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ON THE COMMUTATIVITY OF SEMI-PRIME RINGS

ABDULLAH H. AL-MOAJIL

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Abstract

It is shown that if R is a 2-torsion-free semi-prime ring such that $[xy, [xy, yx]] = 0$ for all $x, y \in R$, then R is commutative.

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A commutativity theorem which was proven by Gupta [2] asserts that a division ring D which satisfies the polynomial identity $xy^2x = yx^2y$ for all $x, y \in D$ must be commutative. This was generalized by Awtar [1] who proves that if R is a semi-prime ring (that it contains no non-zero nilpotent ideals) and if $xy^2x - yx^2y$ is central for all $x, y \in R$, then R is commutative. In this paper we give a further generalization of this result for the case of 2-torsion-free rings. We will prove the following theorem.

THEOREM. *Let R be a 2-torsion-free semi-prime ring. If xy commutes with $xy^2x - yx^2y$ for all $x, y \in R$ then R is commutative.*

The proof we give is elementary and does not make use of any of the previously known commutativity theorems. First we need to prove some lemmas. We will let $[x, y]$ denote $xy - yx$ as usual.

LEMMA 1. *If R is a ring and $x, y \in R$ satisfy $[x,[x,y]] = 0$ then $[x^2, y] = 2x[x,y]$.*

PROOF. We have $[x^2, y] - 2x[x,y] = -yx^2 - x^2y + 2xyx = [x,[x,y]] = 0$.

LEMMA 2. *If x, y satisfy the hypothesis of Lemma 1 then $[x,[x,y^2]] = 2[x,y]^2$.*

PROOF. We have

$$\begin{aligned}[x,[x,y^2]] - 2[x,y]^2 &= x^2y^2 + y^2x^2 - 2(xy)^2 - 2(yx)^2 + 2yx^2y \\ &= (x^2y + yx^2 - 2xyx)y + y(x^2y + yx^2 - 2xyx) \\ &= [x,[x,y]]y + y[x,[x,y]] = 0.\end{aligned}$$

LEMMA 3. *Let R be a ring satisfying the identity $[xy,[xy,yx]] = 0$ for all $x, y \in R$. If there exists a non-zero element $x \in R$ such that $x^2 = 0$ then R is not semi-prime.*

PROOF. Let y be an arbitrary element of R. Applying the identity above to x and $y - yx$, and using the fact that $x^2 = 0$ we get

$$\begin{aligned}0 &= [x(y - yx),[x(y - yx),(y - yx)x]] = [x(y - yx),[x(y - yx), yx]] \\ &= [x(y - yx), xy^2x - (xy)^2x] = (xy)^2yx - (xy)^3x.\end{aligned}$$

Therefore,

$$(xy)^2yx = (xy)^3x. \tag{1}$$

Now apply the identity to xyx and y to get

$$\begin{aligned}0 &= [xyxy,[xyxy, yxyx]] \\ &= [xyxy, xyxy^2xyx] = (xy)^4(yx)^2.\end{aligned} \tag{2}$$

Using (1) to substitute $(xy)^3x$ for $(xy)^2yx$ in (2), we obtain $(xy)^6x = 0$. Therefore, $(xy)^7 = 0$.

Since y was arbitrary this proves that $z^7 = 0$ for all z in the right ideal xR. Therefore, it follows by [4, Lemma 1.1], that R contains a non-zero nilpotent ideal.

PROOF OF THE THEOREM. Since R is semi-prime we may assume, in view of Lemma 3, that R contains no nilpotent elements. Let $x, y \in R$ be arbitrary. Then $[xy,[xy, yx]] = 0$ by assumption. This obviously implies that

$[(xy)^2, [xy, yx]] = 0$. Moreover, by Lemma 1, $[(xy)^2, yx] = 2xy[xy, yx]$. Therefore, $(xy)^2$ commutes with $[(xy)^2, yx]$. That is,

(3) $$\left[(xy)^2, \left[(xy)^2, yx\right]\right] = 0.$$

Using (3) and Lemma 2 we get,

$$\begin{aligned} 2\left[(xy)^2, yx\right]^2 &= \left[(xy)^2, \left[(xy)^2, (yx)^2\right]\right] \\ &= \left[(xyx)y, \left[(xyx)y, y(xyx)\right]\right] = 0 \end{aligned}$$

by taking $z = xyx$ and applying the assumption on elements of R.

Since R is 2-torsion-free and contains no nilpotent elements this implies that $[(xy)^2, yx] = 0$. Therefore, since $[yx, [yx, xy]] = 0$, Lemma 2 implies that $2[yx, xy]^2 = [yx, 0] = 0$. Hence, by the assumption on R, $[yx, xy] = 0$, that is

(4) $$xy^2x = yx^2y.$$

Since x and y were arbitrary, this holds for all $x, y \in R$. Therefore, replacing y with $x + y$ in (4) we get $x^2yx + xyx^2 = x^3y + yx^3$, that is

(5) $$\left[x^2, \left[x, y\right]\right] = 0.$$

Since $[x^2, y] = x[x, y] + [x, y]x$ and x^2 commutes with $[x, y]$ by (5), we get $[x^2, [x^2, y]] = 0$. Moreover, replacing y with y^2 we obtain $[x^2, [x^2, y^2]] = 0$. Hence, by Lemma 2, $2[x^2, y]^2 = [x^2, [x^2, y^2]] = 0$, which implies that $[x^2, y] = 0$ or

(6) $$x^2y = yx^2.$$

Now replacing y with $x^2 + y$ in (4) we obtain $[x^3, [x, y]] = 0$ which implies that $[x^3, [x^3, y]] = 0$, since $[x^3, y] = x^2[x, y] + x[x, y]x + [x, y]x^2$. Repeating the argument above for x^3 and y^2 we obtain,

(7) $$x^3y = yx^3.$$

Applying (6) and (7) we get $(xyx - x^2y)^2 = 0$. Thus $xyx = x^2y = yx^2$. Replacing y with y^2 we get $xy^2x = x^2y^2 = y^2x^2$. Therefore, $(xy - yx)^2 = 0$ which implies that $xy = yx$. Since x and y were arbitrary we conclude that R is commutative.

At the end we point out that one could have quoted Gupta's result [2] after equation (4) or Herstein's theorem [5] after equation (6) to conclude the proof. This would have been on the expense of the self-containment of this paper. Moreover, the part of the proof that starts after (4) gives an alternative proof to Gupta's theorem.

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Gulf Organization for Industrial Consulting
P.O. Box 5114
Doha, Quatar

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