$. Hence, for almost all $X \in \Sigma$, there is a Drinfeld modular subvariety X' with $X \subsetneq X' \subset Z$.

So we now assume that $\mathcal{K}$ is amply small. By Theorem 7.3.2 with $N = 2(r-1)\cdot(2^s-1) + r^2 \cdot 2^{s+1}$ for $s := \dim Z - d$, for almost all $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'}) \in \Sigma$, there exists a prime $\mathfrak{p}$ of F with the following properties.

(i) There is a prime $\mathfrak{p}'$ of F' above $\mathfrak{p}$ with local degree $[F'_{\mathfrak{p}'}/F_{\mathfrak{p}}] = 1$.

(ii) We have $\mathcal{K} = \mathcal{K}_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}$ with $\mathcal{K}_{\mathfrak{p}} \subset \mathrm{GL}_r(F_{\mathfrak{p}})$ a maximal compact subgroup and $\mathcal{K}^{(\mathfrak{p})} \subset \mathrm{GL}_r(\mathbb{A}^{f,\mathfrak{p}}_F)$.

(iii) The subgroup $\mathcal{K}'_{\mathfrak{p}} := (b_{\mathfrak{p}}\mathcal{K}_{\mathfrak{p}}b_{\mathfrak{p}}^{-1}) \cap \mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ of $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ is maximal compact.

(iv) We have $|k(\mathfrak{p})|^{2(r-1)\cdot(2^s-1)+r^2\cdot 2^{s+1}} < D(X)$ for $s := \dim Z - d$.

Furthermore, by Theorem 4.3.2 we have:

(v) $D(X) > \deg(Z)^{2^{s+1}}/C^2$,

for almost all $X \in \Sigma$ with C the constant from Proposition 4.3.1.

By Proposition 7.3.1, for all $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ and $\mathfrak{p}$ with (i)–(v) there is a subgroup $\tilde{\mathcal{K}} \subset \mathcal{K}$ and a Drinfeld modular subvariety $\tilde{X} \subset S^r_{F,\tilde{\mathcal{K}}}$ such that:

(a) $\pi_1(\tilde{X}) = X$ for the canonical projection $\pi_1 : S^r_{F,\tilde{\mathcal{K}}} \to S^r_{F,\mathcal{K}}$;

(b) $\mathfrak{p}$ is good for $\tilde{X} \subset S^r_{F,\tilde{\mathcal{K}}}$;

(c) $[\mathcal{K} : \tilde{\mathcal{K}}] < |k(\mathfrak{p})|^{r^2}$.

Furthermore, for such an $\tilde{X} \subset S^r_{F,\tilde{\mathcal{K}}}$, we choose an F-irreducible component $\tilde{Z}$ of $\pi_1^{-1}(Z)$ with $\tilde{X} \subset \tilde{Z}$. Since π_1 is finite of degree $[\mathcal{K} : \tilde{\mathcal{K}}]$ by Theorem 3.1.3, we have $\dim \tilde{Z} = \dim Z > \dim X = \dim \tilde{X}$ and

$$\begin{aligned} \deg \tilde{Z} &\leqslant \deg \pi_1^{-1} Z = [\mathcal{K} : \tilde{\mathcal{K}}] \cdot \deg Z < |k(\mathfrak{p})|^{r^2} \cdot \deg Z, \\ \deg \tilde{X} &\geqslant \deg \pi_1(\tilde{X}) = \deg X \end{aligned}$$

by Proposition 4.2.3. Therefore, using Proposition 4.3.1, we get the inequality

$$\begin{aligned} \deg \tilde{X} \quad &\geqslant \quad \deg X \geqslant C \cdot D(X) = D(X)^{1/2} \cdot (C \cdot D(X)^{1/2}) \\ &\overset{\text{(iv),(v)}}{>} |k(\mathfrak{p})|^{(r-1)\cdot(2^s-1)+r^2\cdot 2^s} \cdot \deg(Z)^{2^s} \geqslant |k(\mathfrak{p})|^{(r-1)\cdot(2^s-1)} \cdot \deg(\tilde{Z})^{2^s}. \end{aligned}$$

Therefore $\tilde{X} \subset \tilde{Z} \subset S^r_{F,\tilde{\mathcal{K}}}$ together with $\mathfrak{p}$ satisfy the assumptions of Theorem 8.2.1. So we find a Drinfeld modular subvariety $\tilde{X}'$ of $S^r_{F,\tilde{\mathcal{K}}}$ with $\tilde{X} \subsetneq \tilde{X}' \subset \tilde{Z}$, and $X' := \pi_1(\tilde{X}')$ is a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$ with $X \subsetneq X' \subset Z$. □

Acknowledgements

The results presented in this article made up my PhD thesis at ETH Zürich in Switzerland. I would like to express my deepest gratitude to my advisor Richard Pink for guiding and supporting me in course of my PhD studies and carefully revising early drafts of this work. In particular I am indebted to him for providing some crucial ideas which ensured the success of this article. I also thank Gebhard Böckle, Florian Breuer and the anonymous referees for going through the manuscript and providing me with useful comments and constructive suggestions.

References

And98 Y. André, *Finitude des couples d'invariants modulaires singuliers sur une courbe algébrique plane non modulaire*, J. Reine Angew. Math. **505** (1998), 203–208.

Bor91 A. Borel, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 126, second edition (Springer, 1991).

Bre05 F. Breuer, *The André–Oort conjecture for products of Drinfeld modular curves*, J. Reine Angew. Math. **579** (2005), 115–144.

Bre07 F. Breuer, *CM points on products of Drinfeld modular curves*, Trans. Amer. Math. Soc. **359** (2007), 1351–1374.

Bre12 F. Breuer, *Special subvarieties of Drinfeld modular varieties*, J. Reine Angew. Math. **668** (2012), 35–57.

BP05 F. Breuer and R. Pink, *Monodromy groups associated to non-isotrivial Drinfeld modules in generic characteristic*, in *Number fields and function fields: two parallel worlds*, Progress in Mathematics, vol. 239 (Birkhäuser, Boston, MA, 2005), 61–69.

CU05 L. Clozel and L. Ullmo, *Equidistribution de sous-variétés spéciales*, Ann. of Math. (2) **161** (2005), 1571–1588.

Con99 B. Conrad, *Irreducible components of rigid spaces*, Ann. Inst. Fourier **49** (1999), 473–541.

DH87 P. Deligne and D. Husemöller, *Survey of Drinfeld modules*, in *Current trends in arithmetical algebraic geometry,* Arcata, CA, 1985, Contemporary Mathematics, vol. 67 (American Mathematical Society, Providence, RI, 1987), 25–91.

Dri74 V. G. Drinfeld, *Elliptic modules*, Mat. Sb. **94** (1974), 594–627, 656.

Edi98 B. Edixhoven, *Special points on the product of two modular curves*, Compositio Math. **114** (1998), 315–328.

Edi01 B. Edixhoven, *On the André–Oort conjecture for Hilbert modular surfaces*, in *Moduli of abelian varieties,* Texel, 1999, Progress in Mathematics, vol. 195 (Birkhäuser, Basel, 2001), 133–155.

Edi05 B. Edixhoven, *Special points on products of modular curves*, Duke Math. J. **126** (2005), 325–348.

EY03 B. Edixhoven and A. Yafaev, *Subvarieties of Shimura varieties*, Ann. of Math. (2) **157** (2003), 621–645.

FJ05 D. Fried and M. Jarden, *Field arithmetic*, second edition (Springer, 2005).

Ful98 W. Fulton, *Intersection theory*, second edition (Springer, 1998).

Gek89 E.-U. Gekeler, *On the de Rham isomorphism for Drinfeld modules*, J. Reine Angew. Math. **401** (1989), 188–208.

Gek90 E.-U. Gekeler, *De Rham cohomology and the Gauss–Manin connection for Drinfeld modules*, in *p-adic analysis,* Trento, 1989, Lecture Notes in Mathematics, vol. 1454 (Springer, Berlin, 1990), 223–255.

GW10 U. Görtz and T. Wedhorn, *Algebraic geometry 1: schemes, with examples and exercises* (Vieweg+Teubner, 2010).

Gos98 D. Goss, *Basic structures of function field arithmetic* (Springer, 1998).

Gro64 A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, Publ. Math. Inst. Hautes Études Sci. **20** (1964), 5–251.

Gro66 A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Publ. Math. Inst. Hautes Études Sci. **28** (1966), 5–248.

Har77 R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52 (Springer, Berlin, 1977).

Hei04 G. J. van der Heiden, *Weil pairing for Drinfeld modules*, Monatsh. Math. **143** (2004), 115–143.

KY12 B. Klingler and A. Yafaev, *The André–Oort conjecture*, Preprint (2012), arXiv:1209.0936 [math.NT].

Koh11 J. Kohlhaase, *Lubin–Tate and Drinfeld bundles*, Tohoku Math. J. (2) **63** (2011), 217–254.

Leh09 T. Lehmkuhl, *Compactification of the Drinfeld modular surfaces*, Mem. Amer. Math. Soc. **197** (2009), 921.

Mum70 D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5 (Oxford University Press, London, 1970).

Neu07 J. Neukirch, *Algebraische Zahlentheorie* (Springer, 2007).

Noo06 R. Noot, *Correspondances de Hecke, action de Galois et la conjecture de André–Oort [d'après Edixhoven et Yafaev]*, Séminaire Bourbaki, Astérisque **307** (2006), 165–197, exp. no. 942.

Pil11 J. Pila, *O-minimality and the André–Oort conjecture for* $\mathbb{C}^n$, Ann. of Math. (2) **173** (2011), 1779–1840.

Pin97 R. Pink, *The Mumford-Tate conjecture for Drinfeld-modules*, Publ. Res. Inst. Math. Sci. **33** (1997), 393–425.

Pin00 R. Pink, *Strong approximation for Zariski dense subgroups over arbitrary global fields*, Comment. Math. Helv. **75** (2000), 608–643.

Pin12 R. Pink, *Compactification of Drinfeld modular varieties and Drinfeld modular forms of arbitrary rank*, Manuscripta Math., to appear; math.AG/1008.0013v4.

Pra77 G. Prasad, *Strong approximation for semi-simple groups over function fields*, Ann. of Math. (2) **105** (1977), 553–572.

Ser88 J-P. Serre, *Algebraic groups and class fields*, Graduate Texts in Mathematics, vol. 117 (Springer, Berlin, 1988).

Sti93 H. Stichtenoth, *Algebraic function fields and codes* (Springer, 1993).

UY12 E. Ullmo and A. Yafaev, Galois orbits and equidistribution: towards the André–Oort conjecture, Preprint (2012), arXiv:1209.0934 [math.NT].

Yaf06 A. Yafaev, *A conjecture of Yves André's*, Duke Math. J. **132** (2006), 393–407.

Patrik Hubschmid patrik.hubschmid@iwr.uni-heidelberg.de
Interdisciplinary Center for Scientific Computing, Universität Heidelberg,
Im Neuenheimer Feld 368, 69120 Heidelberg, Germany