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FINE SPECTRA AND LIMIT LAWS I. FIRST-ORDER LAWS

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ABSTRACT. Using Feferman-Vaught techniques we show a certain property of the fine spectrum of an admissible class of structures leads to a first-order law. The condition presented is best possible in the sense that if it is violated then one can find an admissible class with the same fine spectrum which does not have a first-order law. We present three conditions for verifying that the above property actually holds.

The first condition is that the count function of an admissible class has regular variation with a certain uniformity of convergence. This applies to a wide range of admissible classes, including those satisfying Knopfmacher's Axiom A, and those satisfying Bateman and Diamond's condition.

The second condition is similar to the first condition, but designed to handle the discrete case, *i.e.*, when the sizes of the structures in an admissible class K are all powers of a single integer. It applies when either the class of indecomposables or the whole class satisfies Knopfmacher's Axiom $A^{\#}$.

The third condition is also for the discrete case, when there is a uniform bound on the number of K-indecomposables of any given size.

1. **Preliminaries.** Throughout the paper we will be working with classes K of finite structures, for a first-order language L. First we give a list of definitions that will be used:

1. $I(\mathsf{K})$ is the closure of K with respect to isomorphism, and
2. $P_{\text{fin}}(\mathsf{K})$ is the closure of K with respect to finite direct products.
3. For classes $\mathsf{K}_1, \ldots, \mathsf{K}_n$, the product of these classes is defined by $\mathsf{K}_1 \cdots \mathsf{K}_n = I\{\mathbf{A}_1 \times \cdots \times \mathbf{A}_n : \mathbf{A}_i \in \mathsf{K}_i\}$. We write K^n if $\mathsf{K} = \mathsf{K}_1 = \cdots = \mathsf{K}_n$.
4. $\mathsf{K}^{\geq r}$ is the class $\bigcup_{i \geq r} \mathsf{K}^i$.
5. An L-structure $\mathbf{A}$ is *trivial* if its universe A has only one element.
6. A member $\mathbf{A}$ of K is K-*indecomposable* if (i) its universe has at least two elements in it, and (ii) it is not isomorphic to the direct product of two nontrivial members of K.
7. K has *unique factorization*[1] if every nontrivial member can be uniquely written, up to isomorphism and the order of the factors, as a direct product of K-

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[1] There is a substantial literature on unique factorization. The best overview is in McKenzie, McNulty and Taylor's book [17], Chapter 5. We can also recommend the two papers: Bigelow and Burris [3], and Willard [25].

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indecomposables.

8. The *fine spectrum* σ_{K} of K is the sequence defined by: $\sigma_{\mathsf{K}}(n)$ is the number of structures of size n in K (up to isomorphism).[2]
9. The *(total) count* function τ_{K} for K is defined by

$$\tau_{\mathsf{K}}(x) = \sum_{n \leq x} \sigma_{\mathsf{K}}(n).$$

10. Given a property P let $\sigma_{\mathsf{K}}(n \mid P)$ be the number of structures in K of size n having the property P. And let $\tau_{\mathsf{K}}(x \mid P) = \sum_{n \leq x} \sigma_{\mathsf{K}}(n \mid P)$.
11. The *cumulative probability*, $\mathrm{Prob}_{\mathsf{K}}(P)$, that a property P holds in K is defined by

$$\mathrm{Prob}_{\mathsf{K}}(P) =: \lim_{n \longrightarrow +\infty} \frac{\tau_{\mathsf{K}}(n \mid P)}{\tau_{\mathsf{K}}(n)},$$

provided this limit exists.

12. K has a *first-order law* if, for every first-order sentence ϕ, $\mathrm{Prob}_{\mathsf{K}}(\phi)$ exists; if the latter is always 0 or 1 then we say K has a *first-order* 0–1 *law*.

REMARK 1.1. In much of the work on laws one considers the proportion of structures of size n which satisfy a given property. See, for example, [5] and [6]. We have adopted Compton's approach to studying direct products, namely consider the proportion of structures of size at most n which satisfy a given property.

As an example let K be the class of groups $I(\{\mathbf{Z}_6^n \times \mathbf{Z}_{15}^n : n \geq 1\})$. There is, up to isomorphism, only one K-indecomposable, namely $\mathbf{Z}_6 \times \mathbf{Z}_{15}$. K has unique factorization; the fine spectrum is: $\sigma_{\mathsf{K}}(n) = 1$ if n is a power of 90, and $= 0$ otherwise; and $\tau_{\mathsf{K}}(x)$ is 0 for $x < 90$, and is n if $90^n \leq x < 90^{n+1}$. In the last section we will see that K has a first-order law; and in Part II [7] we show it has a first-order 0–1 law.

2. Loaded classes.

DEFINITION 2.1. A class K of finite structures is *admissible*[3] if

- $\mathsf{K} = IP_{\mathrm{fin}}(\mathsf{K})$,
- $\sigma_{\mathsf{K}}(n) < +\infty$ for all n, $\sigma_{\mathsf{K}}(1) = 1$,
- K has unique factorization, and
- a trivial structure $\mathbf{A}$ in K acts as a multiplicative identity, *i.e.*, $\mathbf{A} \times \mathbf{B} \cong \mathbf{B}$ for all $\mathbf{B} \in \mathsf{K}$.

REMARK 2.2. This definition guarantees that if K is an admissible class then we can determine its fine spectrum from the fine spectrum of the class of K-indecomposables. Furthermore, the fine-spectrum σ_{K} can be recovered from the count function τ_{K}.

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[2] The fine spectrum was introduced in 1975 by Walter Taylor [22] for the case that K is an equationally defined class of algebras.

[3] Admissible classes form a select part of the general framework of *arithmetical categories* used by Knopfmacher [13] in his study of generalized prime number theorems.

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DEFINITION 2.3. Let K be a class of structures, and let F be the class of K-indecomposables. We say that K is *loaded* if, for every partition $\mathsf{F}_1, \ldots, \mathsf{F}_k$ of F into classes closed under isomorphism, and for every sequence $r_1, \ldots, r_k$ of nonnegative integers, the set $\mathsf{F}_1^{\geq r_1} \cdots \mathsf{F}_k^{\geq r_k}$ of structures $\mathbf{A} \in \mathsf{K}$ with at least r_i factors from F_i, $1 \leq i \leq k$, has an asymptotic density, *i.e.*,

$$\mathrm{Prob}_{\mathsf{K}}\ (\text{is in } \mathsf{F}_1^{\geq r_1} \cdots \mathsf{F}_k^{\geq r_k}) \text{ exists.}$$

REMARK 2.4. Note that K is loaded also implies

$$\mathsf{F}_1^{r_1} \cdots \mathsf{F}_i^{r_i} \mathsf{F}_{i+1}^{\geq r_{i+1}} \cdots \mathsf{F}_k^{\geq r_k}$$

has an asymptotic density as, for any property P,

$$\begin{aligned} \tau_{\mathsf{K}}(n \mid\ & \text{has exactly } r_i \text{ factors from } \mathsf{F}_i \text{ and } P) \\ &= \tau_{\mathsf{K}}(n \mid \text{ has at least } r_i \text{ factors from } \mathsf{F}_i \text{ and } P) \\ &\quad - \tau_{\mathsf{K}}(n \mid \text{ has at least } r_i + 1 \text{ factors from } \mathsf{F}_i \text{ and } P). \end{aligned}$$

3**. Logical aspects.** The Feferman-Vaught methods played a key role in [4] where it is proved that every directly representable variety has a first-order law. Now we apply them to admissible classes.

LEMMA 3.1. *Let ϕ be a first-order sentence with a Feferman-Vaught sequence $\langle \Phi, \phi_1, \ldots, \phi_k \rangle$. Then there is a positive integer c_ϕ such that if H is a class of structures with $\mathsf{H} \models \phi_i$ or $\mathsf{H} \models \neg\phi_i$, $1 \leq i \leq k$, then*

(a) for each positive integer n either $\mathsf{H}^n \models \phi$ or $\mathsf{H}^n \models \neg\phi$, and

(b) $n \geq c_\phi$ implies $\mathsf{H}^n \models \phi$, or $n \geq c_\phi$ implies $\mathsf{H}^n \models \neg\phi$.

PROOF. Let $\mathbf{A}_0, \ldots, \mathbf{A}_{n-1} \in \mathsf{H}$. Then

$$\mathbf{A}_0 \times \cdots \times \mathbf{A}_{n-1} \models \phi \quad \text{iff} \quad \mathbf{2}^n \models \Phi([\![\phi_1]\!], \ldots, [\![\phi_k]\!]), \tag{1}$$

where $[\![\phi_i]\!]$ is the characteristic function of the set of coordinates where ϕ_i holds. Each $[\![\phi_i]\!]$ is either the 1 or 0 of $\mathbf{2}^n$, for $n \geq 1$, depending solely on whether $\mathsf{H} \models \phi_i$ or $\mathsf{H} \models \neg\phi_i$. Thus $\Phi([\![\phi_1]\!], \ldots, [\![\phi_k]\!])$ is a sentence which does not depend on n. Part (a) now follows from (1). For (b) first recall that Skolem's elimination of quantifiers [20] for Boolean algebras gives, for each Boolean algebra sentence Ψ, the existence of a constant c_Ψ with the property

$$\begin{aligned} n \geq c_\Psi \Rightarrow \mathbf{2}^n \models \Psi, \quad &\text{or} \\ n \geq c_\Psi \Rightarrow \mathbf{2}^n \models \neg\Psi.& \end{aligned}$$

Let $c_\phi = \max\{c_\Psi : \Psi = \Phi(\lambda_1, \ldots, \lambda_k), \lambda_i \in \{0, 1\}\}$. ■

Next we introduce a simple tool which has been popular with universal algebraists.

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DEFINITION 3.2. The *ternary discriminator function* t on a set S is the mapping $t: S^3 \longrightarrow S$ defined by

$$t(a, b, c) = \begin{cases} c & \text{if } a = b \\ a & \text{otherwise.} \end{cases}$$

For $\mathbf{A}$ a structure let $\mathbf{A}^t$ denote the expansion by the ternary discriminator. And for H a class of structures let H^t be the class of $\mathbf{A}^t$, for $\mathbf{A}$ in H.

We are only going to use the simplest properties of the ternary discriminator, namely the ability to define the indecomposable factors and to define factor congruences.

LEMMA 3.3. *Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be nontrivial structures.*
(a) $\mathbf{A}_1^t \times \cdots \times \mathbf{A}_n^t$ satisfies the sentence

$$\phi_{ind} := \forall x \forall y \forall z \Big(\big(x = y \longrightarrow t(x, y, z) = z\big) \wedge \big(x \neq y \longrightarrow t(x, y, z) = x\big)\Big)$$

iff $n = 1$.
(b) Given a, b from $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ the binary relation

$$\{(c, d) : t(a, b, c) = t(a, b, d)\}$$

is the kernel of the projection map

$$\pi_J: \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \longrightarrow \prod_{j \in J} \mathbf{A}_j$$

where $J = \{i : a_i = b_i, 1 \leq i \leq n\}$.

PROOF. (Straightforward.) ■

THEOREM 3.4. *Suppose that K is admissible.*
(a) If K is loaded then K has a first-order law.
(b) If K is not loaded then there is an admissible K' with the same fine spectrum as K such that K' does not have a first-order law.

PROOF. (a) Let ϕ be a first-order sentence with a Feferman-Vaught sequence $\langle \Phi, \phi_1, \ldots, \phi_k \rangle$; and let each ϕ_i have a Feferman-Vaught sequence $\langle \Phi_i, \phi_{i,1}, \ldots, \phi_{i,k_i} \rangle$. Let $\mathsf{F}_0, \ldots, \mathsf{F}_{\ell-1}$ be the equivalence classes of F, the class of K-indecomposables, obtained by defining two members of F to be equivalent when they agree on the same $\phi_{i,j}$'s. Let $c = \max\{c_{\phi_j} : 1 \leq j \leq k\}$.

For $0 \leq i < \ell$ and $0 \leq j \leq c$ define

$$\mathsf{H}_{ij} = \begin{cases} \mathsf{F}_i^j & \text{if } j < c \\ \mathsf{F}_i^{\geq c} & \text{if } j = c. \end{cases}$$

Then for $0 \leq j_i \leq c$ let

$$p_{j_0, \ldots, j_{\ell-1}} = \mathrm{Prob}_{\mathsf{K}}(\text{is in } \mathsf{H}_{0, j_0} \cdots \mathsf{H}_{\ell-1, j_{\ell-1}}).$$

For $\mathbf{A} \in \mathsf{K}$ let $\gamma(\mathbf{A})$ be the function in $(c+1)^\ell$, *i.e.*, $\gamma(\mathbf{A})\colon \{0,\ldots,\ell-1\} \longrightarrow \{0,\ldots,c\}$, defined by: $\gamma(\mathbf{A})(i)$ is the minimum of c and the number of factors of $\mathbf{A}$ from F_i (in a complete factorization of $\mathbf{A}$). A consequence of Lemma 3.1 is that if $\gamma(\mathbf{A}) = \gamma(\mathbf{B})$ then either both $\mathbf{A}$ and $\mathbf{B}$ satisfy ϕ, or neither do.

Thus for $g \in (c+1)^\ell$ one has $\gamma^{-1}(g) \subseteq \mathsf{K}$ consists of all structures $\mathbf{A}$ in K such that, for $0 \le i < \ell$, $\mathbf{A}$ has exactly $g(i)$ factors from F_i if $g(i) < c$, and at least c factors from F_i if $g(i) = c$. Consequently

$$\mathrm{Prob}_{\mathsf{K}}\,(\text{is in } \gamma^{-1}(g)) = p_{g(0),\ldots,g(\ell-1)}.$$

Define a function $f\colon (c+1)^\ell \longrightarrow \{0,1\}$ by $f(g) = 1$ iff members of $\gamma^{-1}(g)$ satisfy ϕ. Then we have

$$\mathrm{Prob}_{\mathsf{K}}(\phi) = \sum_{g\in(c+1)^\ell} f(g)\cdot p_{g(0),\ldots,g(\ell-1)}.$$

(b) Now let us suppose that K is not loaded. Let F be the class of K-indecomposables. Let $\mathsf{F}_1,\ldots,\mathsf{F}_k$ be a partition of F into classes closed under isomorphism, and let $r_1,\ldots,r_k$ be a sequence of nonnegative integers such that

$$\mathrm{Prob}_{\mathsf{K}}\,(\text{is in } \mathsf{F}_1^{\ge r_1}\cdots\mathsf{F}_k^{\ge r_k})$$

is not defined. Let F' be an expansion of F^t by $2k$ constants a_i, b_i, such that we have $a_i = b_i$ in members of F_i, and $a_i \ne b_i$ in members of $\mathsf{F}\setminus\mathsf{F}_i$. Let $\mathsf{K}' = IP_{\mathrm{fin}}(\mathsf{F}')$. Then

- K' is admissible;
- the set of K'-indecomposables is F';
- the fine spectrum of K' is the same as that of K;
- the K'-indecomposables F' are defined in K' by ϕ_{ind} (from Lemma 3.3);
- as $\mathrm{Prob}_{\mathsf{K}}$ (is in $\mathsf{F}_1^{\ge r_1}\cdots\mathsf{F}_k^{\ge r_k}$) does not exist, it follows that $\mathrm{Prob}_{\mathsf{K}'}$ (is in $(\mathsf{F}_1')^{\ge r_1}\cdots(\mathsf{F}_k')^{\ge r_k}$) does not exist;
- $(\mathsf{F}_1')^{\ge r_1}\cdots(\mathsf{F}_k')^{\ge r_k}$ is defined in K' by the sentence ϕ which expresses "has a least r_i indecomposable factors, for $1 \le i \le k$, which satisfy $a_i = b_i$" (such a sentence can be constructed using Lemma 3.3).

As $\mathrm{Prob}_{\mathsf{K}'}$ (is in $(\mathsf{F}_1')^{\ge r_1}\cdots(\mathsf{F}_k')^{\ge r_k}$) does not exist, it follows that K' does not have a first-order law. ∎

Thus, among the admissible classes K, the ones for which knowledge of the fine spectrum alone is sufficient to conclude a first-order law are precisely the ones that are loaded.

4. Monotone functions and regular variation at infinity.

In this section we collect some basic facts about the behavior of $f(xt)/f(t)$ as $t \longrightarrow +\infty$.

DEFINITION 4.1. Let $f\colon (0,+\infty) \longrightarrow [0,+\infty)$ be given. Define the functions ρ_f and $\rho_f^\star$ by

$$\rho_f(x) = \lim_{t\longrightarrow+\infty} \frac{f(xt)}{f(t)},$$

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provided this limit exists and is in $(0, +\infty)$;

$$\rho_f^\star(x) = \lim_{t \to +\infty} \frac{f(xt)}{f(t)},$$

provided this limit exists and is in $[0, +\infty]$;

f is said to have *regular variation (at infinity) with index* α (see [9], p. 3), also indicated by writing $f \in RV_\alpha$, if

(a) $\mathrm{dom}(\rho_f) = (0, +\infty)$, and

(b) $\rho_f(x) = x^\alpha$ on its domain.

PROPOSITION 4.2. *Suppose* $f\colon (0, +\infty) \longrightarrow [0, +\infty)$ *is* nondecreasing. *Then*

- ρ_f *and* $\rho_f^\star$ *are nondecreasing on their domains,*
- ρ_f *is a multiplicative function on its domain, a multiplicative subgroup of the positive reals,*
- *and we have only the following three possibilities:*
  - *(a)* $\mathrm{dom}(\rho_f) = \mathrm{dom}(\rho_f^\star) = (0, +\infty)$. *In this case* $f \in RV_\alpha$ *for some* $\alpha \geq 0$*;*
  - *(b)* $\mathrm{dom}(\rho_f) = \mathrm{dom}(\rho_f^\star) = \{c^n : n \in Z\}$ *for some* $c \geq 1$. *In this case there is an* $\alpha > 0$ *such that* $\rho_f(x) = x^\alpha$ *for* $x \in \mathrm{dom}(\rho_f)$.
  - *(c)* $\mathrm{dom}(\rho_f) = \{1\} \neq \mathrm{dom}(\rho_f^\star)$. *In this case there is a* $c \geq 1$ *such that*

$$\rho_f^\star(x) = \begin{cases} 0 & \text{for } x \in (0, 1/c) \\ 1 & \text{for } x = 1 \\ +\infty & \text{for } x \in (c, +\infty). \end{cases}$$

Furthermore, no point of $(1/c, 1) \cup (1, c)$ *is in* $\mathrm{dom}(\rho_f^\star)$.

PROOF. The fact that ρ_f is nondecreasing on its domain follows simply from the monotonicity of f since $0 < c < d$ implies

$$\frac{f(ct)}{f(t)} \leq \frac{f(dt)}{f(t)}.$$

If $c, d \in \mathrm{dom}(\rho_f)$, then

$$\frac{f(\frac{c}{d}t)}{f(t)} = \frac{f(\frac{c}{d}t)}{f(\frac{1}{d}t)} \Big/ \frac{f(t)}{f(\frac{1}{d}t)}$$

suffices to show the limit as $t \longrightarrow +\infty$ of the left side exists, and that

$$\rho_f(c/d) = \rho_f(c)/\rho_f(d).$$

Thus $\mathrm{dom}(\rho_f)$ is a multiplicative subgroup of the positive reals, and ρ_f is a multiplicative function.

Now the multiplicative subgroups of the positive reals are either dense in the positive reals, or they are cyclic. In the former case, as ρ_f is multiplicative and monotone on its domain, it follows that ρ_f is continuous at $x = 1$; and then one can show $\mathrm{dom}(\rho_f) = (0, +\infty)$, and ρ_f is continuous on this domain. Thus it must be x^α for some $\alpha \geq 0$.

The other claims are now relatively straightforward to verify. ∎

COROLLARY 4.3. *Let* f *be nondecreasing and nonnegative on* $(0, +\infty)$. *If* $\rho_f(c) = 1$ *for a single positive* $c \neq 1$ *then* $f \in RV_0$.

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PROOF. From $\rho_f(c) = 1$ follows $\rho_f(c^n) = 1$ for $n \in Z$. Given $x > 0$ choose $n \in Z$ such that $c^n \leq x < c^{n+1}$. Then we have, from the monotonicity of f,

$$\frac{f(c^n t)}{f(t)} \leq \frac{f(xt)}{f(t)} \leq \frac{f(c^{n+1} t)}{f(t)};$$

and, as the limit as $t \longrightarrow +\infty$ of the extremes is 1, it follows the same holds for the middle quotient. Thus $\rho_f(x) = 1$ for $x > 0$, *i.e.*, $f \in RV_0$. ∎

5. **Generalized integers.** The interesting questions about loaded admissible classes K are now mainly questions about developing a better understanding of this concept—such questions are really in the domain of number theory, in the study of Beurling's *generalized integers*, as described in the 1969 article [1] by Bateman and Diamond.

DEFINITION 5.1. Let

$$\mathbf{d} = (d_1, d_2, \ldots)$$

be a (possibly finite) sequence of integers[4] with $1 < d_1 \leq d_2 \leq \cdots$. If the sequence is infinite we require $d_n \longrightarrow +\infty$.

(a) The *generalized integers* $\mathsf{N}_{\mathbf{d}}$ consists of the set of sequences $\mathbf{n} = (n_1, n_2, \ldots)$ of nonnegative integers, of the same length as $\mathbf{d}$, which are eventually 0 if $\mathbf{d}$ is an infinite sequence.[5]

(b) The product $\mathbf{m} \cdot \mathbf{n}$ of two generalized integers $\mathbf{m}$ and $\mathbf{n}$ is given by coordinatewise addition, *i.e.*,

$$\mathbf{m} \cdot \mathbf{n} = (m_i + n_i).$$

Note that the sequence $(0, 0, \ldots)$ of zeros in $\mathsf{N}_{\mathbf{d}}$ is the multiplicative identity $\mathbf{1}$ of $\mathsf{N}_{\mathbf{d}}$.

(c) The *generalized primes* P consist of the set of elements of $\mathsf{N}_{\mathbf{d}}$ of the form $(0, \ldots, 0, 1, 0, \ldots)$, *i.e.*, members of $\mathsf{N}_{\mathbf{d}}$ which are 0 except for one coordinate which is 1. $\mathbf{p}_i$, the i-th generalized prime, has a 1 only in the i-th coordinate.

Note that the sets $\mathsf{N}_{\mathbf{d}}$ and P depend only on the length of the sequence $\mathbf{d}$.

(d) Every element of $\mathsf{N}_{\mathbf{d}}$ different from $(0, 0, \ldots)$ has a *unique factorization* into members of P.

(e) The *size* $|\mathbf{n}|$ of a generalized integer $\mathbf{n}$ does depend on $\mathbf{d}$, and is given by

$$|\mathbf{n}| = \prod d_i^{n_i}.$$

(f) The notation $\sum_{\mathbf{p} \in \mathsf{Q}}$, where $\mathsf{Q} \subseteq \mathsf{P}$, means $\sum_{n=1}^{+\infty} \sum_{\substack{\mathbf{p} \in \mathsf{Q} \\ |\mathbf{p}| = n}}$. We use the notation $\prod_{\mathbf{p} \in \mathsf{Q}}$ in a similar manner.

[4] Bateman and Diamond [1] allow the d_n to be positive real numbers. Our definition of generalized integers reflects the context of working with admissible classes. Essentially all of the results of this section carry over verbatim to their setting.

[5] Bateman and Diamond only consider the case that the sequence $\mathbf{d}$ is infinite. For generalized integers they use the sizes $\prod d_i^{n_i}$ instead of the sequences $(n_1, n_2, \ldots)$. Of course several sequences may have the same size, so they speak of their generalized integers as having a certain *multiplicity*.

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DEFINITION 5.2. (a) a_n is the number of generalized integers of size n, namely

$$a_n = \big|\{\mathbf{m} \in \mathsf{N}_{\mathbf{d}} : |\mathbf{m}| = n\}\big|.$$

Note that $a_1 = 1$.

(b) $A_{\mathbf{d}}(x)$ is the number of generalized integers of size at most x, *i.e.*,

$$A_{\mathbf{d}}(x) = \sum_{n \le x} a_n.$$

For any property P, $A_{\mathbf{d}}(x \mid P)$ is the number of generalized integers of size at most x which satisfy P.

The following simple observation is quite important.

LEMMA 5.3. *The number of generalized integers of size at most x which are multiples of a given generalized integer* $\mathbf{m}$ *is precisely* $A_{\mathbf{d}}(x/|\mathbf{m}|)$*.*

PROOF. We observe that

$$|\mathbf{m} \cdot \mathbf{n}| \le x \quad \text{iff } |\mathbf{m}| \cdot |\mathbf{n}| \le x \quad \text{iff } |\mathbf{n}| \le x/|\mathbf{m}|.$$

From the unique factorization property in $\mathsf{N}_{\mathbf{d}}$ we know that $\mathbf{m} \cdot \mathbf{n}_1 = \mathbf{m} \cdot \mathbf{n}_2$ iff $\mathbf{n}_1 = \mathbf{n}_2$. Thus the set of multiples of $\mathbf{m}$ which have size at most x is in one-one correspondence with the set of generalized integers of size at most $x/|\mathbf{m}|$ using the mapping $\mathbf{m} \cdot \mathbf{n} \longmapsto \mathbf{n}$.
■

In terms of Dirichlet series and Euler products we have, at least formally,

$$\sum_{n=1}^{+\infty} a_n/n^s = \prod_{\mathbf{p} \in \mathsf{P}} \Big(\sum_n \frac{1}{|\mathbf{p}|^{ns}} \Big) = \prod_{\mathbf{p} \in \mathsf{P}} \Big(1 - \frac{1}{|\mathbf{p}|^s} \Big)^{-1}. \tag{2}$$

REMARK 5.4. Every admissible class K of finite structures has a system of generalized integers $\mathsf{N}_{\mathbf{d}}$ associated with it, namely let $\mathbf{D}_1, \mathbf{D}_2, \ldots$ be representatives of the K-indecomposables, ordered by increasing size, and let $d_i = |D_i|$. Define the map $\nu \colon \mathsf{K} \longrightarrow \mathsf{N}_{\mathbf{d}}$ by $\nu(\mathbf{A}) = \mathbf{n}$, where, given the complete factorization

$$\mathbf{A} \cong \mathbf{D}_{i_1}^{a_1} \times \cdots \times \mathbf{D}_{i_k}^{a_k},$$

$\mathbf{n}$ is the member of $\mathsf{N}_{\mathbf{d}}$ with the nonzero entries $n_{i_j} = a_j$. This map satisfies $\nu(\mathbf{A} \times \mathbf{B}) = \nu(\mathbf{A}) \cdot \nu(\mathbf{B})$; and its kernel is the equivalence relation of '*is isomorphic to*'.

Conversely, given generalized integers $\mathsf{N}_{\mathbf{d}}$, one can find an admissible class K such that $\mathsf{N}_{\mathbf{d}}$ is the set of generalized integers associated with K. Thus counting problems for admissible classes, where one counts up to isomorphism, are identical to counting problems in generalized integers.

Now let us translate the basic definitions regarding loaded classes to the language of generalized integers.

DEFINITION 5.5. (a) The (asymptotic) density of $\mathsf{S} \subseteq \mathsf{N}_{\mathbf{d}}$ is given by

$$\Delta(\mathsf{S}) = \lim_{x \longrightarrow +\infty} \frac{A_{\mathbf{d}}(x \mid \text{is in } \mathsf{S})}{A_{\mathbf{d}}(x)},$$

provided this limit exists.

(b) For $\mathsf{H}_i \subseteq \mathsf{N}_{\mathbf{d}}$ define

$$\mathsf{H}_1 \cdots \mathsf{H}_k = \{\mathbf{n}_1 \cdots \mathbf{n}_k \mid \mathbf{n}_1 \in \mathsf{H}_1, \ldots, \mathbf{n}_k \in \mathsf{H}_k\}.$$

We write H^k for the set of generalized integers which can be factored as the product of exactly k members of H. And let

$$\mathsf{H}^{\geq r} = \bigcup_{i \geq r} \mathsf{H}^i,$$

the set of generalized integers which can be written as the product of at least r members from H.

DEFINITION 5.6. $\mathsf{N}_{\mathbf{d}}$ is *loaded* if, for every partition $\mathsf{P}_1, \ldots, \mathsf{P}_k$ of the generalized primes P and for every sequence $r_1, \ldots, r_k$ of nonnegative integers, the set

$$\mathsf{P}_1^{\geq r_1} \cdots \mathsf{P}_k^{\geq r_k}$$

of generalized integers which have, for each i, at least r_i factors from P_i (including repeats), has an asymptotic density.

PROPOSITION 5.7. *Suppose* $\mathsf{N}_{\mathbf{d}}$ *is loaded. Then*

(a) for $\mathsf{P}_1, \ldots, \mathsf{P}_k$ *a partition of* P *and for* $r_i \geq 0$ *the set*

$$\mathsf{P}_1^{r_1} \cdots \mathsf{P}_i^{r_i} \cdot \mathsf{P}_{i+1}^{\geq r_{i+1}} \cdots \mathsf{P}_k^{\geq r_k}$$

has a density;

(b) for $\mathsf{Q} \subseteq \mathsf{P}$ *and r a nonnegative integer, the following sets have a density:*

1. $\mathsf{Q}^{\geq r} \cdot (\mathsf{P} \setminus \mathsf{Q})^{\geq 0}$, *i.e., the set of generalized integers which have at least r factors from* Q*;*
2. $\mathsf{Q}^{r} \cdot (\mathsf{P} \setminus \mathsf{Q})^{\geq 0}$, *i.e., the set of generalized integers which have exactly r factors from* Q*;*
3. Q^{r}, *i.e., the set of generalized integers which factor into exactly r factors, all from* Q*;*
4. $\mathsf{Q}^{\geq r}$, *i.e., the set of generalized integers which factor into at least r factors, and all factors come from* Q*;*

(c) $\Delta(\mathsf{P}) = 0$, *i.e., the set of generalized primes has density* 0*, and*

(d) $\Delta(\mathbf{m} \cdot \mathsf{N}_{\mathbf{d}}) = \lim_{x \longrightarrow +\infty} \frac{A_{\mathbf{d}}(x / |\mathbf{m}|)}{A_{\mathbf{d}}(x)} = \rho_{A_{\mathbf{d}}}(1 / |\mathbf{m}|)$ *exists and is* > 0*, i.e., the set of multiples of any generalized integer has positive density.*

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PROOF. Item (a) follows essentially as in Remark 2.4. For example if P is partitioned into two classes P_1 and P_2 then

$$\mathsf{P}_1^{r_1} \cdot \mathsf{P}_2^{\geq r_2} = \mathsf{P}_1^{\geq r_1} \cdot \mathsf{P}_2^{\geq r_2} \setminus \mathsf{P}_1^{\geq r_1+1} \cdot \mathsf{P}_2^{\geq r_2};$$

and as $\mathsf{P}_1^{\geq r_1} \cdot \mathsf{P}_2^{\geq r_2} \supseteq \mathsf{P}_1^{\geq r_1+1} \cdot \mathsf{P}_2^{\geq r_2}$ it follows that $\Delta(\mathsf{P}_1^{r_1} \cdot \mathsf{P}_2^{\geq r_2})$ exists and

$$\Delta(\mathsf{P}_1^{r_1} \cdot \mathsf{P}_2^{\geq r_2}) = \Delta(\mathsf{P}_1^{\geq r_1} \cdot \mathsf{P}_2^{\geq r_2}) - \Delta(\mathsf{P}_1^{\geq r_1+1} \cdot \mathsf{P}_2^{\geq r_2}).$$

Item (b) is a special case of item (a).

For item (c) first note that by (b) *every* set of generalized primes has a density. Now suppose $\Delta(\mathsf{P}) = a > 0$. This clearly implies that the number of generalized primes is infinite as the density of any finite set is 0. One can choose a sequence of integers $0 = n_0 < n_1 < n_2 < \cdots$ such that if we set

$$\mathsf{Q} = \{\mathbf{p} \in \mathsf{P} : n_{2i} < |\mathbf{p}| \leq n_{2i+1}\}$$

then, with $Q(x)$ being the counting function for Q, we have

$$\frac{Q(n_i)}{A_{\mathbf{d}}(n_i)} \text{ is } \begin{cases} > \frac{2}{3}a & \text{if } i \text{ is odd} \\ < \frac{1}{3}a & \text{if } i \text{ is even} \end{cases}.$$

But this guarantees that the set of primes Q does not have a density, contradicting item (b). Thus $\Delta(\mathsf{P}) = 0$.

For item (d) we use Lemma 5.3 to see that the number of generalized integers of size at most x which are multiples of $\mathbf{m}$ is precisely $A_{\mathbf{d}}(x/|\mathbf{m}|)$. To show that this limit exists we simply factor $\mathbf{m}$ to obtain $\mathbf{m} = \mathbf{p}_1^{r_1} \cdots \mathbf{p}_k^{r_k}$, where the $\mathbf{p}_i$ are distinct generalized primes. Then let $\mathsf{P}_i = \{\mathbf{p}_i\}$, for $1 \leq i \leq k$; and let $\mathsf{P}_{k+1} = \mathsf{P} \setminus \{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$. Then $\mathsf{P}_1, \ldots, \mathsf{P}_{k+1}$ is a finite partition of the primes, and

$$\mathbf{m} \cdot \mathsf{N}_{\mathbf{d}} = \mathsf{P}_1^{\geq r_1} \cdots \mathsf{P}_k^{\geq r_k} \cdot \mathsf{P}_{k+1}^{\geq 0}.$$

The right hand side has a density by the definition of loaded. Now we need to show this density is positive.

First, suppose $\mathbf{n} \cdot \mathsf{N}_{\mathbf{d}}$ has positive density for some $\mathbf{n} \neq \mathbf{1}$. Then, by the monotonicity of $A_{\mathbf{d}}$, we can apply Proposition 4.2 to claim $\rho_{A_{\mathbf{d}}}(|\mathbf{n}|^k) = \rho_{A_{\mathbf{d}}}(|\mathbf{n}|)^k > 0$ for $k \in Z$. As $\rho_{A_{\mathbf{d}}}$ is nondecreasing it follows that $\rho_{A_{\mathbf{d}}}(1/|\mathbf{n}|) > 0$ for all $\mathbf{n} \in \mathsf{N}_{\mathbf{d}}$. Thus the density of each $\mathbf{n} \cdot \mathsf{N}_{\mathbf{d}}$ is positive.

So now suppose that $\Delta(\mathbf{n} \cdot \mathsf{N}_{\mathbf{d}}) = 0$ for every $\mathbf{n} \in \mathsf{N}_{\mathbf{d}}$, $\mathbf{n} \neq \mathbf{1}$. Then for any finite subset F of $\mathsf{N}_{\mathbf{d}}$ we have $\Delta(\mathsf{F} \cdot \mathsf{N}_{\mathbf{d}}) = 0$, *i.e.*, the density of the set of multiples of any finite subset of generalized integers is 0. Now we can use this information to construct a sequence of integers $0 = n_0 < n_1 < n_2 < \cdots$ such that if we set

$$\mathsf{Q} = \{\mathbf{p} \in \mathsf{P} : n_{2i} < |\mathbf{p}| \leq n_{2i+1}\}$$

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then, with $Q^{\geq 0}(x)$ being the counting function for $\mathsf{Q}^{\geq 0}$, we have

$$\frac{Q^{\geq 0}(n_i)}{A_{\mathbf{d}}(n_i)} \text{ is } \begin{cases} > \frac{2}{3} & \text{if } i \text{ is odd} \\ < \frac{1}{3} & \text{if } i \text{ is even} \end{cases}.$$

But this guarantees that the set $\mathsf{Q}^{\geq 0}$ does not have a density, contradicting item (b). Thus $\Delta(\mathbf{n}\cdot\mathsf{N}_{\mathbf{d}}) > 0$ for all $\mathbf{n} \in \mathsf{N}_{\mathbf{d}}$. ∎

REMARK 5.8 (BATEMAN AND DIAMOND ([1], THEOREM 4C)). prove that $\Delta(\mathsf{P}) = 0$ under the hypotheses (a) $A_{\mathbf{d}} \in RV_\alpha$ for some $\alpha > 0$, and (b) $\sum_{\mathbf{p}\in\mathsf{P}} \frac{1}{|\mathbf{p}|^\alpha} = +\infty$.

The definition of regular variation is too restrictive for some of the cases we will encounter. Thus we introduce the following:

DEFINITION 5.9. Let $f\colon(0,+\infty) \longrightarrow [0,+\infty)$ be given. f is said to have *d-regular variation (at infinity) with index α*, also indicated by writing $f \in RV_\alpha(d)$, if

$$\lim_{n\longrightarrow+\infty} \frac{f(d^m\cdot d^n)}{f(d^n)} = (d^m)^\alpha$$

for m a non-negative integer, and for n restricted to the integers.

COROLLARY 5.10. *If $\mathsf{N}_{\mathbf{d}}$ is loaded then either*

(a) $A_{\mathbf{d}} \in RV_\alpha$ for some $\alpha \geq 0$, or

(b) there is a positive integer d such that all sizes $|\mathbf{m}|$ of the generalized integers from N_d are powers of d, and $A_{\mathbf{d}} \in RV_\alpha(d)$ for some $\alpha \geq 0$.

PROOF. The function $A_{\mathbf{d}}$ is nonnegative and monotone, so Proposition 4.2 applies. By Proposition 5.7(d) we see that the case (c) in 4.2 cannot occur. 4.2(a) gives our (a).

The other possibility is 4.2(b). So let $\mathrm{dom}(\rho_{A_{\mathbf{d}}}) = \{c^n : n \in Z\}$. Let $\{d^n : n \in Z\}$ be the cyclic subgroup generated by $\{|\mathbf{p}| : \mathbf{p} \in \mathsf{P}\}$, the set of sizes of the generalized primes (which must be in $\mathrm{dom}(\rho_{A_{\mathbf{d}}})$ by 5.7(d) and 4.2). We can assume $d > 1$. As d is rational, it follows that it must be an integer. Thus all the sizes of the primes, the d_i's, are powers of a single integer d, and then it follows that all the sizes of the integers in $\mathsf{N}_{\mathbf{d}}$ are also powers of d. Now let

$$\lim_{n\longrightarrow+\infty} \frac{A_{\mathbf{d}}(d^{n+1})}{A_{\mathbf{d}}(d^n)} = \beta,$$

and let

$$\alpha = \log_d(\beta).$$

Then it is easy to check that $A_{\mathbf{d}} \in RV_\alpha(d)$. ∎

Because of the importance of the case (b) in Corollary 5.10 we introduce the following:

DEFINITION 5.11. $\mathsf{N}_{\mathbf{d}}$ is a *discrete* generalized number system if the sizes of the generalized integers are all powers of some integer d.

From Corollary 5.10 we see the study of loaded $\mathsf{N}_{\mathbf{d}}$'s naturally splitting into two (overlapping!) cases:

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- $\mathsf{N}_{\mathbf{d}}$ is *regular* if $A_{\mathbf{d}} \in RV_\alpha$, and
- $\mathsf{N}_{\mathbf{d}}$ is *discretely regular* if the sizes of the generalized integers are all powers of some integer d, and $A_{\mathbf{d}} \in RV_\alpha(d)$ for some $\alpha \geq 0$.

In both cases we are able to obtain wide ranging results that apply to many examples of generalized integers (and admissible classes) studied in the literature. Some of the open questions in the discrete case appear to involve difficult additive number theory. Questions regarding both cases are formulated at the end of this section.

EXAMPLE 5.12. A system of generalized integers with exactly one generalized prime is an example of a system that is both regular and discrete.

It is no accident that this example is actually slowly varying at infinity.

PROPOSITION 5.13. *Let $\mathsf{N}_{\mathbf{d}}$ be a system of generalized integers that is both regular and discrete. Then $A_{\mathbf{d}} \in RV_0$.*

PROOF. Let the sizes of the generalized primes in P form a subset of $\{d, d^2, \ldots, d^k, \ldots\}$, where d is the positive integer that generates the domain of $\rho_{A_{\mathbf{d}}}$. Then one can find arbitrarily large reals t such that no sizes of members of $\mathsf{N}_{\mathbf{d}}$ are in the interval $[t, \frac{3}{2}t]$. Thus one can find arbitrarily large t such that $A_{\mathbf{d}}(\frac{3}{2}t)/A_{\mathbf{d}}(t) = 1$. By the regularity assumption it follows that $\lim_{t\to+\infty} A_{\mathbf{d}}(\frac{3}{2}t)/A_{\mathbf{d}}(t) = 1$. Then from Lemma 4.3 one has $A_{\mathbf{d}}(x) \in RV_0$. ∎

REMARK 5.14. For any system $\mathsf{N}_{\mathbf{d}}$ of generalized integers with P infinite, Bateman and Diamond ([1], p. 158) show that $\sum_{\mathbf{n}\in\mathsf{N}_{\mathbf{d}}} |\mathbf{n}|^{-\sigma}$, $\prod_{\mathbf{p}\in\mathsf{P}}(1-|\mathbf{p}|^{-\sigma})^{-1}$, and $\sum_{\mathbf{p}\in\mathsf{P}} |\mathbf{p}|^{-\sigma}$ converge for the same positive real σ. The infimum of such σ will be called the *abscissa of convergence* of $\mathsf{N}_{\mathbf{d}}$ (or of P). If, furthermore, $A_{\mathbf{d}} \in RV_\alpha$, then they note on p. 166 of [1] that α is the abscissa of convergence of $\mathsf{N}_{\mathbf{d}}$.

PROPOSITION 5.15. *Let $\mathsf{N}_{\mathbf{d}}$ be a system of generalized integers, and let α be the abscissa of convergence of $\sum a_n/n^s$.*

(a) The abscissa of convergence is given by

$$\alpha = \limsup_{n\to+\infty} \frac{\log A_{\mathbf{d}}(n)}{\log n}.$$

(b) If $\alpha < +\infty$ then, for $s > \alpha$,

$$A_{\mathbf{d}}(n) = O(n^s).$$

(c) If $\mathsf{N}_{\mathbf{d}}$ is loaded then $\alpha < +\infty$.

PROOF. For (a) and (b) see Titchmarsh [23], §9.14, 292–293. For (c) choose $\mathbf{a} \in \mathsf{N}_{\mathbf{d}}$ with $\mathbf{a} \neq \mathbf{1}$. Then $\mathsf{N}_{\mathbf{d}}$ is loaded implies $\mathbf{a}\cdot\mathsf{N}_{\mathbf{d}}$ has positive density γ. Choose $\lambda > \gamma^{-1}$, and let $a = |\mathbf{a}|$. Then

$$\gamma = \Delta(\mathbf{a}\cdot\mathsf{N}_{\mathbf{d}}) = \lim_{n\to+\infty} \frac{A_{\mathbf{d}}(a^n)}{A_{\mathbf{d}}(a^{n+1})},$$

so $A_{\mathbf{d}}(a^n) = O(\lambda^n)$. Since $A_{\mathbf{d}}$ is nondecreasing, $A_{\mathbf{d}}(x) = O(x^{\log_a \lambda})$. Thus by part (a) the abscissa of convergence is at most $\log_a \lambda$. ∎

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THE REGULAR CASE.

THEOREM 5.16. *Let* $\mathsf{N}_{\mathbf{d}}$ *be a system of generalized integers such that*
(a) $A_{\mathbf{d}}$ *has regular variation (at infinity) with index* $\alpha(\geq 0)$*, i.e.,*

$$\rho_{A_{\mathbf{d}}}(x) = \lim_{t\longrightarrow+\infty} \frac{A_{\mathbf{d}}(xt)}{A_{\mathbf{d}}(t)} = x^{\alpha}$$

for $x > 0$ *;*
(b) there is a positive constant C *such that*

$$A_{\mathbf{d}}(xt) \geq Cx^{\alpha} A_{\mathbf{d}}(t)$$

for $t, x \geq 1$.
Then $\mathsf{N}_{\mathbf{d}}$ *is loaded.*

PROOF. Let $\mathsf{P}_1, \ldots, \mathsf{P}_k$ be a partition of the generalized primes P and $r_1, \ldots, r_k$ a sequence of nonnegative integers. Let

$$\mathsf{B} = \mathsf{P}_1^{\geq r_1} \cdots \mathsf{P}_k^{\geq r_k}$$

have the Dirichlet series $\sum_{n=1}^{+\infty} b_n / n^s$, *i.e.*, b_n is the number of elements of B of size n; and let $B(x)$ be the counting function for B. Let $\{1, \ldots, k\} = J_1 \cup J_2$ where

$$J_1 = \{j : 1 \leq j \leq k, \sum_{\mathbf{p}\in\mathsf{P}_j} \frac{1}{|\mathbf{p}|^{\alpha}} < +\infty\},$$

$$J_2 = \{j : 1 \leq j \leq k, \sum_{\mathbf{p}\in\mathsf{P}_j} \frac{1}{|\mathbf{p}|^{\alpha}} = +\infty\}.$$

The class P_j is *large* if $j \in J_2$; and *small* otherwise.

CASE 1. First we consider the case that $r_j = 0$ for all $j \in J_2$; thus $\mathsf{P}_1^{\geq r_1} \cdots \mathsf{P}_k^{\geq r_k}$ puts restrictions only on the number of generalized primes belonging to the 'small' classes P_j. Then clearly we have

$$\begin{aligned}\sum_{n=1}^{+\infty} \frac{b_n}{n^s} &= \prod_{j\in J_1} \Bigg(\sum_{\substack{\ell\geq r_j \\ i_1\leq\cdots\leq i_\ell \\ \mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_\ell}\in\mathsf{P}_j}} \frac{1}{(|\mathbf{p}_{i_1}| \cdots |\mathbf{p}_{i_\ell}|)^s} \Bigg) \cdot \prod_{j\in J_2} \prod_{\mathbf{p}\in\mathsf{P}_j} (1 - \frac{1}{|\mathbf{p}|^s})^{-1} \\ &= \prod_{j\in J_1} g_j(s) \cdot \prod_{j=1}^{k} \prod_{\mathbf{p}\in\mathsf{P}_j} \left(1 - \frac{1}{|\mathbf{p}|^s}\right)^{-1}, \qquad (3)\end{aligned}$$

where

$$g_j(s) = \prod_{\mathbf{p}\in\mathsf{P}_j} \left(1 - \frac{1}{|\mathbf{p}|^s}\right) \cdot \sum_{\substack{\ell\geq r_j \\ i_1\leq\cdots\leq i_\ell \\ \mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_\ell}\in\mathsf{P}_j}} \frac{1}{(|\mathbf{p}_{i_1}| \cdots |\mathbf{p}_{i_\ell}|)^s}. \qquad (4)$$

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Let X_α denote the set of the Dirichlet series $\sum_{n=1}^{+\infty} e_n/n^s$ that are absolutely convergent for $s = \alpha$, *i.e.*,

$$\sum_{n=1}^{+\infty} \frac{|e_n|}{n^\alpha} < +\infty.$$

Clearly X_α is closed under the Dirichlet convolution product, *i.e.*, if $\sum_{n=1}^{+\infty} e_n/n^s$ and $\sum_{n=1}^{+\infty} f_n/n^s$ belong to X_α then so does $\sum_{n=1}^{+\infty} g_n/n^s$, where $g_n = \sum_{k|n} e_k f_{n/k}$. Now let us write

(5) $$\prod_{\mathbf{p}\in \mathsf{P}_j} \left(1 - \frac{1}{|\mathbf{p}|^s}\right) = \sum_{n=1}^{+\infty} \frac{x_n}{n^s},$$

(6) $$\prod_{\mathbf{p}\in \mathsf{P}_j} \left(1 + \frac{1}{|\mathbf{p}|^s}\right) = \sum_{n=1}^{+\infty} \frac{y_n}{n^s},$$

(7) $$\sum_{\substack{\ell\geq r_j \\ i_1\leq\cdots\leq i_\ell \\ \mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_\ell}\in \mathsf{P}_j}} \frac{1}{(|\mathbf{p}_{i_1}|\cdots|\mathbf{p}_{i_\ell}|)^s} = \sum_{n=1}^{+\infty} \frac{z_n}{n^s},$$

and

(8) $$\prod_{\mathbf{p}\in \mathsf{P}_j} \left(1 - \frac{1}{|\mathbf{p}|^s}\right)^{-1} = \prod_{\mathbf{p}\in \mathsf{P}_j} \left(\sum_{k=0}^{+\infty} \frac{1}{|\mathbf{p}|^{ks}}\right) = \sum_{n=1}^{+\infty} \frac{u_n}{n^s}.$$

Clearly we have

(9) $$|x_n| \leq y_n \leq u_n$$

and

(10) $$0 \leq z_n \leq u_n.$$

It follows from $j \in J_1$ that the Dirichlet series in (8) belongs to X_α, thus by (9) and (10), the Dirichlet series in (5) and (7) belong to X_α, too. Thus their product, the Dirichlet series representing $g_j(s)$, belongs to X_α (for all $j \in J_1$). Thus writing

(11) $$\prod_{j\in J_1} g_j(s) = \sum_{n=1}^{+\infty} \frac{v_n}{n^s},$$

this Dirichlet series belongs to X_α, *i.e.*,

(12) $$\sum_{n=1}^{+\infty} \frac{|v_n|}{n^\alpha} < +\infty.$$

By (2), (3) and (11) we have

(13) $$\sum_{n=1}^{+\infty} \frac{b_n}{n^s} = \sum_{n=1}^{+\infty} \frac{v_n}{n^s} \cdot \sum_{n=1}^{+\infty} \frac{a_n}{n^s},$$

whence

(14) $$B(x) = \sum_{n\leq x} b_n = \sum_{k\ell\leq x} v_k a_\ell = \sum_{k\leq x}\Big(v_k \sum_{\ell\leq x/k} a_\ell\Big) = \sum_{k\leq x} v_k A_{\mathbf{d}}(x/k).$$

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Fix an $\epsilon > 0$. Then, for $1 < \Omega < x$ we have, by (14),

(15)
$$\begin{aligned}
&\Big|B(x) - \Big(\sum_{k=1}^{+\infty} \frac{v_k}{k^\alpha}\Big) A_{\mathbf{d}}(x)\Big| \\
&\quad \le \Big|B(x) - \Big(\sum_{k\le\Omega} \frac{v_k}{k^\alpha}\Big) A_{\mathbf{d}}(x)\Big| + A_{\mathbf{d}}(x) \sum_{\Omega<k} \frac{|v_k|}{k^\alpha} \\
&\quad \le \Big|B(x) - \sum_{k\le\Omega} v_k A_{\mathbf{d}}(x/k)\Big| + \sum_{k\le\Omega} |v_k| \cdot \Big|A_{\mathbf{d}}(x/k) - \frac{A_{\mathbf{d}}(x)}{k^\alpha}\Big| + A_{\mathbf{d}}(x) \sum_{\Omega<k} \frac{|v_k|}{k^\alpha} \\
&\quad \le \Big|\sum_{\Omega<k\le x} v_k A_{\mathbf{d}}(x/k)\Big| + \sum_{k\le\Omega} |v_k| \cdot \Big|A_{\mathbf{d}}(x/k) - \frac{A_{\mathbf{d}}(x)}{k^\alpha}\Big| + A_{\mathbf{d}}(x) \sum_{\Omega<k} \frac{|v_k|}{k^\alpha}.
\end{aligned}$$

Denote the three terms in the last expression by R_1, R_2, and R_3. It follows from (12) that there is an $\Omega_0(\epsilon)$ such that

$$R_3 < \epsilon A_{\mathbf{d}}(x) \tag{16}$$

if $\Omega \ge \Omega_0(\epsilon)$. Next, by the hypothesis (b) of the theorem, there is a positive constant C such that

$$R_1 \le \sum_{\Omega<k\le x} |v_k| \cdot 1/C \cdot \frac{A_{\mathbf{d}}(x)}{k^\alpha} \le 1/C \cdot A_{\mathbf{d}}(x) \cdot \sum_{\Omega<k} \frac{|v_k|}{k^\alpha}, \tag{17}$$

so that by (12) there is an $\Omega_1(\epsilon)$ such that

$$R_1 < \epsilon A_{\mathbf{d}}(x)$$

if $\Omega \ge \Omega_1(\epsilon)$. Finally, it follows from the hypothesis (a) that, writing $\sum_{k\le\Omega} |v_k| = L_\Omega$, there is an $x_0(\epsilon, \Omega)$ such that

$$\Big|A_{\mathbf{d}}(x/k) - \frac{A_{\mathbf{d}}(x)}{k^\alpha}\Big| < \frac{\epsilon}{L_\Omega} A_{\mathbf{d}}(x) \quad \text{if } x \ge x_0(\epsilon, \Omega),$$

uniformly for $1 \le k \le \Omega$. Thus, for $x \ge x_0(\epsilon, \Omega)$, we have

$$R_2 < \sum_{k\le\Omega} |v_k| \frac{\epsilon}{L_\Omega} A_{\mathbf{d}}(x) = \epsilon A_{\mathbf{d}}(x). \tag{18}$$

Now let $\Omega = \max\big(\Omega_0(\epsilon), \Omega_1(\epsilon)\big)$. Then for $x \ge x_0(\epsilon, \Omega)$ we have from (15), (16), (17) and (18) that

$$R_1 + R_2 + R_3 < 3\epsilon A_{\mathbf{d}}(x).$$

Since ϵ is arbitrary it thus follows that

$$\lim_{x\to+\infty} \frac{B(x)}{A_{\mathbf{d}}(x)} = \sum_{k=1}^{+\infty} \frac{v_k}{k^\alpha} \tag{19}$$

(by (12) this infinite series is absolutely convergent), which completes the proof in the first case.

CASE 2. Consider now the general case when $r_j > 0$ may occur for some (or all) $j \in J_2$, *i.e.*, for some 'large' sets P_j. We will reduce this case to the previous one.

Recall that $\mathsf{B} = \mathsf{P}_1^{\geq r_1} \cdots \mathsf{P}_k^{\geq r_k}$, and let $\mathsf{B}^\star = \prod_{j\in J_1} \mathsf{P}_j^{\geq r_j} \cdot \prod_{j \in J_2} \mathsf{P}_j^{\geq 0}$, the subset of $\mathsf{N}_{\mathbf{d}}$ whose members have at least r_j factors from P_j for each $j \in J_1$, with no restriction on the number of factors from P_j for $j \in J_2$. Then $\mathsf{B} \subseteq \mathsf{B}^\star$. We define

$$\overline{\mathsf{B}} = \mathsf{B}^\star \setminus \mathsf{B}. \tag{20}$$

Let $\mathsf{E}_j = \mathsf{P}_j^{<r_j} \cdot (\mathsf{P} \setminus \mathsf{P}_j)^{\geq 0}$, the subset of $\mathsf{N}_{\mathbf{d}}$ for which the number of factors from P_j is less than r_j, and there are no restrictions on the number of factors from the other classes P_i. Denote the counting functions associated with $\mathsf{B}^\star$, $\overline{\mathsf{B}}$ and E_j by $B^\star(x)$, $\overline{B}(x)$ and $E_j(x)$, respectively. Clearly,

$$\overline{\mathsf{B}} \subseteq \bigcup_{\substack{j \in J_2 \\ r_j > 0}} \mathsf{E}_j. \tag{21}$$

Thus we have

$$|B^\star(x) - B(x)| \leq \sum_{\substack{j\in J_2\\ r_j>0}} E_j(x). \tag{22}$$

We know from the first half of the proof that the density associated with $\mathsf{B}^\star$, *i.e.*, the limit

$$\lim_{x \to +\infty} \frac{B^\star(x)}{A_{\mathbf{d}}(x)}$$

exists. Thus, in view of (22), it suffices to show that the density associated with E_j is 0 for each $j \in J_2$ for which $r_j > 0$, *i.e.*, if $j \in J_2$ and $r_j > 0$ then, for all $\epsilon > 0$, there is an $x_0(\epsilon, j)$ such that

$$E_j(x) < \epsilon A_{\mathbf{d}}(x) \text{ for } x > x_0(\epsilon, j). \tag{23}$$

For $j \in J_2$, with $r_j > 0$, and for $\Delta > 0$, consider a *finite* subset F of P_j with

$$\sum_{\mathbf{p} \in \mathsf{F}} \frac{1}{|\mathbf{p}|^\alpha} > \Delta. \tag{24}$$

(As $j \in J_2$, such a subset F exists.) Now consider the partition of P given by F, $\mathsf{P} \setminus \mathsf{F}$, and let $\mathsf{H} = \mathsf{F}^{<r_j} \cdot (\mathsf{P} \setminus \mathsf{F})^{\geq 0}$, the subset of $\mathsf{N}_{\mathbf{d}}$ whose members have less than r_j factors from F. Let $\overline{\mathsf{H}} = \mathsf{F}^{\geq r_j}(\mathsf{P} \setminus \mathsf{F})^{\geq 0}$, the complement of H in $\mathsf{N}_{\mathbf{d}}$, namely the generalized integers with at least r_j factors from F. Denote the counting functions associated with H and $\overline{\mathsf{H}}$ by $H(x)$ and $\overline{H}(x)$, respectively. Then clearly,

$$\mathsf{E}_j \subseteq \mathsf{H} \tag{25}$$

and

$$H(x) + \overline{H}(x) = A_{\mathbf{d}}(x). \tag{26}$$

By (25) and (26), we see that (23) would follow from

$$\overline{H}(x) > (1 - \epsilon) A_{\mathbf{d}}(x). \tag{27}$$

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To prove this actually holds, observe that the Case 1 part of the proof of this theorem can be applied to this situation to show the density associated with $\overline{\mathsf{H}}$ exists, and by (4), (11), and (19), this density is

$$\lim_{x\longrightarrow+\infty}\frac{\overline{H}(x)}{A_{\mathbf{d}}(x)}=\sum_{k=1}^{+\infty}\frac{v_k}{k^\alpha}, \tag{28}$$

where now the sequence (v_n) is defined by

$$\begin{aligned}
g_1(x) &= \sum_{n=1}^{+\infty}\frac{v_n}{n^s}\\
&= \prod_{\mathbf{p}\in\mathsf{F}}\Big(1-\frac{1}{|\mathbf{p}|^s}\Big)\sum_{\substack{\ell\geq r_j\\ i_1\leq\cdots\leq i_\ell\\ \mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_\ell}\in\mathsf{F}}}\frac{1}{(|\mathbf{p}_{i_1}|\cdots|\mathbf{p}_{i_\ell}|)^s}\\
&= \prod_{\mathbf{p}\in\mathsf{F}}\Big(1-\frac{1}{|\mathbf{p}|^s}\Big)\left(\prod_{\mathbf{p}\in\mathsf{F}}\Big(1-\frac{1}{|\mathbf{p}|^s}\Big)^{-1}-1-\sum_{\substack{1\leq\ell<r_j\\ i_1\leq\cdots\leq i_\ell\\ \mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_\ell}\in\mathsf{F}}}\frac{1}{(|\mathbf{p}_{i_1}|\cdots|\mathbf{p}_{i_\ell}|)^s}\right)\\
&= 1-\prod_{\mathbf{p}\in\mathsf{F}}\Big(1-\frac{1}{|\mathbf{p}|^s}\Big)\Big(1+\sum_{\substack{1\leq\ell<r_j\\ i_1\leq\cdots\leq i_\ell\\ \mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_\ell}\in\mathsf{F}}}\frac{1}{(|\mathbf{p}_{i_1}|\cdots|\mathbf{p}_{i_\ell}|)^s}\Big).
\end{aligned}$$

Thus the density in (28) is

$$g_1(\alpha)=\sum_{n=1}^{+\infty}\frac{v_n}{n^\alpha}=1-\prod_{\mathbf{p}\in\mathsf{F}}\Big(1-\frac{1}{|\mathbf{p}|^\alpha}\Big)\cdot\Big(1+\sum_{\substack{1\leq\ell<r_j\\ i_1\leq\cdots\leq i_\ell\\ \mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_\ell}\in\mathsf{F}}}\frac{1}{(|\mathbf{p}_{i_1}|\cdots|\mathbf{p}_{i_\ell}|)^\alpha}\Big). \tag{29}$$

Hence, writing

$$\sum_{\mathbf{p}\in\mathsf{F}}\frac{1}{|\mathbf{p}|^\alpha}=\delta,$$

and using the fact that $1-x<e^{-x}$ for $x>0$, we have

$$\begin{aligned}
1-\lim_{x\longrightarrow+\infty}\frac{\overline{H}(x)}{A_{\mathbf{d}}(x)} &= 1-g_1(\alpha)\\
&= \prod_{\mathbf{p}\in\mathsf{F}}\Big(1-\frac{1}{|\mathbf{p}|^\alpha}\Big)\cdot\Big(1+\sum_{\ell=1}^{r_j-1}\sum_{\substack{i_1\leq\cdots\leq i_\ell\\ \mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_\ell}\in\mathsf{F}}}\frac{1}{(|\mathbf{p}_{i_1}|\cdots|\mathbf{p}_{i_\ell}|)^\alpha}\Big)\\
&\leq \exp\Big(-\sum_{\mathbf{p}\in\mathsf{F}}\frac{1}{|\mathbf{p}|^\alpha}\Big)\cdot\left(1+\sum_{\ell=1}^{r_j-1}\Big(\sum_{\mathbf{p}\in\mathsf{F}}\frac{1}{|\mathbf{p}|^\alpha}\Big)^\ell\right)\\
&= \exp(-\delta)\cdot\Big(\sum_{\ell=0}^{r_j-1}\delta^\ell\Big).
\end{aligned} \tag{30}$$

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Clearly,

$$\lim_{x\to+\infty} \exp(-x)\cdot\left(\sum_{\ell=0}^{r_j-1} x^{\ell}\right) = 0.$$

Thus if Δ in (24) is large enough it follows from (30) that

$$1 - \lim_{x\to+\infty} \frac{\bar{H}(x)}{A_{\mathbf{d}}(x)} < \epsilon,$$

which proves (27); and this completes the proof of the theorem. ∎

REMARK 5.17. Let $\mathsf{N}_{\mathbf{d}}$ be a system of generalized integers with Dirichlet series $\zeta_{\mathbf{d}}(s) = \sum_{n=1}^{+\infty} a_n/n^s$. For $\mathsf{P}_1,\ldots,\mathsf{P}_k$ a partition of the generalized primes P, and $r_1,\ldots,r_k$ a sequence of nonnegative integers, let $\mathsf{B} = \mathsf{P}_1^{\geq r_1}\cdots\mathsf{P}_k^{\geq r_k}$ have the Dirichlet series $\zeta_{\mathsf{B}}(s) = \sum_{n=1}^{+\infty} b_n/n^s$. Assuming the hypotheses of Theorem 5.16 we have from (13) the equation $\zeta_{\mathsf{B}}(s) = \sum_{n=1}^{+\infty} \frac{v_n}{n^s}\cdot\zeta_{\mathbf{d}}(s)$, valid for $s > \alpha$. Let $\mathsf{B}_i = \mathsf{P}_i^{\geq r_i}\cdot\mathsf{N}_{\mathbf{d}}$, the set of generalized integers that have at least r_i factors (counting repeats) from P_i. Then from (3, 11, 19) we can deduce

(a) $\Delta(\mathsf{B}) = \lim_{s\to\alpha+} \zeta_{\mathsf{B}}(s)/\zeta_{\mathbf{d}}(s)$.

(b) $\Delta(\mathsf{B}) = \Delta(\mathsf{B}_1)\cdots\Delta(\mathsf{B}_k)$.

REMARK 5.18. Suppose that the assumptions (a), (b) of Theorem 5.16 hold. Furthermore suppose $\mathsf{Q}\subseteq\mathsf{P}$ satisfies

$$\sum_{\mathbf{p}\in\mathsf{Q}} \frac{1}{|\mathbf{p}|^{\alpha}} = +\infty.$$

Then, for $r \geq 0$, from the proof of Theorem 5.16 we have

(a) $\Delta\left(\mathsf{Q}^{\geq r}\cdot(\mathsf{P}\setminus\mathsf{Q})^{\geq 0}\right) = 1$, *i.e.*, the set of generalized integers with at least r factors from Q has density 1;

(b) $\Delta\left(\mathsf{Q}^{r}\cdot(\mathsf{P}\setminus\mathsf{Q})^{\geq 0}\right) = 0$, *i.e.*, the set of generalized integers with exactly r factors from Q has density 0.

Now we derive some corollaries from Theorem 5.16 that are easy to apply to numerous situations. We will be using the condition $A_{\mathbf{d}}(x) = \left(1+o(1)\right)x^{\alpha}S(x)$. As $A_{\mathbf{d}}(x) \geq 1$ for $x \geq 1$ it follows that $S(x)$ is eventually positive, and that $A_{\mathbf{d}}(x)\in RV_{\alpha}$ iff $S(x)\in RV_0$.

COROLLARY 5.19. *Let $\mathsf{N}_{\mathbf{d}}$ be a system of generalized integers, and suppose $\alpha \geq 0$ and $S(x)\in RV_0$ are such that*

(a) $A_{\mathbf{d}}(x) = \left(1+o(1)\right)x^{\alpha}S(x)$, and

(b) there is a positive constant C_0 such that $S(xt) \geq C_0 S(t)$ for $t, x \geq 1$.

Then $\mathsf{N}_{\mathbf{d}}$ is loaded.

PROOF. Let $C_1 = \min\left(1, \liminf_{x\to+\infty} S(x)\right)$. Then $C_1 > 0$ as $S(x)$ is eventually positive, and in view of condition (b).

Choose $\Omega \geq 1$ such that $A_{\mathbf{d}}(\Omega) > 0$, and for $u, v \geq \Omega$,

$$|A_{\mathbf{d}}(u) - u^{\alpha}S(u)| \leq \frac{1}{2}u^{\alpha}S(u) \tag{31}$$

$$S(u) \geq \frac{1}{2}C_1. \tag{32}$$

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Then for $u \geq \Omega$ we have

$$\frac{1}{2}u^{\alpha}S(u) \leq A_{\mathbf{d}}(u) \leq \frac{3}{2}u^{\alpha}S(u). \tag{33}$$

Let

$$C = \min\left(\frac{C_0}{3}, \frac{1}{\Omega^{\alpha}}, \frac{C_1}{4A_{\mathbf{d}}(\Omega)}\right).$$

CLAIM. *$A_{\mathbf{d}}(xt) \geq Cx^{\alpha}A_{\mathbf{d}}(t)$ for $x, t \geq 1$.*

Once we have proved this claim then applying Theorem 5.16 will conclude the proof of our corollary. We break the proof of the claim into three steps.

(I) Suppose $x \geq 1$ and $t \geq \Omega$. Then

$$\begin{aligned}
A_{\mathbf{d}}(xt) &\geq \frac{1}{2}(xt)^{\alpha}S(xt) \quad \text{by (33)}\\
&\geq \frac{1}{2}x^{\alpha}t^{\alpha}\big(C_0S(t)\big) \quad \text{by (b)}\\
&= (C_0/2)x^{\alpha}\big(t^{\alpha}S(t)\big)\\
&\geq (C_0/2)x^{\alpha}\Big(\frac{2}{3}A_{\mathbf{d}}(t)\Big) \quad \text{by (33)}\\
&= (C_0/3)x^{\alpha}A_{\mathbf{d}}(t)\\
&\geq Cx^{\alpha}A_{\mathbf{d}}(t).
\end{aligned}$$

(II) Suppose $1 \leq x, t \leq \Omega$. Then

$$\begin{aligned}
A_{\mathbf{d}}(xt) &\geq A_{\mathbf{d}}(t) \quad \text{as } A_{\mathbf{d}} \text{ is nondecreasing}\\
&\geq (x^{\alpha}/\Omega^{\alpha})A_{\mathbf{d}}(t) \quad \text{as } x^{\alpha} \leq \Omega^{\alpha}\\
&= (1/\Omega^{\alpha})x^{\alpha}A_{\mathbf{d}}(t)\\
&\geq Cx^{\alpha}A_{\mathbf{d}}(t).
\end{aligned}$$

(III) Suppose $1 \leq t \leq \Omega \leq x$. Then

$$\begin{aligned}
A_{\mathbf{d}}(xt) &\geq \big(A_{\mathbf{d}}(t)/A_{\mathbf{d}}(\Omega)\big)A_{\mathbf{d}}(xt) \quad \text{as } A_{\mathbf{d}}(t) \leq A_{\mathbf{d}}(\Omega) \neq 0\\
&\geq \big(A_{\mathbf{d}}(t)/A_{\mathbf{d}}(\Omega)\big)\Big(\frac{1}{2}(xt)^{\alpha}S(xt) \quad \text{by (33)}\\
&\geq \big(2A_{\mathbf{d}}(\Omega)\big)^{-1}A_{\mathbf{d}}(t)x^{\alpha}S(xt)\\
&\geq \big(2A_{\mathbf{d}}(\Omega)\big)^{-1}A_{\mathbf{d}}(t)x^{\alpha}\frac{1}{2}C_1 \quad \text{by (32)}\\
&= C_1\big(4A_{\mathbf{d}}(\Omega)\big)^{-1}A_{\mathbf{d}}(t)x^{\alpha}\\
&\geq Cx^{\alpha}A_{\mathbf{d}}(t).
\end{aligned}$$

This concludes the proof. ■

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COROLLARY 5.20. *Suppose* $\mathsf{N}_{\mathbf{d}}$ *is a system of generalized integers. If* $\alpha \geq 0$ *and* $S(x) \in RV_0$ *are such that*
(a) $A_{\mathbf{d}}(x) = \big(1 + o(1)\big)x^{\alpha}S(x)$*, and*
(b) $S(x)$ *is eventually nondecreasing,*
then $\mathsf{N}_{\mathbf{d}}$ *is loaded.*

PROOF. Suppose $S(x)$ is nondecreasing on $[b, \infty)$. Let

$$S'(x) = \begin{cases} S(b) & \text{on } (0, b] \\ S(x) & \text{on } [b, \infty). \end{cases}$$

Then, noting that $S'(xt) \geq S'(t)$ for $x, t \geq 1$, $S'(x)$ can be used for the $S(x)$ of Corollary 5.19 (with $C_0 = 1$). ∎

COROLLARY 5.21. *Let* $\mathsf{N}_{\mathbf{d}}$ *be a system of generalized integers such that the count function* $A_{\mathbf{d}}$ *satisfies*

$$A_{\mathbf{d}}(x) = \big(1 + o(1)\big)Cx^{\alpha},$$

for some $C > 0$ *and* $\alpha \geq 0$*. Then* $\mathsf{N}_{\mathbf{d}}$ *is loaded.*

PROOF. Let $S(x) = C$, and apply Corollary 5.20. ∎

REMARK 5.22. Finding prime number theorems for various classes of objects with multiplication has been an important area of research throughout the century, starting with Landau's 1903 work [16] on prime ideals in algebraic integers, through Bateman and Diamond's exposition [1] of Beurling's 1937 work [2] on generalized integers and Knopfmacher's book [13] on arithmetical semigroups. In each case they needed conditions on the growth of the count function $A_{\mathbf{d}}$. What is particularly interesting for us is that their conditions are covered by $A_{\mathbf{d}}(x) = \big(1 + o(1)\big)Cx^{\alpha}$, and hence always give loaded systems. We will comment further on this in the final section.

THE DISCRETELY REGULAR CASE. Now we consider the following special case of the problem studied so far: assume that there is an integer d greater than 1 such that the sizes of the generalized primes in P form a subset of $\{d, d^2, \ldots, d^k, \ldots\}$. We will see that the results from the previous section can be carried over to this new setting.

The next result shows that the sizes of members of a discrete number system are rather well-behaved.

PROPOSITION 5.23. *Let* $\mathsf{N}_{\mathbf{d}}$ *be a discrete system of generalized integers, and let the positive integer d be the generator of the domain of* $\rho_{A_{\mathbf{d}}}$ *(as a cyclic subgroup of the positive reals under multiplication). Then there is an* N *such that for any integer* $m \geq N$ *one has a generalized integer* $\mathbf{n} \in \mathsf{N}_{\mathbf{d}}$ *with* $|\mathbf{n}| = d^m$.

PROOF. Let $E(\mathsf{N}_{\mathbf{d}}) = \{\log_d(|\mathbf{n}|) : \mathbf{n} \in \mathsf{N}_{\mathbf{d}}\}$. Then $E(\mathsf{N}_{\mathbf{d}})$ is a subsemigroup of the nonnegative integers under addition, and thus (by a well-known result of Schur—see Wilf [24], p. 97) it is eventually the set of multiples of the period $g = \gcd\big(E(\mathsf{N}_{\mathbf{d}})\big)$ of $E(\mathsf{N}_{\mathbf{d}})$. As d is a generator of the domain of $\rho_{A_{\mathbf{d}}}$ one must have $g = 1$. ∎

For convenience in writing up the results below we designate the following sentence:

($\star$) $\begin{cases}\text{The domain of } \rho_{A_{\mathbf{d}}} \text{ is the cyclic subgroup (of the positive reals under} \\ \text{multiplication) generated by the positive integer } d.\end{cases}$

THEOREM 5.24. *Let* $\mathsf{N}_{\mathbf{d}}$ *be a discrete system of generalized integers such that* ($\star$) *holds. If*

(a) $A_{\mathbf{d}} \in RV_\alpha(d)$ *for some* $\alpha \geq 0$, *and*

(b) *there is a positive constant* C *such that*

$$A_{\mathbf{d}}(d^m \cdot d^n) \geq C(d^m)^\alpha A_{\mathbf{d}}(d^n)$$

for integers $m, n \geq 0$,

then $\mathsf{N}_{\mathbf{d}}$ *is loaded.*

PROOF. The proof is essentially the same as for Theorem 5.16 after restricting variables running over the reals or integers to integer powers of d, *i.e.*, to the domain of $\rho_{A_{\mathbf{d}}}$. Note, for example, that in this situation the a_n, b_n, x_n, y_n, z_n, v_n from the proof of Theorem 5.16 are 0 unless n is a power of d. Thus, expressions like $A_{\mathbf{d}}(x/k)$ in the proof are such that both x and k are powers of d. ∎

COROLLARY 5.25. *Let* $\mathsf{N}_{\mathbf{d}}$ *be a discrete system of generalized integers such that* ($\star$) *holds. If there are* $\alpha \geq 0$ *and* $S(x) \in RV_0(d)$ *such that*

(a) $A_{\mathbf{d}}(d^n) = \big(1 + o(1)\big)(d^n)^\alpha S(d^n)$, *and*

(b) *there is a positive constant* C_0 *such that* $S(d^m \cdot d^n) \geq C_0 S(d^n)$ *for integers* $m, n \geq 0$,

then $\mathsf{N}_{\mathbf{d}}$ *is loaded.*

PROOF. The proof is essentially the same as for Corollary 5.19, again restricting variables as in the previous proof. ∎

COROLLARY 5.26. *Let* $\mathsf{N}_{\mathbf{d}}$ *be a discrete system of generalized integers such that* ($\star$) *holds. If* $\alpha \geq 0$ *and* $S(x) \in RV_0(d)$ *are such that*

(a) $A_{\mathbf{d}}(d^n) = \big(1 + o(1)\big)(d^n)^\alpha S(d^n)$, *and*

(b) $S(d^n)$ *is eventually nondecreasing,*

then $\mathsf{N}_{\mathbf{d}}$ *is loaded.*

PROOF. Again modify the range of the variables in the proof of Corollary 5.20. ∎

COROLLARY 5.27. *Let* $\mathsf{N}_{\mathbf{d}}$ *be a discrete system of generalized integers such that* ($\star$) *holds, and such that the count function* $A_{\mathbf{d}}$ *satisfies*

$$A_{\mathbf{d}}(d^n) = \big(1 + o(1)\big)C(d^n)^\alpha,$$

for some $C > 0$ *and* $\alpha \geq 0$. *Then* $\mathsf{N}_{\mathbf{d}}$ *is loaded.*

PROOF. Let S be the constant function with value C, and apply Corollary 5.26. ∎

Now we consider what happens when one bounds the multiplicities of the primes. Let $\mathsf{N}_{\mathbf{d}}$ be a discrete system of generalized integers, and let d be a positive integer such that the sizes of members of $\mathsf{N}_{\mathbf{d}}$ are all powers of d. For each $k \in \mathsf{N}$ suppose the size d^k occurs with multiplicity α_k. In this case, the generating function is

$$\sum_{n=1}^{+\infty} \frac{a_n}{n^s} = \prod_{k=1}^{+\infty}\Big(1 - \frac{1}{d^{ks}}\Big)^{-\alpha_k} = \prod_{k=1}^{+\infty}\Big(\sum_{j=0}^{+\infty} \frac{1}{d^{jks}}\Big)^{\alpha_k} = \sum_{n=0}^{+\infty} \frac{f(n)}{d^{ns}}$$

where $f(n)$ denotes the number of non-negative integer solutions (in the unknowns $x_i^{(k)}$, $k = 1, 2, \ldots, i = 1, 2, \ldots, \alpha_k$) of the linear additive equation

$$\sum_{k=1}^{+\infty} k(x_1^{(k)} + x_2^{(k)} + \cdots + x_{\alpha_k}^{(k)}) = n. \tag{34}$$

(Note that if $\alpha_k = 1$ for all $k \in \mathsf{N}$, then $f(n)$ is equal to the well-known and intensively studied partition function $p(n)$.) It follows that

$$A_{\mathbf{d}}(x) = \sum_{n \le x} a_n = \sum_{m \le \frac{\log x}{\log d}} f(m).$$

THEOREM 5.28. *Let $\mathsf{N}_{\mathbf{d}}$ be a given system of generalized integers. Suppose there is a $d \in \mathsf{N}$ such that the size of every generalized prime is in $\{d, d^2, \ldots, d^k, \ldots\}$. Moreover, suppose the multiplicities α_k with which the sizes d^k of the generalized primes occur are uniformly bounded, i.e., there is a $U \in \mathsf{N}$ such that for every k there are at most U generalized primes whose size is d^k. Then $A_{\mathbf{d}} \in RV_0$, and hence is loaded. (In Part II [7], the condition $A_{\mathbf{d}} \in RV_0$ is referred to as front-loaded).*

PROOF. The case that there are only *finitely* many generalized primes is handled by precisely the arguments used to prove Corollary 9 of [4]. So now we assume that there are *infinitely* many generalized primes.

Let z denote the least integer with $\alpha_z > 0$. By Corollary 4.3 and Lemma 5.3, it suffices to show that

$$\rho_{A_{\mathbf{d}}}(d^{-z}) = 1,$$

i.e.,

$$\begin{aligned}
\lim_{t \to +\infty} \frac{A_{\mathbf{d}}(t/d^z)}{A_{\mathbf{d}}(t)} &= \lim_{t \to +\infty} \Big(\sum_{m \le \frac{\log t}{\log d} - z} f(m)\Big)\Big(\sum_{m \le \frac{\log t}{\log d}} f(m)\Big)^{-1} \\
&= \lim_{N \in \mathsf{N}, N \to +\infty} \Big(\sum_{n=0}^{N-z} f(n)\Big)\Big(\sum_{n=0}^{N} f(n)\Big)^{-1} \\
&= 1 - \lim_{N \in \mathsf{N}, N \to +\infty} \Big(\sum_{n=N-z+1}^{N} f(n)\Big)\Big(\sum_{n=0}^{N} f(n)\Big)^{-1} \\
&= 1
\end{aligned}$$

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or, in equivalent form,

$$\lim_{N\in\mathsf{N},N\longrightarrow+\infty}\Big(\sum_{n=N-z+1}^{N}f(n)\Big)\Big(\sum_{n=0}^{N}f(n)\Big)^{-1}=0. \tag{35}$$

First we will show that the denominator in (35) grows faster than any power of N:

$$\lim_{N\longrightarrow+\infty}N^{-\alpha}\sum_{n=0}^{N}f(n)=+\infty \quad \text{(for all } \alpha>0). \tag{36}$$

To show this, let H be a positive integer with

$$H>\alpha,$$

and consider H generalized primes with different sizes $d^{k_1}<d^{k_2}<\cdots<d^{k_H}$. Then each solution (in nonnegative integers) of

$$\sum_{j=1}^{H}k_jx^{(k_j)}\le N \tag{37}$$

is counted in the sum in (36), so that the number of solutions of (37) gives a lower bound for this sum. Moreover, each H-tuple $(x^{(k_1)},x^{(k_2)},\ldots,x^{(k_H)})$ with $x^{(k_j)}\in\{0,1,\ldots,[N/(k_jH)]\}$ provides a solution of (37). The number of these H-tuples is

$$\prod_{j=1}^{H}\big([N/(k_jH)]+1\big)>N^H\Big(H^H\prod_{j=1}^{H}k_j\Big)^{-1}$$

so that

$$\sum_{n=0}^{N}f(n)>N^H\Big(H^H\prod_{j=1}^{H}k_j\Big)^{-1}.$$

This holds for any fixed H and all N, which clearly implies (36).

Assume now that $\epsilon>0$, and let L be a positive integer large enough in terms of ϵ (to be fixed later). We split the number of solutions of (34) into two parts: let $f_1(n)$ denote the number of those solutions where there are at most L pairs $(k,x_i^{(k)})$ with $k\in\mathsf{N}$, $1\le i\le k$, $x_i^{(k)}>0$, *i.e.*, there are $\le L$ non-zero terms $kx_i^{(k)}$ on the left-hand side of (34), and let $f_2(n)$ denote the number of solutions with more than L non-zero terms.

First we will give an upper bound for

$$\sum_{n=N-z+1}^{N}f_1(n)\le\sum_{n=1}^{N}f_1(n). \tag{38}$$

Clearly we obtain an upper bound for the right hand side if we count those sums $\sum_{k,i}kx_i^{(k)}$ where all $k,x_i^{(k)}$ satisfy $1\le k,x_i^{(k)}\le N$, and the number, say t, of terms satisfies $t\le L$. For fixed t, the k-values in these terms can be chosen in at most N^t ways (since repetition is allowed). If the k-values have been fixed, then for each k, the subscript i with $x_i^{(k)}>0$

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can be chosen in at most $\alpha_k \leq U$ ways, and if both k and i are fixed, then $x_i^{(k)}(\leq N)$ can be chosen in at most N ways. Thus we obtain the upper bound

$$\sum_{n=1}^{N} f_1(n) \leq \sum_{t=0}^{L} N^t U^t N^t = O(N^{2L}) \tag{39}$$

(for fixed L, and $N \longrightarrow +\infty$). It follows from (36), (38) and (39) that

$$\sum_{n=N-z+1}^{N} f_1(n) = o\Big(\sum_{n=0}^{N} f(n)\Big) \quad \text{(for any fixed } L\text{, and } N \longrightarrow +\infty). \tag{40}$$

In order to give an upper bound for

$$\sum_{n=N-z+1}^{N} f_2(n), \tag{41}$$

consider a solution counted in this sum, *i.e.*, consider an integer n with $N - z < n \leq N$ and a representation of n in the form (34) with more than L non-zero terms. Starting out from each of these solutions, we construct several new equations in the following way: to each non-zero term $kx_i^{(k)}$ we assign the equation obtained by subtracting k so that $x_i^{(k)}$ is replaced by $x_i^{(k)} - 1$:

$$\sum_{j \neq k} j\Big(x_1^{(j)} + x_2^{(j)} + \cdots + x_{\alpha_j}^{(j)}\Big) + k\Big(x_1^{(k)} + \cdots + x_{i-1}^{(k)} + (x_i^{(k)} - 1) + x_{i+1}^{(k)} + \cdots + x_{\alpha_k}^{(k)}\Big) = n - k. \tag{42}$$

In this way, from each solution of (34) counted in (41) we obtain as many solutions counted in

$$\sum_{m=1}^{N} f(m) \tag{43}$$

as the number of the non-zero terms in (34), *i.e.*, more than L solutions, so that we obtain more than

$$M = L \cdot \sum_{n=N-z+1}^{N} f_2(n) \tag{44}$$

solutions of type (42) counted in (43). However, we may get each of these solutions more than once, so that we need an upper bound for their multiplicity. If we start out from a fixed solution of type (42) counted in (43), and we replace $n - k$ on the right hand side by m:

$$\sum_{j=1}^{+\infty} j(y_1^{(j)} + y_2^{(j)} + \cdots + y_{\alpha_j}^{(j)}) = m, \tag{45}$$

then to reconstruct the initial solution (counted in (41)), we have to add one of the numbers $k = N - z + 1 - m, N - z + 2 - m, \ldots, N - m$ to the equation (45) so that on the left hand side we add 1 to one of the terms $y_i^{(k)}$ with $1 \leq i \leq \alpha_k$. We may choose k and i in at most z, resp. $\alpha_k \leq U$ ways, so that we may get each of the at least M solutions

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(where M is defined in (44)) of type (45) counted in (43) with multiplicity at most zU. Thus we have

$$\sum_{m=1}^{N} f(m) > \frac{L}{zU} \sum_{n=N-z+1}^{N} f_2(n)$$

whence

$$\sum_{n=N-z+1}^{N} f_2(n) < \frac{zU}{L} \sum_{m=1}^{N} f(m).$$

If L is large enough in terms of ϵ, z and U, then $zU/L < \epsilon$ so that

$$\sum_{n=N-z+1}^{N} f_2(n) < \epsilon \sum_{m=1}^{N} f(m) \quad (\text{for } N > N_0(\epsilon)). \tag{46}$$

Since ϵ in (46) is arbitrary and $f(n) = f_1(n) + f_2(n)$, thus (35) follows from (40) and (46), and this completes the proof of the theorem. ∎

Note that some upper bound on the sizes of the multiplicities α_k is necessary. To see this, consider a sequence α_k growing so fast that α_k is greater than the number of generalized composite integers of size at most d^k. Then the density of P, the set of generalized primes, is not zero. But then, by Proposition 5.7(c), $\mathsf{N}_{\mathbf{d}}$ is not loaded. This example can be easily generalized to the following setting. Suppose $\phi(n)$ is an arithmetic function such that $\limsup_{n\to+\infty} \phi(n) = +\infty$. Then there is a sequence (α_n) of nonnegative integers such that

$$\beta_n := \frac{\log \alpha_n}{\log n} < \phi(n)$$

for n sufficiently large, and $\mathsf{N}_{\mathbf{d}}$ is not loaded for $\mathbf{d}$ being the sequence of powers of d where the multiplicity of d^k is α_k.

Although our result that *bounded multiplicities in the discrete case guarantees front-loaded* covers an important part of the discrete generalized number systems, it would be very interesting to know if the condition

$$\beta_n = O(1)$$

is the precise upper bound condition to guarantee $\mathsf{N}_{\mathbf{d}}$ is (front-)loaded. (See Question 8.3 of Compton's survey paper [5].) As just stated, examples to show the relevance of this condition are rather easy to construct. But to show that it is sufficient appears to require tools that go well beyond what we have developed here.

We close this section with several questions that we think are of interest:

1. Are 'loaded' and 'regular' equivalent for systems of generalized integers $\mathsf{N}_{\mathbf{d}}$?
2. Do the two conditions $\Delta(\mathsf{P}) = 0$ and $\Delta(\mathbf{m} \cdot \mathsf{N}_{\mathbf{d}}) > 0$ for $\mathbf{m} \in \mathsf{N}_{\mathbf{d}}$, imply that $\mathsf{N}_{\mathbf{d}}$ is loaded?
3. Can one simplify the definition of loaded?
   (a) If the set of multiples of H^r has a density for every $\mathsf{H} \subseteq \mathsf{P}$ and positive integer r, does it follow that $\mathsf{N}_{\mathbf{d}}$ is loaded?

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(b) If the set of multiples of H has a density for every $\mathsf{H} \subseteq \mathsf{P}$, does it follow that $\mathsf{N}_{\mathbf{d}}$ is loaded?

4. Do the partition blocks of the generalized primes behave 'independently' in loaded systems, *i.e.*, if $\mathsf{P}_1, \ldots, \mathsf{P}_k$ is a partition of the generalized primes P and $r_i \geq 0$ does it follow that

$$\Delta(\mathsf{P}_1^{\geq r_1} \cdots \mathsf{P}_k^{\geq r_k}) = \prod_{i=1}^{k} \Delta(\mathsf{P}_i^{\geq r_i} \cdot \mathsf{N}_{\mathbf{d}}) \ ?$$

5. Is the Dirichlet convolution product of the Dirichlet series for two loaded generalized integer systems again loaded?
6. How fast can the multiplicities grow in the discrete case and still yield a (front-)loaded system?

6**. Applications to admissible classes.** Now we translate the key results from the previous section into the setting of admissible classes of structures, and give examples where these results apply. First, from Proposition 5.15, we have the following restriction on loaded classes.

PROPOSITION 6.1. *If* K *is an admissible class with* $\limsup_{x \longrightarrow +\infty} \frac{\log \tau_{\mathsf{K}}(x)}{\log x} = +\infty$ *then* K *is* not *loaded.*

REMARK 6.2. Higman [10] gives lower bounds on the number of (nilpotent) groups of order p^n, and Knopfmacher gives related bounds for various classes of groups, rings, and algebras in [11] and [12]. The lower bounds for these classes K state that, for some positive constant C,

$$\log \tau_{\mathsf{K}}(n) \geq C(\log n)^2$$

on an infinite subset of the positive integers. By Proposition 6.1 we see that such classes K are not loaded. If they should happen to have a first-order limit law, then one must prove it by means other than those used in this paper.

In the following we will say that K is *regular*, *discrete*, respectively *discretely regular*, if the corresponding system of generalized integers is regular, discrete, respectively discretely regular.

REGULAR ADMISSIBLE CLASSES.

THEOREM 6.3. *Let* K *be a regular admissible class of structures such that*

(a) $\tau_{\mathsf{K}} \in RV_\alpha$*, and*

(b) there is a positive constant C such that

$$\tau_{\mathsf{K}}(xt) \geq Cx^\alpha \tau_{\mathsf{K}}(t)$$

for $t, x \geq 1$.

Then K *is loaded, and hence has a first-order limit law.*

COROLLARY 6.4. *Let* K *be an admissible class of structures. If* $\alpha \geq 0$ *and* $S(x) \in RV_0$ *are such that*

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(a) $\tau_{\mathsf{K}}(x) = \big(1 + o(1)\big)x^\alpha S(x)$*, and*
(b) there is a positive constant C_0 *such that* $S(xt) \geq C_0 S(t)$ *for* $t, x \geq 1$*,*
then K *is loaded, and hence has a first-order law.*

COROLLARY 6.5. *Let* K *be an admissible class of structures. If* $\alpha \geq 0$ *and* $S(x) \in RV_0$ *are such that*
(a) $\tau_{\mathsf{K}}(x) = \big(1 + o(1)\big)x^\alpha S(x)$*, and*
(b) $S(x)$ *is eventually nondecreasing,*
then K *is loaded, and hence has a first-order law.*

COROLLARY 6.6. *Let* K *be an admissible class of structures such that the count function* τ_{K} *satisfies*

$$\tau_{\mathsf{K}}(x) = \big(1 + o(1)\big)Cx^\alpha,$$

for some $C > 0$ *and* $\alpha \geq 0$*. Then* K *is loaded, and hence has a first-order law.*

Knopfmacher's Axiom A, mentioned in the abstract, is

$$\tau_{\mathsf{K}}(x) = Cx^\alpha + O(x^\beta),$$

where $0 \leq \beta < \alpha$. We can apply Corollary 6.6 to claim that *all admissible classes satisfying Axiom A are loaded, and hence have a first-order law*.[6]

EXAMPLE 6.7. The main examples of *admissible* classes satisfying Axiom A from Knopfmacher's 1975 book [13], along with the instance of the Axiom A which they satisfy, are given in the table below.[7] Z_K means the integers belonging to the number field K.

K	τ_{K}
Sets	$x + O(1)$
Semisimple Rings	$Cx + O(\sqrt{x})$
Semisimple Z_K Algebras or Z_K Modules	$Cx + \begin{cases} O(\sqrt{x}) & \text{if } [K:Q] < 3 \\ O(\sqrt{x}\log x) & \text{if } [K:Q] = 3 \\ O(x^{1-2(1+[K:Q])^{-1}}) & \text{if } [K:Q] > 3 \end{cases}$

EXAMPLE 6.8. The study of the number of (unordered) factorizations of an integer n by Oppenheim [18], [19] in 1926/1927 show that if we have an admissible class K with exactly one indecomposable of each size, *i.e.*, $\sigma_{\mathsf{F}}(n) = 1$ for all n, then we have the following asymptotics[8] for τ_{K}:

$$\tau_{\mathsf{K}}(x) = \big(1 + o(1)\big)x\frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}.$$

[6] K. Compton was the first to discover the existence of a first-order law for Abelian Groups. His work was based on the use of Tauberian theorems, as found in Knopfmacher's book [13].
[7] He gives numerous other examples of arithmetical categories, but either they are not admissible classes (*e.g.*, topological spaces), or the Dirichlet series has its abscissa of convergence at $+\infty$ (*e.g.*, p-groups).
[8] This result was found independently by Szekeres and Turán [21] in 1933.

As the function

$$S(x) = \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}$$

is slowly varying, and increasing for $x > e$, we can apply Corollary 6.5 to conclude that K is loaded, and thus has a first-order limit law.

One can easily find examples of such classes among lattice based structures, *e.g.*, let K be the class of finite lattices which can be expressed as direct products of chains.

DISCRETELY REGULAR ADMISSIBLE CLASSES. Again, for convenience in writing up the results in this part we have the following designated sentence:

$(\star\star)$ $\begin{cases} d \text{ is the positive integer that generates the cyclic subgroup (of the positive} \\ \text{reals under multiplication) generated by the sizes of the members of } \mathsf{K}. \end{cases}$

THEOREM 6.9. *Let K be a discrete admissible class such that $(\star\star)$ holds. If*
(a) $\tau_{\mathsf{K}} \in RV_\alpha(d)$ for some $\alpha \geq 0$, and
(b) there is a positive constant C such that

$$\tau_{\mathsf{K}}(d^m \cdot d^n) \geq C(d^m)^\alpha \tau_{\mathsf{K}}(d^n)$$

for integers $m, n \geq 0$, then K is loaded, and thus has a first-order limit law.

COROLLARY 6.10. *Let K be a discrete admissible class such that $(\star\star)$ holds. If $\alpha \geq 0$ and $S(x) \in RV_0(d)$ are such that*
(a) $\tau_{\mathsf{K}}(d^n) = \big(1 + o(1)\big)(d^n)^\alpha S(d^n)$, and
(b) there is a positive constant C_0 such that $S(d^m \cdot d^n) \geq C_0 S(d^n)$ for integers $m, n \geq 0$, then K is loaded, and thus has a first-order limit law.

COROLLARY 6.11. *Let K be a discrete admissible class such that $(\star\star)$ holds. If $\alpha \geq 0$ and $S(x) \in RV_0(d)$ are such that*
(a) $\tau_{\mathsf{K}}(d^n) = \big(1 + o(1)\big)(d^n)^\alpha S(d^n)$, and
(b) $S(d^n)$ is eventually nondecreasing,
then K is loaded, and thus has a first-order limit law.

COROLLARY 6.12. *Let K be a discrete admissible class such that $(\star\star)$ holds. If*

$$\tau_{\mathsf{K}}(d^n) = \big(1 + o(1)\big)C(d^n)^\alpha,$$

for some $C > 0$ and $\alpha \geq 0$, then K is loaded, and thus has a first-order limit law.

REMARK 6.13. In Theorem 6.9 and its corollaries one can replace the count function τ_{K} by the fine spectrum σ_{K} and conclude that K is loaded by showing that the hypotheses of Theorem 6.9 are satisfied. The key steps are (1) $\sigma_{\mathsf{K}} \in RV_\alpha(d)$ implies $\tau_{\mathsf{K}} \in RV_\alpha(d)$, and (2) $\sigma_{\mathsf{K}}(d^m \cdot d^n) \geq C(d^m)^\alpha \sigma_{\mathsf{K}}(d^n)$ implies $\tau_{\mathsf{K}}(d^m \cdot d^n) \geq C(d^m)^\alpha \tau_{\mathsf{K}}(d^n)$.

For our purposes, Knopfmacher's Axiom $A^{\#}$, mentioned in the abstract, can be written in the form (see [14], p. 7)

$$\sigma_{\mathsf{K}}(d^n) = B(d^n)^\alpha + O\big((d^n)^\beta\big),$$

or, see [14], p. 16, the equivalent form

$$\tau_{\mathsf{K}}(d^n) = \begin{cases} C(d^n)^\alpha + O\big((d^n)^\beta\big) & \text{if } 0 < \beta < \alpha \\ C(d^n)^\alpha + O(n) & \text{if } 0 = \beta < \alpha, \end{cases}$$

where n runs over integers. By Corollary 6.12 *all admissible classes satisfying Axiom $A^{\#}$ are loaded, and hence have a first-order law.*

EXAMPLE 6.14. The main examples of *admissible* classes satisfying Axiom $A^{\#}$ from Knopfmacher's 1979 book [14], along with the instance of the Axiom $A^{\#}$ which they satisfy, are given below.[9] D_q means the ring of integral functions in an algebraic function field in one variable over $GF(q)$.

K	$\sigma_{\mathsf{K}}(d^n)$
$GF[q,t]$ Modules	$Bd^n + O(\sqrt{d^n})$
Semisimple $GF[q,t]$ Algebras	$Bd^n + O(\sqrt{d^n})$
D_q Modules	$Bd^n + O(\sqrt{d^n})$
Semisimple D_q Algebras	$Bd^n + O(\sqrt{d^n})$

EXAMPLE 6.15. In 1992 A. Knopfmacher, J. Knopfmacher and R. Warlimont [15] looked at extensions of the Oppenheim result mentioned in Example 6.8 to the general setting of arithmetical semigroups. What they showed, formulated in terms of a discrete admissible class K, and where d is as in ($\star\star$), is that if the fine spectrum σ_{F} of the class of indecomposables F satisfies axiom $A^{\#}$, *i.e.*, if $\sigma_{\mathsf{F}}(d^n) = \hat{B}(d^n)^\alpha + O\big((d^n)^\beta\big)$, with $0 \le \beta < \alpha$, then the fine spectrum σ_{K} of K has asymptotics given by

$$\sigma_{\mathsf{K}}(d^n) = \big(1 + o(1)\big)(d^n)^\alpha \frac{Be^{D\sqrt{n}}}{n^{3/4}}.$$

As the function

$$S(n) = \frac{Be^{D\sqrt{\log_d(n)}}}{\big(\log_d(n)\big)^{3/4}}$$

satisfies the conditions of Corollary 6.11, we conclude from Remark 6.13 that K is loaded, and thus has a first-order limit law.

THEOREM 6.16. *Let K be a discrete admissible class of structures. Suppose there is a positive integer d such that the size of every K-indecomposable is a power of d. Moreover, suppose the multiplicities α_k with which the sizes d^k of the K-indecomposables occur are*

[9] There are further examples of discrete arithmetical categories in [14] that satisfy Axiom $A^{\#}$, but they are not admissible classes (*e.g.*, the monic polynomials of $GF[q,t]$).

uniformly bounded. Then τ_{K} *is slowly varying, and thus* K *is loaded and has a first-order law. (Note: In Part II [7] such a* K *will be called front-loaded, and will be shown to have a first-order* 0–1 *law.)*

EXAMPLE 6.17. Many varieties K of algebras from algebraic logic are discrete and have a uniformly bounded number of indecomposables of each size, *e.g.*, Boolean algebras, monadic algebras, n-valued Post algebras. From group theory one has the class A(p) of abelian p-groups. From ring theory there are the classes defined by $x^m = x$ (for $m > 1$).

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