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HYPERTRANSCENDENCE OF L-FUNCTIONS FOR $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$

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Abstract

We generalise a result of Hilbert which asserts that the Riemann zeta-function $\zeta(s)$ is hypertranscendental over $\mathbb{C}(s)$. Let π be any irreducible cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ with unitary central character. We establish a certain type of functional difference–differential independence for the associated L-function $L(s,\pi)$. This result implies algebraic difference–differential independence of $L(s,\pi)$ over $\mathbb{C}(s)$ (and more strongly, over a certain field $\mathcal{F}_s$ which contains $\mathbb{C}(s)$). In particular, $L(s,\pi)$ is hypertranscendental over $\mathbb{C}(s)$. We also extend a result of Ostrowski on the hypertranscendence of ordinary Dirichlet series.

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1. Introduction and statement of results

The study of differential independence for Dirichlet series and other allied functions has a long history. Let F be a field of meromorphic functions on the complex plane. A meromorphic function $f(s)$ is called *hypertranscendental* over F if $y = f(s)$ does not satisfy any nontrivial algebraic differential equation over F (that is, any equation of the form $P(y, y', \ldots, y^{(n)}) = 0$, where n is a nonnegative integer and P is a nonzero polynomial in $y, y', \ldots, y^{(n)}$ whose coefficients belong to F). The field F is usually required to be a differential field (that is, F is closed under differentiation). It was proved by Hölder [8] that the Gamma function $\Gamma(s)$ is hypertranscendental over the field $\mathbb{C}(s)$ of rational functions. The following result was stated by Hilbert [7, page 428] in 1900 in his famous lecture at the ICM in Paris (without detailed proof).

THEOREM A [7]. *The Riemann zeta-function $\zeta(s)$ is hypertranscendental over $\mathbb{C}(s)$.*

Hilbert's proof is based on Hölder's result and the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin(\tfrac{1}{2}\pi s)\zeta(1-s), \tag{1.1}$$

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as mentioned in [7, pages 428–429]. A detailed proof was written up by Stadigh in his dissertation [22] and we can also refer to [14, Example 3.2]. Another proof of Theorem A and a general result for a wide class of Dirichlet series were given by Ostrowski [17].

Much later, from the viewpoint of the value-distribution of $\zeta(s)$, Voronin [25] (see also [10, page 254]) obtained yet another proof of Theorem A and the following stronger theorem, which is called *functional independence (in the sense of Voronin)* of $\zeta(s)$ and its derivatives (see [23, page 196]). Voronin's proof is based on a result in his paper [24], which asserts that if σ is a real number with $1/2 < \sigma < 1$, then the set

$$\{(\zeta(\sigma+it), \zeta'(\sigma+it), \ldots, \zeta^{(K)}(\sigma+it)) \in \mathbb{C}^{K+1} : t \in \mathbb{R}\} \tag{1.2}$$

is dense in $\mathbb{C}^{K+1}$. We note that this denseness result can be obtained as a consequence of the so-called *universality theorem* for $\zeta(s)$ (see [23, Section 10.1]).

THEOREM B [25]. *Let K and N be nonnegative integers. Let $H_0, \ldots, H_N : \mathbb{C}^{K+1} \to \mathbb{C}$ be continuous functions, not all identically zero. Then*

$$\sum_{n=0}^{N} s^n H_n(\zeta(s), \zeta'(s), \ldots, \zeta^{(K)}(s)) = 0$$

does not hold identically for $s \in \mathbb{C}\backslash\{1\}$.

Let $\pi = \otimes'_{p\le\infty}\pi_p$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ with unitary central character, where $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{A}_{\mathbb{Q}}$ its ring of adeles. The L-function $L(s,\pi)$ attached to π is defined as the Euler product of local factors $L(s,\pi_p)$:

$$L(s,\pi) = \prod_{p<\infty} L(s,\pi_p) \tag{1.3}$$

(see, for example, [4] and [21]). For each $p < \infty$, we write

$$L(s,\pi_p) = \prod_{j=1}^{m}\Big(1 - \frac{\alpha_\pi(p,j)}{p^s}\Big)^{-1}.$$

Here $\alpha_\pi(p,j)$ $(1 \le j \le m)$ are complex numbers defined in terms of suitable parameters of π_p (Satake parameters if π_p is unramified and Langlands parameters in general). The Euler product (1.3) converges absolutely for $\Re s > 1$ (see [9, Section 5]).

The generalised Ramanujan conjecture (at non-archimedean places) for π predicts that if π_p $(p < \infty)$ is unramified, then

$$|\alpha_\pi(p,j)| = 1 \quad \text{for all } 1 \le j \le m. \tag{1.4}$$

This is verified for certain representations π (for example, π of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ corresponding to a holomorphic Hecke eigen cusp form for $\mathrm{SL}_2(\mathbb{Z})$) but not in general.

As mentioned in [23, page 283], if π satisfies the generalised Ramanujan conjecture (including the archimedean place), then $L(s,\pi)$ has the universality property on a certain strip, which implies an analogous result to Theorem B (Voronin's functional differential independence) for $L(s,\pi)$. We note that for π corresponding to a Maass cusp form $\varphi(z)$ of $\mathrm{SL}_2(\mathbb{Z})$, the Ramanujan conjecture (1.4) is still open, but the author [15, 16] could show the universality theorem and Voronin's functional differential independence for the associated L-function $L(s,\pi) = L(s,\varphi)$. However, at present, the case of higher rank $m = 3, 4, 5, \ldots$ is still open in general for the universality property, a denseness property as in (1.2) and Voronin's functional differential independence, as well as the generalised Ramanujan conjecture.

In this paper, without any assumptions (such as assuming the generalised Ramanujan conjecture) on π, we show in Theorem 1.1 a certain type of functional difference–differential independence for the L-function $L(s,\pi)$. As in Corollary 1.2, this theorem implies the hypertranscendence of $L(s,\pi)$ over $\mathbb{C}(s)$, which is a generalisation of Theorem A. In the following, let μ be any nonnegative integer, $h_0, h_1, \ldots, h_\mu$ any real numbers with $h_0 < h_1 < \cdots < h_\mu$ and $\nu_0, \nu_1, \ldots, \nu_\mu$ any nonnegative integers. We set

$$M := \sum_{j=0}^{\mu} (\nu_j + 1).$$

Following Reich [19, page 1352] (see also [20, page 29]), we say that a function $\Phi : \mathbb{C}^n \to \mathbb{C}$ is 'locally not trivial' if, for every nonempty open set $U \subset \mathbb{C}^n$, the restriction of Φ to U is not identically zero. For example, every holomorphic function $\Phi : \mathbb{C}^n \to \mathbb{C}$ which is not identically zero is 'locally not trivial', by the identity theorem [6, page 6, Theorem 6].

Theorem 1.1. *Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ with unitary central character, where m is any positive integer. Let N be a nonnegative integer. Let $\Phi_N : \mathbb{C}^M \to \mathbb{C}$ be a continuous and 'locally not trivial' function. When $N \geq 1$, for each integer $0 \leq n \leq N-1$, let $\Phi_n : \mathbb{C}^M \to \mathbb{C}$ be a continuous function. Then*

$$\sum_{n=0}^{N} s^n \Phi_n(L(s+h_0,\pi), L'(s+h_0,\pi), \ldots, L^{(\nu_0)}(s+h_0,\pi), L(s+h_1,\pi), \\ \ldots, L^{(\nu_1)}(s+h_1,\pi), \ldots, L(s+h_\mu,\pi), \ldots, L^{(\nu_\mu)}(s+h_\mu,\pi)) = 0$$

does not hold identically for $s \in \mathbb{C}$ with $\Re s + h_0 > 1$.

From this theorem we immediately obtain the following result of algebraic difference–differential independence over $\mathbb{C}(s)$ for $L(s,\pi)$.

Corollary 1.2. *Let π be as in Theorem 1.1. Let*

$$P(s; z_1, \ldots, z_M) = \sum_{a_1,\ldots,a_M} C_{a_1,\ldots,a_M}(s) z_1^{a_1} \cdots z_M^{a_M}$$

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be a nonzero polynomial in M variables $z_1, \ldots, z_M$ whose coefficients $C_{a_1,\ldots,a_M}(s)$ belong to $\mathbb{C}(s)$. Then

$$P(s; L(s+h_0,\pi), L'(s+h_0,\pi), \ldots, L^{(\nu_0)}(s+h_0,\pi), L(s+h_1,\pi), \ldots, L^{(\nu_1)}(s+h_1,\pi), \ldots, L(s+h_\mu,\pi), \ldots, L^{(\nu_\mu)}(s+h_\mu,\pi)) = 0$$

does not hold identically for $s \in \mathbb{C}$ with $\Re s + h_0 > 1$. In particular, $L(s,\pi)$ is hypertranscendental over $\mathbb{C}(s)$.

In order to obtain Theorem 1.1, we will first prove a general result (Theorem 1.4). For a Dirichlet series $F(s) = \sum_{n=1}^\infty a_n n^{-s}$, let $\sigma_a(F(s))$ denote its abscissa of absolute convergence. Reich [20] introduced the following class of Dirichlet series.

Definition 1.3. Let $\mathcal{D}$ denote the class of all Dirichlet series $F(s) = \sum_{n=1}^\infty a_n n^{-s}$ satisfying the following two conditions:

(i) $\sigma_a(F(s)) < \infty$;
(ii) the set of divisors of indices n with $a_n \neq 0$ contains infinitely many primes.

As mentioned in [20, pages 27, 42], the conditions (i) and (ii) of the class $\mathcal{D}$ are exactly the same as Ostrowski's [17, Satz 1] when we consider the case of (ordinary) Dirichlet series. The Riemann zeta-function $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ and the Dirichlet series

$$\log \zeta(s) = \sum_p \sum_{k=1}^\infty \frac{1}{k} p^{-ks} = \sum_{n=2}^\infty \frac{\Lambda(n)}{\log n} n^{-s}$$

belong to $\mathcal{D}$, where $\Lambda(n)$ is the von Mangoldt function ($\Lambda(n) = \log p$ if $n = p^k$ and $\Lambda(n) = 0$ otherwise). As shown in (3.3), without any assumptions, we will prove that $\log L(s,\pi)$ belongs to $\mathcal{D}$.

The next theorem is an extension of Reich's result [20, Satz 1] (the case $N = 0$).

Theorem 1.4. *Let $F(s) \in \mathcal{D}$. Let N be a nonnegative integer. Let $\Phi_N : \mathbb{C}^M \to \mathbb{C}$ be a continuous and 'locally not trivial' function. When $N \geq 1$, for each integer $0 \leq n \leq N-1$ let $\Phi_n : \mathbb{C}^M \to \mathbb{C}$ be a continuous function. Then*

$$\sum_{n=0}^N s^n \Phi_n(F(s+h_0), F'(s+h_0), \ldots, F^{(\nu_0)}(s+h_0), F(s+h_1), \ldots, F^{(\nu_1)}(s+h_1), \ldots, F(s+h_\mu), \ldots, F^{(\nu_\mu)}(s+h_\mu)) = 0 \tag{1.5}$$

does not hold identically for $s \in \mathbb{C}$ with $\Re s + h_0 > \sigma_a(F(s))$.

We note that Theorem 1.4 implies the following result, which is similar to Corollary 1.2. This result is already known and exactly Ostrowski's result [17, Satz 1] in the case of (ordinary) Dirichlet series.

COROLLARY 1.5 [17]. *Let* $F(s) \in \mathcal{D}$. *Let*

$$P(s; z_1, \dots, z_M) = \sum_{a_1,\dots,a_M} C_{a_1,\dots,a_M}(s)\, z_1^{a_1} \cdots z_M^{a_M}$$

be a nonzero polynomial in M *variables* $z_1, \dots, z_M$ *whose coefficients* $C_{a_1,\dots,a_M}(s)$ *belong to* $\mathbb{C}(s)$. *Then*

$$\begin{aligned} &P(s; F(s+h_0), F'(s+h_0), \dots, F^{(\nu_0)}(s+h_0), F(s+h_1), \\ &\quad \dots, F^{(\nu_1)}(s+h_1), \dots, F(s+h_\mu), \dots, F^{(\nu_\mu)}(s+h_\mu)) = 0 \end{aligned}$$

does not hold identically for $s \in \mathbb{C}$ *with* $\Re s + h_0 > \sigma_a(F(s))$. *In particular,* $F(s)$ *is hypertranscendental over* $\mathbb{C}(s)$.

We will also extend Corollaries 1.2 and 1.5 in a certain direction, as in Theorem 1.6 below. Let $\mathcal{F}_s$ denote the field of all meromorphic functions $\phi(s)$ on $\mathbb{C}$ satisfying

$$T(r, \phi) = o(r) \quad \text{as } r \to \infty,$$

where $T(r, \phi)$ is the Nevanlinna characteristic function of $\phi(s)$ (for its definition, see for example [11, Definition 2.1.9]). It is well known (see [11, Theorem 2.2.3]) that $\phi(s)$ is a rational function if and only if $T(r, \phi) = O(\log r)$ as $r \to \infty$. Thus,

$$\mathbb{C}(s) \subset \mathcal{F}_s. \tag{1.6}$$

We note that $\mathbb{C}(s)$ is a proper subfield of $\mathcal{F}_s$. For example, the entire function

$$\phi_0(s) = \sum_{n=0}^{\infty} \frac{s^n}{(2n)!} = \cosh(s^{1/2}) = \frac{1}{2}(\exp(s^{1/2}) + \exp(-s^{1/2}))$$

belongs to $\mathcal{F}_s$ but not to $\mathbb{C}(s)$, since $T(r, \phi_0) \asymp r^{1/2}$. See also [5, page 86].

Extending Hölder's result mentioned before, Bank and Kaufman [1, 2] showed that the Gamma function $\Gamma(s)$ is hypertranscendental over $\mathcal{F}_s$. Using this result and (1.1), Liao and Yang [12, Theorem 3.3] deduced that the Riemann zeta-function $\zeta(s)$ is hypertranscendental over $\mathcal{F}_s$, extending Theorem A. See also [3]. In the next theorem, we further extend the result of Liao and Yang. Our proof is different from theirs. This theorem is also an extension of Corollaries 1.2 and 1.5, according to (1.6).

THEOREM 1.6.

(1) *Let* π *be as in Theorem 1.1. Let*

$$P(s; z_1, \dots, z_M) = \sum_{a_1,\dots,a_M} \phi_{a_1,\dots,a_M}(s)\, z_1^{a_1} \cdots z_M^{a_M}$$

be a nonzero polynomial in M *variables* $z_1, \dots, z_M$ *whose coefficients* $\phi_{a_1,\dots,a_M}(s)$ *belong to* $\mathcal{F}_s$. *Then*

$$\begin{aligned} &P(s; L(s+h_0,\pi), L'(s+h_0,\pi), \dots, L^{(\nu_0)}(s+h_0,\pi), L(s+h_1,\pi), \\ &\quad \dots, L^{(\nu_1)}(s+h_1,\pi), \dots, L(s+h_\mu,\pi), \dots, L^{(\nu_\mu)}(s+h_\mu,\pi)) = 0 \end{aligned}$$

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does not hold identically for $s \in \mathbb{C}$ with $\Re s + h_0 > 1$. In particular, $L(s,\pi)$ is hypertranscendental over $\mathcal{F}_s$.
(2) *We have a similar result for any $F(s) \in \mathcal{D}$. Let*

$$P(s; z_1, \ldots, z_M) = \sum_{a_1,\ldots,a_M} \phi_{a_1,\ldots,a_M}(s)\, z_1^{a_1} \cdots z_M^{a_M}$$

be a nonzero polynomial in M variables $z_1, \ldots, z_M$ whose coefficients $\phi_{a_1,\ldots,a_M}(s)$ belong to $\mathcal{F}_s$. Then

$$P(s; F(s+h_0), F'(s+h_0), \ldots, F^{(\nu_0)}(s+h_0), F(s+h_1), \\ \ldots, F^{(\nu_1)}(s+h_1), \ldots, F(s+h_\mu), \ldots, F^{(\nu_\mu)}(s+h_\mu)) = 0$$

does not hold identically for $s \in \mathbb{C}$ with $\Re s + h_0 > \sigma_a(F(s))$. In particular, $F(s)$ is hypertranscendental over $\mathcal{F}_s$.

2. Proof of Theorem 1.4

The next lemma is a slight modification of [20, Lemma 2] and [19, Lemma 2].

Lemma 2.1. *Let $F(s) \in \mathcal{D}$ and $\sigma_0 \in \mathbb{R}$ with $\sigma_0 + h_0 > \sigma_a(F(s))$. Define a $\mathbb{C}^M$-valued function $g(s)$ by*

$$g(s) := (F(s+h_0), F'(s+h_0), \ldots, F^{(\nu_0)}(s+h_0), F(s+h_1), \ldots, F^{(\nu_1)}(s+h_1), \\ \ldots, F(s+h_\mu), \ldots, F^{(\nu_\mu)}(s+h_\mu)).$$

Then there exist a real number $\sigma_1 > \sigma_0$ and a nonempty open set $V \subset \mathbb{C}^M$ such that $V \cap \{g(\sigma_1 + in) : n \in \mathbb{N}\}$ is dense in V.

Proof. This lemma can be proved by modifying slightly the proof of [20, Lemma 2] and [19, Lemma 2]. The continuous version of Kronecker's approximation theorem was used in the proof of [20, Lemma 2]. Instead of it, we use the discrete version (as in the proof of [19, Lemma 2]). Thus, we obtain the lemma. □

Proof of Theorem 1.4. Let $g(s)$, σ_1 and V be as in Lemma 2.1. By assumption, Φ_N is continuous and not identically zero on V. Hence, there exist a nonempty bounded open set $V_0 \subset V$ and a constant $c > 0$ such that

$$|\Phi_N(\boldsymbol{v}_0)| > c \quad \text{for any } \boldsymbol{v}_0 \in V_0. \tag{2.1}$$

By Lemma 2.1, there exists a sequence of positive integers $\{n_j : j = 1, 2, \ldots\}$ with

$$\lim_{j\to\infty} n_j = \infty \tag{2.2}$$

such that for any j, the value $g(\sigma_1 + in_j)$ satisfies

$$g(\sigma_1 + in_j) \in V_0 (\subset V). \tag{2.3}$$

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We now consider the case $N \geq 1$. By assumption, each Φ_n $(0 \leq n \leq N-1)$ is bounded on the bounded set V_0. Hence, it follows from the triangle inequality, (2.3), (2.1) and (2.2) that, for all large j,

$$\begin{aligned}
&\Big|\sum_{\ell=0}^{N}(\sigma_1 + in_j)^{\ell}\Phi_\ell(g(\sigma_1 + in_j))\Big| \\
&\quad = |(\sigma_1 + in_j)^N|\Big|\Phi_N(g(\sigma_1 + in_j)) + \frac{\Phi_{N-1}(g(\sigma_1 + in_j))}{\sigma_1 + in_j} + \cdots + \frac{\Phi_0(g(\sigma_1 + in_j))}{(\sigma_1 + in_j)^N}\Big| \\
&\quad \geq |(\sigma_1 + in_j)^N|\Big(|\Phi_N(g(\sigma_1 + in_j))| - \Big|\frac{\Phi_{N-1}(g(\sigma_1 + in_j))}{\sigma_1 + in_j}\Big| - \cdots - \Big|\frac{\Phi_0(g(\sigma_1 + in_j))}{(\sigma_1 + in_j)^N}\Big|\Big) \\
&\quad > |(\sigma_1 + in_j)^N|\Big(c - \Big|\frac{\Phi_{N-1}(g(\sigma_1 + in_j))}{\sigma_1 + in_j}\Big| - \cdots - \Big|\frac{\Phi_0(g(\sigma_1 + in_j))}{(\sigma_1 + in_j)^N}\Big|\Big) > |(\sigma_1 + in_j)^N|c_1,
\end{aligned}$$

where c_1 is a positive constant with $c \geq c_1$. Therefore, in the case $N \geq 1$, (1.5) does not hold identically.

Next we consider the case $N = 0$. By (2.3) and (2.1), $|\Phi_0(g(\sigma_1 + in_j))| > c > 0$ for every j. Thus, Theorem 1.4 is proved. □

We remark that, according to the above proof, we actually have the following stronger result than Theorem 1.4. See also [19, pages 1351–1352].

THEOREM 2.2. *Let $F(s) \in \mathcal{D}$. Let σ_0 be a real number with $\sigma_0 + h_0 > \sigma_a(F(s))$. Let N be a nonnegative integer. Let $\Phi_N : \mathbb{C}^M \to \mathbb{C}$ be a continuous and 'locally not trivial' function. When $N \geq 1$, for each integer $0 \leq n \leq N-1$ let $\Phi_n : \mathbb{C}^M \to \mathbb{C}$ be a continuous function. Then there exist a real number σ_1 with $\sigma_1 > \sigma_0$ and a sequence $\{n_j \in \mathbb{N} : j = 1, 2, \ldots\}$ such that*

$$\begin{aligned}
\sum_{n=0}^{N} s^n \Phi_n(&F(s + h_0), F'(s + h_0), \ldots, F^{(\nu_0)}(s + h_0), F(s + h_1), \\
&\ldots, F^{(\nu_1)}(s + h_1), \ldots, F(s + h_\mu), \ldots, F^{(\nu_\mu)}(s + h_\mu)) = 0
\end{aligned}$$

does not hold identically for $s \in \{\sigma_1 + in_j : j = 1, 2, \ldots\}$.

3. Proof of Theorem 1.1

For a prime p and a positive integer k, we set

$$a_\pi(p^k) := \sum_{j=1}^{m} \alpha_\pi(p, j)^k. \tag{3.1}$$

We write

$$G(s, \pi) := \log L(s, \pi) = \sum_{p,k} \frac{a_\pi(p^k)}{k\, p^{ks}} = \sum_{n=2}^{\infty} \frac{\Lambda(n) a_\pi(n)}{(\log n) n^s},$$

the series being absolutely convergent for $\Re s > 1$ (see [9, page 556]), where $\Lambda(n)$ is the von Mangoldt function.

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The next result towards the generalised Ramanujan conjecture (1.4) is due to Rudnick and Sarnak [21, (2.3), Proposition A.1]. This is sharper than the earlier result of Jacquet and Shalika that $|\alpha_\pi(p, j)| < p^{1/2}$.

Lemma 3.1. *Let $\delta = (m^2 + 1)^{-1}$. For all primes p and $1 \le j \le m$,*

$$|\alpha_\pi(p, j)| \le p^{\frac{1}{2}-\delta}.$$

The next lemma, which is a prime number theorem for π, is obtained in [13, Lemma 5.1] without any assumption. See also Remark 3.3 below.

Lemma 3.2. *We have*

$$\sum_{n\le x} \Lambda(n)|a_\pi(n)|^2 \sim x \quad \text{as } x \to \infty.$$

Proof of Theorem 1.1. We write $\delta := (m^2 + 1)^{-1}$. By Lemma 3.1 and (3.1),

$$|a_\pi(p^k)| \le m\, p^{k(\frac{1}{2}-\delta)}. \tag{3.2}$$

Let $\mathbb{P}$ denote the set of all primes. Let $\mathbb{P}_\pi$ denote the set of all primes p such that $a_\pi(p^k) \neq 0$ for some positive integer k (which depends on p). Then, using (3.2),

$$\begin{aligned}
&\sum_{n\le x} \Lambda(n)|a_\pi(n)|^2 \\
&\quad = \sum_{p^k\le x, p\in\mathbb{P}, k\in\mathbb{N}} (\log p)|a_\pi(p^k)|^2 = \sum_{p^k\le x, p\in\mathbb{P}_\pi, k\in\mathbb{N}} (\log p)|a_\pi(p^k)|^2 \\
&\quad \le \sum_{k\le\log x/\log 2}\ \sum_{p\in\mathbb{P}_\pi, p\le x^{1/k}} m^2(\log p)\, p^{k(1-2\delta)} \le \sum_{k\le\log x/\log 2}\ \sum_{p\in\mathbb{P}_\pi, p\le x^{1/k}} m^2(\log x)\, x^{1-2\delta} \\
&\quad \le \sum_{k\le\log x/\log 2} \Big(m^2(\log x)\, x^{1-2\delta} \sum_{p\in\mathbb{P}_\pi, p\le x} 1\Big) \ll m^2(\log x)^2 x^{1-2\delta} \sum_{p\in\mathbb{P}_\pi, p\le x} 1.
\end{aligned}$$

This and Lemma 3.2 imply that

$$\sum_{p\in\mathbb{P}_\pi, p\le x} 1 \gg \frac{x^{2\delta}}{m^2(\log x)^2} \quad \text{as } x \to \infty.$$

Hence, in particular, $\mathbb{P}_\pi$ is an infinite set of primes. Therefore, since $\sigma_a(G(s, \pi)) \le 1$, it follows from the definitions of $\mathcal{D}$ and $\mathbb{P}_\pi$ that

$$G(s, \pi) \in \mathcal{D}. \tag{3.3}$$

By means of (3.3) and Lemma 2.1 (in which we take σ_0 so large that $\sigma_0 + h_0 > 1$), there exist a real number $\sigma_1 > -h_0 + 1$ and a nonempty open set $V \subset \mathbb{C}^M$ such that $V \cap \{g_G(\sigma_1 + i n) : n \in \mathbb{N}\}$ is dense in V, where

$$\begin{aligned}
g_G(s) := (&G(s + h_0, \pi), G'(s + h_0, \pi), \ldots, G^{(\nu_0)}(s + h_0, \pi), G(s + h_1, \pi), \\
&\ldots, G^{(\nu_1)}(s + h_1, \pi), \ldots, G(s + h_\mu, \pi), \ldots, G^{(\nu_\mu)}(s + h_\mu, \pi)).
\end{aligned}$$

Therefore, similarly to the proof of Lemma 2.4 of [18], we find that there exists a nonempty open set $U \subset \mathbb{C}^M$ such that $U \cap \{g_L(\sigma_1 + in) : n \in \mathbb{N}\}$ is dense in U, where

$$g_L(s) := (L(s+h_0,\pi), L'(s+h_0,\pi), \ldots, L^{(\nu_0)}(s+h_0,\pi), L(s+h_1,\pi), \\ \ldots, L^{(\nu_1)}(s+h_1,\pi), \ldots, L(s+h_\mu,\pi), \ldots, L^{(\nu_\mu)}(s+h_\mu,\pi)).$$

Using the argument in the proof of Theorem 1.4 completes the proof. □

REMARK 3.3. Expanding the Euler product (1.3) of $L(s,\pi)$, we write

$$L(s,\pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}.$$

Then

$$\lambda_\pi(p) = a_\pi(p) \quad \text{for every prime } p. \tag{3.4}$$

Let $\mathbb{P}_\pi^*$ denote the set of all primes p such that $a_\pi(p) \neq 0$. We do not know from our argument above whether $\mathbb{P}_\pi^*$ is an infinite set or not. Rudnick and Sarnak [21, page 281] introduced a hypothesis named 'Hypothesis H' for π. This hypothesis is much weaker than the generalised Ramanujan conjecture (1.4) and is true when $1 \le m \le 4$ (see [13, page 137]). If we assume the truth of Hypothesis H for π, then, as in [13, pages 136–137] and [21, pages 281–282],

$$\sum_{p \le x} \frac{|a_\pi(p)|^2}{p} = \log\log x + O(1) \quad \text{as } x \to \infty$$

and, in particular, $\mathbb{P}_\pi^*$ is an infinite set, which, with (3.4), implies that $L(s,\pi) \in \mathcal{D}$.

4. Proof of Theorem 1.6

We use the next lemma, which is [3, Lemma 4].

LEMMA 4.1. *Let K be a positive integer. Suppose that the Dirichlet series*

$$G_j(s) = \sum_{n=1}^{\infty} \frac{a_j(n)}{n^s} \quad j = 1, \ldots, K$$

are convergent in a half-plane $\Re s > \sigma_0$ and that, for each $j = 1, \ldots, K$, $\phi_j(s)$ is a meromorphic function on $\mathbb{C}$ satisfying $T(r,\phi_j) = o(r)$ as $r \to \infty$. Suppose that

$$\sum_{j=1}^{K} \phi_j(s) G_j(s) = 0$$

identically for $\Re s > \sigma_0$. Then, for each positive integer n,

$$\sum_{j=1}^{K} a_j(n) \phi_j(s) = 0$$

identically on $\mathbb{C}$.

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The next proposition is essentially obtained in Chiang and Feng [3]. The weaker assertion when $\mathcal{F}_s$ is replaced by $\mathbb{C}(s)$ was given in Ostrowski [17, page 246]. We recall that $\mathbb{C}(s) \subset \mathcal{F}_s$, as mentioned in (1.6).

Proposition 4.2. *Let N be a positive integer. Let $F_1(s), \dots, F_N(s)$ be Dirichlet series which are convergent in a half-plane $\Re s > \sigma_0$. Then the following are equivalent:*

(i) *$F_1(s), \dots, F_N(s)$ are algebraically dependent over $\mathcal{F}_s$;*
(ii) *$F_1(s), \dots, F_N(s)$ are algebraically dependent over $\mathbb{C}$.*

Proof. Since $\mathbb{C} \subset \mathcal{F}_s$, it is trivial that (ii) implies (i).

Next we shall prove that (i) implies (ii). Assume that (i) holds, that is, there exists a nonzero polynomial $P(s; z_1, \dots, z_N)$ in N variables $z_1, \dots, z_N$ whose coefficients belong to $\mathcal{F}_s$, such that

$$P(s; F_1(s), \dots, F_N(s)) = 0 \tag{4.1}$$

identically for $\Re s > \sigma_0$. We write

$$P(s; F_1(s), \dots, F_N(s)) = \sum_{j=1}^{D} \phi_j(s)\, F_1(s)^{k_1(j)} \cdots F_N(s)^{k_N(j)}, \tag{4.2}$$

where D (the number of terms) is a positive integer, $(k_1(j), \dots, k_N(j))\,(1 \le j \le D)$ are distinct elements in $\mathbb{N}_0^N$ with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\phi_j(s)\,(1 \le j \le D)$ are meromorphic functions in $\mathcal{F}_s$ which are not identically zero. For each $j = 1, \dots, D$, we write

$$F_1(s)^{k_1(j)} \cdots F_N(s)^{k_N(j)} = \sum_{n=1}^{\infty} \frac{A_j(n)}{n^s}. \tag{4.3}$$

Then, from (4.1), (4.2) and Lemma 4.1 (with $K = D$ and $G_j(s) = F_1(s)^{k_1(j)} \cdots F_N(s)^{k_N(j)}$), it follows that for every positive integer n,

$$\sum_{j=1}^{D} A_j(n)\phi_j(s) = 0 \tag{4.4}$$

identically on $\mathbb{C}$. By recalling that each $\phi_j(s)\,(1 \le j \le D)$ is not identically zero and noting that the set of zeros and poles of $\phi_j(s)$ is discrete, we can find a complex number s_0 which is not a pole of any $\phi_j(s)$ and which satisfies

$$\phi_j(s_0) \neq 0 \quad \text{for any } 1 \le j \le D. \tag{4.5}$$

By (4.4),

$$\sum_{j=1}^{D} A_j(n)\phi_j(s_0) = 0 \quad \text{for every positive integer } n. \tag{4.6}$$

From (4.3) and (4.6),

$$\sum_{j=1}^{D} \phi_j(s_0) F_1(s)^{k_1(j)} \cdots F_N(s)^{k_N(j)} = \sum_{n=1}^{\infty} \frac{\sum_{j=1}^{D} \phi_j(s_0) A_j(n)}{n^s} = 0$$

identically for $\Re s > \sigma_0$. This and (4.5) imply that $F_1(s), \dots, F_N(s)$ are algebraically dependent over $\mathbb{C}$, giving (ii). □

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Proof of Theorem 1.6. According to Corollary 1.2, the Dirichlet series $L(s+h_0,\pi)$, $L'(s+h_0,\pi), \ldots, L^{(\nu_0)}(s+h_0,\pi), L(s+h_1,\pi), \ldots, L^{(\nu_1)}(s+h_1,\pi), \ldots, L(s+h_\mu,\pi)$, $\ldots, L^{(\nu_\mu)}(s+h_\mu,\pi)$ are algebraically independent over $\mathbb{C}$. Hence, by Proposition 4.2, they are algebraically independent over $\mathcal{F}_s$. Thus, we have the assertion (1) of Theorem 1.6.

Similarly, the assertion (2) is obtained from Corollary 1.5 and Proposition 4.2. □

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