ON TWO FUNCTIONAL EQUATIONS FOR THE TRIGONOMETRIC FUNCTIONS

Hiroshi Haruki

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1. Introduction. We consider the following cosine and sine functional equations:

$$(1) \quad f(x + y) + f(x - y) = 2f(x)f(y),$$

$$(2) \quad f(x + y)f(x - y) = f(x)^2 - f(y)^2,$$

where f is an entire function of a complex variable z and x, y are complex variables [1; 2; 3]. Furthermore, we consider the following two functional equations:

$$(3) \quad |f(x + y)|^2 + |f(x - y)|^2 = 2|f(x)|^2|f(y)|^2 + 2|g(x)|^2|g(y)|^2,$$

$$(4) \quad |f(x + y)|^2 + |f(x - y)|^2 = 2|f(x)|^2|g(y)|^2 + 2|f(y)|^2|g(x)|^2,$$

where $f(z)$, $g(z)$ are entire functions of a complex variable z and x, y are complex variables. In Sections 2, 3 we shall prove the following theorem

THEOREM 1. (i) If $f(z)(\not\equiv 1)$ is a complex-valued function of a complex variable z and satisfies (1), then f satisfies (3) with $g(z) = \frac{1}{2}(f(z + \gamma) - f(z - \gamma))(1 - f(\gamma)^2)^{-\frac{1}{2}}$ where γ is a complex constant such that $f(2\gamma) \neq 1$ and $\sqrt{1 - f(\gamma)^2}$ denotes one square root of $1 - f(\gamma)^2$.

(ii) If $f(z)(\not\equiv 0)$ is a complex-valued function of a complex variable z and satisfies (2), then f satisfies (4) with $g(z) = (f(z + \gamma) - f(z - \gamma))/(2f(\gamma))$ where γ is a complex constant such that $f(\gamma) \neq 0$.

In Section 4 we shall solve (3), (4), that is, prove the following result.

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THEOREM 2. (i) The only entire solutions of (3) are

$$\begin{cases} f(z) \equiv a \\ g(z) \equiv b, \end{cases}$$

where a, b are arbitrary complex constants with $|a|^2 = |a|^4 + |b|^4$ and

$$\begin{cases} f(z) = \exp(i\alpha)\cos kz \\ g(z) = \exp(i\beta)\sin kz, \end{cases}$$

where α, β are arbitrary real constants and k is an arbitrary complex constant.

(ii) The only entire solutions of (4) are

$$\begin{cases} f(z) \equiv 0 \\ g(z) = \text{arbitrary}, \end{cases}$$

and

$$\begin{cases} f(z) \equiv a \\ g(z) \equiv \frac{1}{\sqrt{2}} \exp(i\theta), \end{cases}$$

$$\begin{cases} f(z) = az \\ g(z) \equiv \exp(i\theta), \end{cases}$$

where a is an arbitrary complex constant and θ is an arbitrary real constant, and

$$\begin{cases} f(z) = a\sin kz \\ g(z) = \exp(i\theta)\cos kz, \end{cases}$$

where a, k are arbitrary complex constants and θ is an arbitrary real constant.

2. Proof of the part (i) of Theorem 1. We may assume that $f \not\equiv 0$. By (1), $f(0) = 1$, so that, putting $y = x$ in (1), we get

(5) $$f(2x) = 2f(x)^2 - 1.$$

Replacing x, y by x + y, x - y, respectively, in (1), and using (5), we deduce that $f(x + y)f(x - y) = f(x)^2 + f(y)^2 - 1$, which, with (1), yields

(6) $$(f(x + y) - f(x - y))^2 = 4(1 - f(x)^2)(1 - f(y)^2).$$

Since $f \not\equiv 1$, there exists a complex number γ such that $f(2\gamma) \neq 1$, so that, by (5), $1 - f(\gamma)^2 \neq 0$. Putting $y = \gamma$ in (6) and setting $g(z) = \frac{1}{2}(f(z + \gamma) - f(z - \gamma))(1 - f(\gamma)^2)^{-\frac{1}{2}}$, we conclude that $g(z)^2 = 1 - f(z)^2$ for $|z| < +\infty$. This, with (6), gives $|f(x + y) - f(x - y)|^2 = 4|g(x)|^2 |g(y)|^2$; also, by (1), $|f(x + y) + f(x - y)|^2 = 4|f(x)|^2 |f(y)|^2$. Adding these two equations and using the parallelogram identity $|a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2$ (a, b complex), we have (3).

COROLLARY TO THEOREM 1 (i). If $f(z)(\not\equiv 1)$ is an entire function of a complex variable z and satisfies (1), then there exists an entire function g of z such that f and g satisfy (3).

Proof. By Theorem 1 (i) and the definition of $g(z)$, this is clear.

3. Proof of the part (ii) of Theorem 1. Since $f \not\equiv 0$, there exists a complex constant γ such that $f(\gamma) \neq 0$. We put $g(z) = (f(z + \gamma) - f(z - \gamma))/(2f(\gamma))$. Replacing x, y by $\frac{1}{2}(x + y)$, $\frac{1}{2}(x - y)$, respectively, in (2), we get $f(x)f(y) = f(\frac{1}{2}(x + y))^2 - f(\frac{1}{2}(x - y))^2$. Thus [1, p. 138; 2],

$$
\begin{aligned}
2f(y)g(x) &= \frac{1}{f(\gamma)}(f(x + \gamma)f(y) - f(x - \gamma)f(y)) \\
&= \frac{1}{f(\gamma)}\left(f\left(\frac{x + y + \gamma}{2}\right)^2 - f\left(\frac{x - y + \gamma}{2}\right)^2 - f\left(\frac{x + y - \gamma}{2}\right)^2 \right. \\
&\quad \left. + f\left(\frac{x - y - \gamma}{2}\right)^2\right) \\
&= \frac{1}{f(\gamma)}\left(\left(f\left(\frac{x + y + \gamma}{2}\right)^2 - f\left(\frac{x + y - \gamma}{2}\right)^2\right) - \left(f\left(\frac{x - y + \gamma}{2}\right)^2 \right.\right. \\
&\quad \left.\left. - f\left(\frac{x - y - \gamma}{2}\right)^2\right)\right) \\
&= \frac{1}{f(\gamma)}(f(x + y)f(\gamma) - f(x - y)f(\gamma)) \\
&= f(x + y) - f(x - y).
\end{aligned}
$$

We therefore have $f(x + y) - f(x - y) = 2f(y)g(x)$, and, interchanging x and y, and using the fact that f is an odd function (which follows from (2)),

$$f(x + y) + f(x - y) = 2f(x)g(y).$$

By these two equations and the parallelogram identity (4) results.

COROLLARY TO THEOREM 1 (ii). If $f(z)(\not\equiv 0)$ is an entire function of a complex variable z and satisfies (2), then there exists an entire function g of z such that f and g satisfy (4).

Proof. By Theorem 1 (ii) this is clear.

4. Proof of Theorem 2.

To prove Theorem 2 we shall use the following two lemmas:

LEMMA 1. *If* $f(z)$, $g(z)$ *are entire functions of a complex variable* z *and if* $|f(z)| \leqq M|g(z)|$ *holds for* $|z| < +\infty$ *where* M *is a non-negative real constant, then* $f(z) = Cg(z)$ *holds for* $|z| < +\infty$ *where* C *is a complex constant with* $|C| \leqq M$.

Proof. By Riemann's Theorem concerning a removable singularity and by Liouville's Theorem this is clear.

LEMMA 2. *If* f *is an entire function of a complex variable* z, *then* $\Delta|f(z)|^2 = 4|f'(z)|^2$ *where* Δ *stands for the Laplacian* $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ $(z = x + iy, x, y$ real) *holds for* $|z| < +\infty$.

Proof. See [4].

Proof of Theorem 2. (i). We may assume that $f(z) \not\equiv$ const. Putting $y = 0$ in (3), we have

$$(7) \qquad |f(x)|^2 = |f(x)|^2|f(0)|^2 + |g(x)|^2|g(0)|^2.$$

We shall show that the assumption $g(0) \neq 0$ leads to a contradiction. If $g(0) \neq 0$, then we have by (7) that $|g(x)|^2 = |g(0)|^{-2}(1 - |f(0)|^2)|f(x)|^2$. Substituting this into (3), we obtain

$$|f(x+y)|^2 + |f(x-y)|^2 = 2(1 + (\frac{1 - |f(0)|^2}{|g(0)|^2})^2)|f(x)|^2|f(y)|^2,$$

whence, by putting $y = x$, we have

$$|f(0)|^2 \leqq 2(1 + (\frac{1 - |f(0)|^2}{|g(0)|^2})^2)|f(x)|^4.$$

By (7) and since $g(0) \neq 0$, also $f(0) \neq 0$; hence, for $|x| < +\infty$, $f(x) \neq 0$ and $|1/f(x)| \leqq K$, where K is a real constant. Thus, by Liouville's Theorem $\frac{1}{f(x)}$ and so $f(x)$ is a complex constant.

This is a contradiction. Hence $g(0) = 0$, so that, by (7) and the fact that $f(z)$ is not a constant, it follows that $|f(0)| = 1$. Putting $x = 0$ in (3), we get $|f(-y)| = |f(y)|$. Since f is an entire function $f(-y) = \exp(i\theta)f(y)$, where θ is a real constant. Putting $y = 0$, we have $\exp(i\theta) = 1$. Hence $f(-y) = f(y)$, so that $f'(0) = 0$.

Taking the Laplacian $\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}$ of both sides of (3) with respect to y ($y = s + it$, s,t real), by Lemma 2 we conclude that

$$(8) \qquad |f'(x+y)|^2 + |f'(x-y)|^2 = 2|f(x)|^2|f'(y)|^2 + 2|g(x)|^2|g'(y)|^2$$

Putting $y = 0$ in (8), we have

$$(9) \qquad f'(x) = \exp(i\theta)g'(0)g(x),$$

where θ is a real constant. If $g'(0) = 0$, by (9) we have $f'(x) = 0$ and so $f(x) \equiv \text{const.}$ This is a contradiction. Hence $g'(0) \neq 0$. By (9), $f''(x) = \exp(i\theta)g'(0)g'(x)$, so that, putting $x = 0$, we obtain $f''(0) \neq 0$.

Taking the Laplacian of both sides of (8) with respect to y ($y = s + it, s,t$ real), by Lemma 2, we have

$$|f''(x+y)|^2 + |f''(x-y)|^2 = 2|f(x)|^2|f''(y)|^2 + 2|g(x)|^2|g''(y)|^2 .$$

Putting $y = 0$, we get for $|x| < +\infty$, $|f''(x)| \geqq |f''(0)f(x)|$. Hence by Lemma 1, for $|x| < +\infty$, $Cf''(x) = f''(0)f(x)$, where C is a complex constant with $|C| \leqq 1$. Setting $x = 0$ and using $|f(0)| = 1$, $f''(0) \neq 0$, we have $C = f(0) \neq 0$. Thus, $f''(x) = (f''(0)/f(0))f(x)$.

Solving this differential equation with the appropriate boundary conditions and using (9), we obtain

$$\begin{cases} f(z) = \exp(i\alpha)\cos kz \\ g(z) = \exp(i\beta)\sin kz , \end{cases}$$

where α, β are real constants and k is a complex constant. Direct substitution shows that the two systems of functions listed in Theorem 2 satisfy (3).

(ii) can be proved similarly.

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Remark. By the two corollaries to Theorem 1 and by Theorem 2, all entire solutions of the functional equations (1), (2) can be easily found.

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University of Waterloo
Waterloo, Ontario

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