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AVERAGES OF TWISTED *L*-FUNCTIONS

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Abstract

We use a relative trace formula on GL(2) to compute a sum of twisted modular L-functions anywhere in the critical strip, weighted by a Fourier coefficient and a Hecke eigenvalue. When the weight k or level N is sufficiently large, the sum is nonzero. Specializing to the central point, we show in some cases that the resulting bound for the average is as good as that predicted by the Lindelöf hypothesis in the k and N aspects.

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1. Introduction

In many situations, the central L-values of modular forms encode information about related algebraic objects. For example, the nonexistence of solutions to certain Diophantine equations can hinge on the existence of cusp forms with nonvanishing central twisted L-value (see [El1, BEN]). Techniques from analytic number theory can then be used to estimate averages of L-values and thereby deduce the existence of such cusp forms. The standard method, introduced by Duke [Du], uses the Petersson trace formula together with Weil's bound for Kloosterman sums. In the present paper, we use a different trace formula to compute the average of twisted L-functions directly at any point in the critical strip. The resulting asymptotic formula has a much better error term (as a function of the level) and follows immediately without any use of regularization, approximate functional equations, or deep results about Kloosterman sums.

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Before stating the main result, we fix the following notation. Let $S_k(N,\psi)$ be the space of cusp forms h on $\Gamma_0(N) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{SL}_2(\mathbf{Z}) \mid c \in N\mathbf{Z}\}$ satisfying

$$h\Big(\frac{az+b}{cz+d}\Big) = \psi(d)(cz+d)^k h(z)$$

for all z in the complex upper half-plane $\mathbf{H}$ and all $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(N)$, where ψ is a Dirichlet character modulo N. We normalize the Petersson inner product on $S_k(N,\psi)$ by

$$\|h\|^2 = \frac{1}{\nu(N)} \iint_{\Gamma_0(N)\backslash\mathbf{H}} |h(z)|^2 y^k \frac{dx\,dy}{y^2}, \tag{1.1}$$

where

$$\nu(N) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)].$$

Given $h \in S_k(N,\psi)$, write $h(z) = \sum_{n>0} a_n(h) q^n$ for $q = e^{2\pi i z}$. Fix an integer D with $(D,N) = 1$, and let χ be a primitive Dirichlet character modulo D. The χ-twisted L-function of h is given for $\mathrm{Re}(s) > k/2 + 1$ by the Dirichlet series

$$L(s,h,\chi) = \sum_{n>0} \frac{\chi(n) a_n(h)}{n^s}.$$

The completed L-function

$$\Lambda(s,h,\chi) = (2\pi)^{-s}\Gamma(s) L(s,h,\chi)$$

has an analytic continuation to the complex plane and satisfies a functional equation relating s to $k-s$, so the central point is $s = k/2$. Taking χ trivial and $D = 1$ gives the usual L-function $\Lambda(s,h)$. When $N = 1$, the functional equation takes the form

$$\Lambda(s,h,\chi) = \frac{i^k}{D^{2s-k}} \frac{\tau(\chi)^2}{D} \Lambda(k-s,h,\overline{\chi}), \tag{1.2}$$

where

$$\tau(\chi) = \sum_{m \in (\mathbf{Z}/D\mathbf{Z})^*} \chi(m) e^{2\pi i m/D} \tag{1.3}$$

is the Gauss sum attached to χ.

Let n be an integer prime to N, and let T_n be the nth Hecke operator, given by

$$T_n h(z) = n^{k-1} \sum_{\substack{ad=n,\\ a>0}} \sum_{b=0}^{d-1} \psi(a) d^{-k} h\Big(\frac{az+b}{d}\Big).$$

Let $\mathcal{F}$ be an orthogonal basis for $S_k(N,\psi)$ consisting of eigenfunctions of T_n. We denote the Hecke eigenvalue by $T_n h = \lambda_n(h) h$, and recall that

$$a_n(h) = a_1(h)\lambda_n(h).$$

Our main result is the following theorem.

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THEOREM 1.1. *With notation as above, assume that $k > 2$, and let $r, n \in \mathbf{Z}^+$ with $(rn, D) = 1$. Then, for all $s = \sigma + i\tau$ in the strip $1 < \sigma < k - 1$,*

$$\begin{aligned}\frac{1}{\nu(N)} \sum_{h\in\mathcal{F}} \frac{\lambda_n(h)\,\overline{a_r(h)}\,\Lambda(s,h,\chi)}{\|h\|^2} &= \frac{2^{k-1}(2\pi rn)^{k-s-1}}{(k-2)!}\Gamma(s) \sum_{d|(n,r)} d^{2s-k+1}\psi\Big(\frac{n}{d}\Big)\chi\Big(\frac{rn}{d^2}\Big) \\ &\quad + \delta_{N,1}\frac{2^{k-1}(2\pi rn)^{s-1}}{(k-2)!}\Gamma(k-s)\frac{i^k}{D^{2s-k}}\frac{\tau(\chi)^2}{D}\sum_{d|(r,n)} d^{k-2s+1}\overline{\chi\Big(\frac{rn}{d^2}\Big)} + E, \qquad (1.4)\end{aligned}$$

where $\delta_{N,1} \in \{0, 1\}$ is nonzero if and only if $N = 1$, and the error term E is an infinite series involving confluent hypergeometric functions (cf. Proposition 8.1) satisfying

$$|E| \le 2\gcd(r,n)\frac{(4\pi rn)^{k-1}D^{k-\sigma-1/2}\varphi(D)B(\sigma,k-\sigma)}{N^{\sigma}(k-2)!}\cosh\Big(\frac{\pi\tau}{2}\Big)\zeta(k-\sigma)\zeta(\sigma). \qquad (1.5)$$

Here, $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is Euler's Beta function and $\varphi(D)$ is Euler's φ-function.

Theorem 1.1 extends first moment estimates of many authors. For prime level N, Duke estimated the average at the central point in the case $k = 2$ and $r = n = 1$ [Du]. Akbary extended his result to allow for arbitrary weight k and summing over newforms [Ak], and Kamiya treated arbitrary level and weight (with oldforms present) and s any point on the critical line [Ka]. These references all make use of Petersson's formula, and obtain an error on the order of $O(N^{-k/4+\varepsilon})$, whereas (1.5) is $O(N^{-k/2})$ for s on the critical line. Ellenberg has shown how to refine Duke's method to improve the error bound to $O(N^{-k/2+\varepsilon})$ [El2].

In the case of twisting by a quadratic Dirichlet character when $N = 1$, a different method was offered by Kohnen and Sengupta. They gave an asymptotic for the average in the weight aspect using Waldspurger's formula relating the central twisted L-values to certain Fourier coefficients of half-integral weight modular forms [KS].

Here, we prove Theorem 1.1 by direct computation of a GL(2) relative trace formula involving integration over $N \times T$, where N is unipotent and T is a torus. This method was introduced in [KL2], in which the untwisted case was treated. The incorporation of twisting is achieved with an adelic twisting operator, which we define in Section 3. Results of this nature have been used to bound the ranks of Jacobians of modular curves (cf. [IS2]).

Several authors have investigated first moments of Rankin–Selberg L-functions, that is, where χ in (1.4) is replaced with a fixed cusp form h on GL(2). In the case where h is dihedral, one can do this via a relative trace formula on $T \times T$ [RR, FW] or by appealing to the Gross–Zagier formula [MR]. Averages for more general h have been studied recently by Nelson [N] and Holowinsky and Templier [HT] (level aspect) and by Li and Masri [LM] (weight aspect). In several of the above references, a 'hybrid'

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subconvexity result is obtained, valid for forms whose level is in some range depending on the level of the fixed form h. Unfortunately, an analogous hybrid subconvexity result (in N and D) is not possible in the present paper because of the poor control of the D-aspect in (1.5).

An immediate application of Theorem 1.1 is the nonvanishing of L-functions, as follows.

COROLLARY 1.2. *Suppose that $N > 1$ and $\gcd(r, n) = 1$. Then, for any s in the critical strip $(k-1)/2 < \mathrm{Re}(s) < (k+1)/2$, the sum (1.4) is nonzero as long as $N + k$ is sufficiently large.*

(See Section 9, where the required size of $N + k$ as a function of D and s may be ascertained.) This can be interpreted as a GRH-on-average for the twisted L-functions, though no distinction is made between points on and off the critical line. When $N = 1$, we cannot prove nonvanishing on the critical line $\mathrm{Re}(s) = k/2$, since the first two terms have the same magnitude there. Indeed, they cancel out at $s = k/2$ if χ is quadratic and conditions on k, D conspire in (1.2) to force the L-functions to vanish. However, by arguments given in [KL2], one can show that when $N = 1$ and $\mathrm{Re}(s) \neq k/2$, the sum (1.4) is nonzero if k is sufficiently large.

We have not made any attempt to go further and address the question of how many forms give a nonvanishing L-value. However, we note that by using estimates for mollified first and second moments, Iwaniec and Sarnak have shown that a positive proportion of cusp forms (in fact 50% in certain families) have nonvanishing quadratic-twisted central L-value [IS1].

According to the generalized Lindelöf hypothesis, for a newform h,

$$L\Big(\frac{k}{2}, h, \chi\Big) \ll (D^2 N k)^{\varepsilon}. \tag{1.6}$$

Let $\mathcal{F}_k(N)^{\mathrm{new}}$ be any orthogonal basis for the span $S_k(N)^{\mathrm{new}}$ of the newforms with trivial character. Using the fact [Ser, page 86] that $\dim S_k(N)^{\mathrm{new}} \sim ((k-1)/12)\nu(N)^{\mathrm{new}}$, where $N^{1-\varepsilon} \ll \nu(N)^{\mathrm{new}} \leq N$, (1.6) implies the following 'averaged' Lindelöf hypothesis:

$$\sum_{h \in \mathcal{F}_k(N)^{\mathrm{new}}} L\Big(\frac{k}{2}, h, \chi\Big) \ll D^{\varepsilon} (Nk)^{1+\varepsilon}. \tag{1.7}$$

(This bound can only be expected to be an accurate prediction of the magnitude of the sum when the L-values are nonnegative.)

One can use Theorem 1.1 to prove certain instances of (1.7) unconditionally, although from (1.5) it is clear that we cannot achieve adequate bounds in the D-aspect. The idea is to set $n = r = 1$ in (1.4) and use well-known bounds for the Petersson norm, along with positivity of the L-values when χ is real. If oldforms are present, the method apparently grinds to a halt because $\overline{a_1(h)}\Lambda(k/2, h, \chi)/\|h\|^2$ may be negative. Indeed, even in the simplest case where $N = p$ is prime, if h is a newform of level 1, and h_p is a nonzero basis element (unique up to scaling) orthogonal to h in $\mathrm{Span}\{h(z), h(pz)\}$,

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then using [ILS, (2.45)–(2.46)] it is not hard to show that

$$\overline{a_1(h_p)}\Lambda\Big(\frac{k}{2}, h_p, \chi\Big) = \mu \cdot \Big(\frac{\lambda_p(h)^2}{(p+1)^2} - \frac{\lambda_p(h)\chi(p)}{p^{1/2}(p+1)}\Big)\Lambda\Big(\frac{k}{2}, h, \chi\Big) \tag{1.8}$$

for some constant $\mu > 0$ depending on the choice of h_p[1]. The above can clearly be negative, for example if $\chi(p) = 1$ and the real eigenvalue $\lambda_p(h)$ is close to 1. Therefore, we have to content ourselves here with cases in which oldforms are not present. We highlight two such cases, one in the k-aspect and one in the N aspect, though the proof applies more generally.

COROLLARY 1.3. *Let $\mathcal{F}_k(1)$ be an orthogonal basis for $S_k(1)$ consisting of newforms, normalized with first Fourier coefficient equal to 1. Then, for any real primitive Dirichlet character χ,*

$$\sum_{h\in\mathcal{F}_k(1)} L\Big(\frac{k}{2}, h, \chi\Big) \ll_{\varepsilon,D} k^{1+\varepsilon}. \tag{1.9}$$

Let $4 \le k_0 \le 14$ be an even integer not equal to 12, let N be a prime not dividing the conductor of χ, and let $\mathcal{F}_{k_0}(N)^{new}$ be an orthogonal basis for $S_{k_0}(N)^{new} = S_{k_0}(N)$ consisting of normalized newforms. Then

$$\sum_{h\in\mathcal{F}_{k_0}(N)^{new}} L\Big(\frac{k_0}{2}, h, \chi\Big) \ll_{\varepsilon,D} N^{1+\varepsilon}. \tag{1.10}$$

REMARKS 1.4. The estimate (1.9) was first proven by Kohnen and Sengupta by different means [KS]. The case of trivial χ was proven earlier by Sengupta by essentially the same method we use here [Sen]. The analog of (1.10) for the second moment was established by Fomenko in case of trivial χ [Fo]. (By Cauchy–Schwarz, the estimate (1.10) is a consequence of its second moment analog.)

PROOF. In Theorem 1.1, suppose that the central character ψ is trivial and that χ is real. We assume that there are no oldforms, so $\mathcal{F}$ can be chosen to consist of newforms h, normalized with $a_1(h) = 1$. By the hypotheses on ψ and χ, we have $L(k/2, h, \chi) \ge 0$ for all $h \in \mathcal{F}$ [Gu]. Furthermore, we have the bound

$$\frac{(4\pi)^{k-1}}{(k-2)!}\nu(N)\|h\|^2 \ll (kN)^{1+\varepsilon}$$

for all newforms $h \in \mathcal{F}$ (see [IM, (2.29)]). Therefore, due to the nonnegativity of the L-values,

$$\begin{aligned}\sum_{h\in\mathcal{F}} L\Big(\frac{k}{2}, h, \chi\Big) &\ll \frac{(kN)^{1+\varepsilon}(k-2)!}{(4\pi)^{k-1}\nu(N)} \sum_{h\in\mathcal{F}} \frac{L(\frac{k}{2}, h, \chi)}{\|h\|^2} \\ &= \frac{(kN)^{1+\varepsilon}(k-2)!}{2^{k-1}(2\pi)^{k/2-1}\Gamma(\frac{k}{2})\nu(N)} \sum_{h\in\mathcal{F}} \frac{\Lambda(\frac{k}{2}, h, \chi)}{\|h\|^2}.\end{aligned}$$

[1] By [ILS, (2.45)], $h_p = w(-(\lambda_p(h)/(p+1))h(z) + p^{(k-1)/2}h(pz))$ for some nonzero $w \in \mathbf{C}$; by (2.46), $a_1(h_p) = -w(\lambda_h(p)/(p+1))$; by an easy manipulation, $L(s, h(pz), \chi) = (\chi(p)/p^s)L(s, h, \chi)$, so $L(k/2, h_p, \chi) = w(-\lambda_p(h)/(p+1) + \chi(p)/p^{1/2})L(k/2, h, \chi)$. Then (1.8) holds with $\mu = |w|^2$.

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Applying the theorem with $n = r = 1$,

$$\sum_{h\in\mathcal{F}} L\Big(\frac{k}{2}, h, \chi\Big) \ll (kN)^{1+\varepsilon}\Big(1 + \delta_{N,1}\frac{i^k\tau(\chi)^2}{D} + \frac{(k-2)!}{2^{k-1}(2\pi)^{k/2-1}\Gamma(\frac{k}{2})}E\Big).$$

It is clear from (1.5) that the third term in the parentheses tends to 0 as $N \to \infty$. Using Stirling's approximation, it is not hard to show that the same is true as $k \to \infty$ (see Section 9 for details), and the corollary follows. $\square$

2. Notation and preliminaries

Let $\mathbf{A}$ and $\mathbf{A}_{\mathrm{fin}}$ be the adeles and finite adeles of $\mathbf{Q}$, respectively. Fix a positive integer N. For $x \in \mathbf{A}^*$, we let x_N denote the idele whose pth component is x_p for all $p|N$ and 1 for all $p \nmid N$. For any integer d, we also write $d_p = \mathrm{ord}_p(d)$ (the p-adic valuation of d). It should be clear from the context which meaning we take when a subscript p appears.

Let ψ be a Dirichlet character modulo N, extended to $\mathbf{Z}$ by $\psi(d) = 0$ if $(d, N) > 1$. We let ψ^* denote its adelic counterpart (a Hecke character), defined via strong approximation $\mathbf{A}^* = \mathbf{Q}^*(\mathbf{R}^+ \times \widehat{\mathbf{Z}}^*)$ by the pullback

$$\psi^* : \mathbf{A}^* \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^*, \tag{2.1}$$

where the first arrows are the canonical projections, and the last arrow is ψ. We drop the $*$ from the notation for the local constituents. Thus, $\psi_p : \mathbf{Q}_p^* \to \mathbf{C}^*$ is given by restricting ψ^* to the embedded image of $\mathbf{Q}_p^*$ in $\mathbf{A}^*$. Note that if d is an integer prime to N, then

$$\psi(d) = \prod_{p|N} \psi_p(d) = \psi^*(d_N). \tag{2.2}$$

Later we will consider a character χ of modulus D, and all of the above notation will apply equally with D in place of N.

We let $\theta : \mathbf{A} \longrightarrow \mathbf{C}^*$ denote the standard character of $\mathbf{A}$, given locally by

$$\theta_p(x) = \begin{cases} e^{-2\pi i x} & \text{if } p = \infty\ (x \in \mathbf{R}), \\ e^{2\pi i r_p(x)} & \text{if } p < \infty\ (x \in \mathbf{Q}_p), \end{cases}$$

where $r_p(x) \in \mathbf{Q}$ is the p-principal part of x, a number with p-power denominator characterized up to $\mathbf{Z}$ by $x \in r_p(x) + \mathbf{Z}_p$. The global character $\theta = \prod_{p\le\infty} \theta_p$ is then trivial on $\mathbf{Q}$ and, for finite p, θ_p is trivial precisely on $\mathbf{Z}_p$. For $r \in \mathbf{Q}$, we define

$$\theta_r(x) = \theta(-rx) = \overline{\theta(rx)}. \tag{2.3}$$

Every character of $\mathbf{Q}\backslash\mathbf{A}$ is of the form θ_r for some $r \in \mathbf{Q}$.

Let G denote the algebraic group GL_2, with center Z, and let $\overline{G}$ denote G/Z. The group $G(\mathbf{A}_{\mathrm{fin}})$ has the following sequences of open compact subgroups of $K = G(\widehat{\mathbf{Z}})$:

$$K_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \,\middle|\, c \in N\widehat{\mathbf{Z}}\right\},$$

$$K_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) \,\middle|\, d \equiv 1 \bmod N\widehat{\mathbf{Z}}\right\}.$$

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Because $\det K_0(N) = \det K_1(N) = \widehat{\mathbf{Z}}^*$, strong approximation holds for both of these and, in particular,

$$G(\mathbf{A}) = G(\mathbf{Q})(G(\mathbf{R})^+ \times K_1(N)), \tag{2.4}$$

where $G(\mathbf{R})^+ = \{g \in G(\mathbf{R}) \mid \det g > 0\}$.

Let $L^2(\psi^*) = L^2(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A}), \psi^*)$ be the space of measurable $\mathbf{C}$-valued functions ϕ on $G(\mathbf{A})$ satisfying $\phi(z\gamma g) = \psi^*(z)\phi(g)$ for all $z \in Z(\mathbf{A}), \gamma \in G(\mathbf{Q}), g \in G(\mathbf{A})$, and which are square integrable over $\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})$. Let $L^2_0(\psi^*)$ denote the subspace of cuspidal functions.

We now normalize Haar measure on each group of interest. Everything is the same as in [KL1, Section 7], where more detail is given. On $\mathbf{R}$ we take Lebesgue measure dx, and we take $dy/|y|$ on $\mathbf{R}^*$. We normalize the additive measure dx on $\mathbf{Q}_p$ by taking $\mathrm{meas}(\mathbf{Z}_p) = 1$, and likewise d^*y on $\mathbf{Q}_p^*$ is normalized by $\mathrm{meas}(\mathbf{Z}_p^*) = 1$. These choices determine Haar measures on $\mathbf{A}$ and $\mathbf{A}^*$ in the usual way, with the property that $\mathrm{meas}(\mathbf{A}/\mathbf{Q}) = 1$. We give the compact abelian group $K_\infty = \mathrm{SO}(2)$ the measure dk of total length 1, and use the above measures to define measures on $N(\mathbf{R}) = \{(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix})\} \cong \mathbf{R}$ and $M(\mathbf{R}) = \{(\begin{smallmatrix}y & \\ & z\end{smallmatrix})\} \cong \mathbf{R}^* \times \mathbf{R}^*$. These choices determine a Haar measure on $G(\mathbf{R})$ by the Iwasawa decomposition: $dg = d(mnk) = dm\,dn\,dk$. In the same way, our fixed measures on $\mathbf{Q}_p$ and $\mathbf{Q}_p^*$ determine measures on $N(\mathbf{Q}_p)$ and $M(\mathbf{Q}_p)$, respectively. We take the unique measure on $G(\mathbf{Q}_p)$ for which the open compact subgroup $K_p = G(\mathbf{Z}_p)$ has measure 1. On $\overline{G}(\mathbf{R})$ we take the measure $d\overline{m}\,dn\,dk$, where $d\overline{m}$ is the measure $dy/|y|$ on $\overline{M}(\mathbf{R}) \cong \{(\begin{smallmatrix}y & \\ & 1\end{smallmatrix})\} \cong \mathbf{R}^*$. These local measures determine a Haar measure on $\overline{G}(\mathbf{A})$ for which $\mathrm{meas}(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})) = \pi/3$.

Having fixed the measure, we note the following lemma.

Lemma 2.1. *Let $D > 0$ and let χ be a Dirichlet character modulo D (not necessarily primitive), with Gauss sum $\tau(\chi)$ as in (1.3). Let χ^* be the adelic realization of χ as in (2.1). Then, for any integer n prime to D,*

$$\int_{\widehat{\mathbf{Z}}^*} \chi^*(u)\theta_{\mathrm{fin}}\Big(\frac{nu}{D}\Big)\,d^*u = \frac{\overline{\chi(n)}}{\varphi(D)}\tau(\chi), \tag{2.5}$$

where φ is Euler's φ-function and $\theta_{\mathrm{fin}} = \prod_{p<\infty}\theta_p$.

Proof. The integrand in (2.5) is invariant under the subgroup

$$U_D = (1 + D\widehat{\mathbf{Z}}) \cap \widehat{\mathbf{Z}}^* = \prod_{p|D}(1 + D\mathbf{Z}_p)\prod_{p\nmid D}\mathbf{Z}_p^*.$$

Note that $\widehat{\mathbf{Z}}^*/U_D \cong (\mathbf{Z}/D\mathbf{Z})^*$, so $\mathrm{meas}(U_D) = \varphi(D)^{-1}$. Therefore,

$$\begin{aligned}\int_{\widehat{\mathbf{Z}}^*} \chi^*(u)\theta_{\mathrm{fin}}\Big(\frac{nu}{D}\Big)\,d^*u &= \frac{1}{\varphi(D)}\sum_{m\in(\mathbf{Z}/D\mathbf{Z})^*}\chi^*(m_D)\theta_{\mathrm{fin}}\Big(\frac{nm}{D}\Big)\\ &= \frac{1}{\varphi(D)}\sum_{m \bmod D}\chi(m)e^{2\pi inm/D} = \overline{\chi(n)}\frac{\tau(\chi)}{\varphi(D)}.\end{aligned}$$

□

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For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbf{R})^+$, we set

$$j(g,z) = \det(g)^{-1/2}(cz+d).$$

Recall that $j(g_1g_2,z) = j(g_1,g_2z)j(g_2,z)$. The group action of $G(\mathbf{R})^+$ on the complex upper half-plane $\mathbf{H}$ by linear fractional transformations extends to a right action on the space of functions $h : \mathbf{H} \longrightarrow \mathbf{C}$ via the weight k slash operator

$$h|_g(z) = j(g,z)^{-k}h(g(z)) \quad (g \in G(\mathbf{R})^+, z \in \mathbf{H}).$$

Fix a Dirichlet character ψ of modulus N, a positive integer k satisfying

$$\psi(-1) = (-1)^k, \tag{2.6}$$

and let $S_k(N,\psi)$ denote the space of cusp forms of level N, weight k, and character ψ. Thus, $h \in S_k(N,\psi)$ satisfies

$$h|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = \psi(d)\, h \tag{2.7}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $z \in \mathbf{H}$.

The adelization of h is the function $\phi_h \in L_0^2(\overline{\psi^*})$ defined using strong approximation (2.4) by

$$\phi_h(\gamma(g_\infty \times g_{\mathrm{fin}})) = j(g_\infty, i)^{-k}h(g_\infty(i)) \tag{2.8}$$

for $\gamma \in G(\mathbf{Q})$, $g_\infty \in G(\mathbf{R})^+$, and $g_{\mathrm{fin}} \in K_1(N)$. The modularity of h makes ϕ_h well defined, and one checks readily that the central character is indeed $\overline{\psi^*}$ (see for example the proof of Proposition 4.5 [KL3]; the complex conjugate is needed here because we have not included it in (2.7)). With the choice of Haar measure on $\overline{G}(\mathbf{A})$ given above, and the normalization (1.1), the map $h \mapsto \phi_h$ is an isometry, that is, $\|h\| = \|\phi_h\|$ (cf. [KL1, (12.20)]).

We recall the meaning of the following 'period integrals'.

Lemma 2.2. *For $r \in \mathbf{Q}$ and $h \in S_k(N,\psi)$,*

$$\int_{\mathbf{Q}\backslash\mathbf{A}} \phi_h\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \overline{\theta_r(x)}\, dx = \begin{cases} e^{-2\pi r}a_r(h) & \text{if } r \in \mathbf{Z}^+, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*} \phi_h\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) |y|^{s-k/2}\, d^*y = \Lambda(s,h).$$

Proof. See for example [KL1, Corollary 12.4] and [KL2, Lemma 3.1], respectively. □

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3. The twisting operator

From now on, we assume that χ is a primitive Dirichlet character modulo D, with $(D,N)=1$. Recall that $L(s,h,\chi)=L(s,h_\chi)$, where $h_\chi\in S_k(D^2N,\chi^2\psi)$ is given by

$$h_\chi(z)=\sum_{r=1}^{\infty}\chi(r)a_r(h)e^{2\pi irz}$$

or, equivalently,

$$h_\chi=\frac{1}{\tau(\overline{\chi})}\sum_{m \bmod D}\overline{\chi(m)}h|_{\left(\begin{smallmatrix}1 & m/D\\ 0 & 1\end{smallmatrix}\right)} \tag{3.1}$$

(see for example [Bu, page 59]). Likewise, it follows from the definitions that for $g_\infty\in G(\mathbf{R})^+$,

$$\phi_{h_\chi}(g_\infty\times 1_{\mathrm{fin}})=\frac{1}{\tau(\overline{\chi})}\sum_{m \bmod D}\overline{\chi(m)}\phi_h\left(\begin{pmatrix}1 & m/D\\ 0 & 1\end{pmatrix}g_\infty\times 1_{\mathrm{fin}}\right). \tag{3.2}$$

Because χ is assumed to be primitive, we have $|\tau(\overline{\chi})|=\sqrt{D}$.

We now define a test function $f^\chi: G(\mathbf{A}_{\mathrm{fin}})\longrightarrow\mathbf{C}$ which essentially realizes the twisting map $h\mapsto h_\chi$ adelically (see also [RR]). It will be supported on the disjoint union

$$\mathrm{Supp}(f^\chi)=\bigcup_{\substack{m \bmod D,\\ (m,D)=1}}\begin{pmatrix}1 & -m/D\\ 0 & 1\end{pmatrix}Z(\mathbf{A}_{\mathrm{fin}})K_1(N), \tag{3.3}$$

where the rational matrix is embedded diagonally in $G(\mathbf{A}_{\mathrm{fin}})$. The value of f^χ on the coset indexed by m is defined to be

$$f^\chi\left(\begin{pmatrix}1 & -m/D\\ 0 & 1\end{pmatrix}zk\right)=\frac{\nu(N)\overline{\chi(m)}\psi^*(z)}{\tau(\overline{\chi})}. \tag{3.4}$$

Here, as before,

$$\nu(N)=[K:K_0(N)]=[\overline{K}:\overline{K_1(N)}]=\mathrm{meas}(\overline{K_1(N)})^{-1}. \tag{3.5}$$

Lemma 3.1. *Consider the open compact subgroup*

$$J=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in K_1(D^2N)\;\middle|\; a\equiv 1 \bmod D\widehat{\mathbf{Z}}\right\}.$$

The function f^χ is right $K_1(N)$-invariant and left J-invariant.

Proof. The first claim is obvious from the definition of f^χ. The second claim follows from the fact that

$$\begin{pmatrix}a & b\\ cD^2N & d\end{pmatrix}\begin{pmatrix}1 & -\frac{m}{D}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -\frac{m}{D}\\ 0 & 1\end{pmatrix}\begin{pmatrix}a+mcDN & b+(d-a)\frac{m}{D}-m^2cN\\ cD^2N & d-mcDN\end{pmatrix},$$

noting that if $a\equiv 1 \bmod D$ and $d\equiv 1 \bmod D^2N$, the matrix on the right belongs to $K_1(N)$. $\square$

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Note that because f^χ has compact support modulo the center, and transforms under $Z(\mathbf{A}_{\mathrm{fin}})$ by ψ^*, it defines an operator on $L^2(\overline{\psi^*})$ by

$$R(f^\chi)\phi(x) = \int_{\overline{G}(\mathbf{A}_{\mathrm{fin}})} f^\chi(g)\phi(xg)\,dg. \tag{3.6}$$

This operator is closely related to the twisting function $h \mapsto h_\chi$, as we now show.

Proposition 3.2. *Let $h \in S_k(N,\psi)$ and let χ^* be the Hecke character attached to χ as in (2.1). Then, for all $x \in G(\mathbf{A}_{\mathrm{fin}})$,*

$$R(f^\chi)\phi_h(x) = \chi^*(a_D)\phi_{h_\chi}(x), \tag{3.7}$$

where a is determined from x using strong approximation by writing

$$x = x_{\mathbf{Q}}\left(x_\infty \times \begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)$$

for $x_{\mathbf{Q}} \in G(\mathbf{Q})$, $x_\infty \in G(\mathbf{R})^+$, and $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in K_1(D^2N)$, and a_D is the finite idele with local components $(a_D)_p = a_p$ (respectively 1) if $p|D$ (respectively $p \nmid D$).

Remark 3.3. If $a \in \widehat{\mathbf{Z}}^*$, then $\chi^*(a_D) = \chi^*(a)$.

Proof. Clearly, $R(f^\chi)\phi_h$ inherits the left $G(\mathbf{Q})$-invariance from ϕ_h. Hence, we can assume that $x_{\mathbf{Q}} = 1$. It is an easy consequence of the above lemma that $R(f^\chi)\phi$ is right J-invariant. Therefore, we may modify x_{fin} on the right by an appropriate element of J to reduce to the case where $x_{\mathrm{fin}} = \left(\begin{smallmatrix} a_D & \\ & 1\end{smallmatrix}\right)$. Hence, it suffices to prove that

$$R(f^\chi)\phi_h\left(x_\infty \times \begin{pmatrix} a_D & \\ & 1\end{pmatrix}\right) = \chi^*(a_D)\phi_{h_\chi}(x_\infty \times 1_{\mathrm{fin}}).$$

Let $\alpha_m = \left(\begin{smallmatrix} 1 & m/D \\ 0 & 1\end{smallmatrix}\right)$. From the preceding definitions and the right $K_1(N)$-invariance of ϕ_h, for any $x \in G(\mathbf{A})$,

$$\begin{aligned} R(f^\chi)\phi_h(x) &= \sum_{m \bmod D} \int_{\alpha_m^{-1}\overline{K_1(N)}} f^\chi(g)\phi_h(xg)\,dg \\ &= \sum_{m \bmod D} \phi_h(x\alpha_m^{-1})\frac{\nu(N)\overline{\chi(m)}}{\tau(\overline{\chi})}\int_{\alpha_m^{-1}\overline{K_1(N)}} dg \\ &= \frac{1}{\tau(\overline{\chi})}\sum_{m \bmod D}\overline{\chi(m)}\phi_h(x\alpha_m^{-1}). \end{aligned}$$

Taking $x = x_\infty \times \left(\begin{smallmatrix} a_D & \\ & 1\end{smallmatrix}\right)$,

$$\begin{aligned} \phi_h(x\alpha_m^{-1}) &= \phi_h\left(x_\infty \times \begin{pmatrix} a_D & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & -\frac{m}{D} \\ 0 & 1\end{pmatrix}\right) \\ &= \phi_h\left(x_\infty \times \begin{pmatrix} 1 & -\frac{a_D m}{D} \\ 0 & 1\end{pmatrix}\begin{pmatrix} a_D & \\ & 1\end{pmatrix}\right) \\ &= \phi_h\left(x_\infty \times \begin{pmatrix} 1 & -\frac{a_D m}{D} \\ 0 & 1\end{pmatrix}\right) \end{aligned}$$

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by the right $K_1(N)$-invariance of ϕ_h. Fix an integer z relatively prime to D such that $z \equiv a_D \bmod D\widehat{\mathbf{Z}}$, and multiply through on the left by α_{zm}. By the left $G(\mathbf{Q})$-invariance this has no effect, so the above is

$$= \phi_h\left(\begin{pmatrix} 1 & \dfrac{zm}{D} \\ 0 & 1 \end{pmatrix} x_\infty \times \begin{pmatrix} 1 & \dfrac{(z-a_D)m}{D} \\ 0 & 1 \end{pmatrix}\right) = \phi_h(\alpha_{zm}x_\infty \times 1_{\mathrm{fin}}),$$

again by right $K_1(N)$-invariance. Therefore,

$$\begin{aligned} R(f^\chi)\phi_h\left(x_\infty \times \begin{pmatrix} a_D & \\ & 1 \end{pmatrix}\right) &= \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod D} \overline{\chi(m)}\phi_h(\alpha_{zm}x_\infty \times 1_{\mathrm{fin}}) \\ &= \frac{\chi(z)}{\tau(\overline{\chi})} \sum_{m \bmod D} \overline{\chi(m)}\phi_h(\alpha_m x_\infty \times 1_{\mathrm{fin}}) = \chi(z)\phi_{h_\chi}(x_\infty \times 1_{\mathrm{fin}}) \end{aligned}$$

by (3.2). This gives the desired result, since $\chi(z) = \chi^*(a_D)$ by (2.1) and (2.2). □

It will be useful to define local components for f^χ. For any prime p, we have defined χ_p to be the character of $\mathbf{Q}_p^*$ attached to the Hecke character χ^*. When $p|D$,

$$\chi_p : \mathbf{Z}_p^* \longrightarrow (\mathbf{Z}_p/D\mathbf{Z}_p)^* \cong (\mathbf{Z}/p^{D_p}\mathbf{Z})^* \longrightarrow \mathbf{C}^*.$$

Still assuming that $p|D$, a local version of (2.5) is the following:

$$\int_{\mathbf{Z}_p^*} \chi_p(u)\theta_p\left(\frac{nu}{D}\right) d^*u = \frac{\overline{\chi_p(n)}}{\varphi(p^{D_p})}\tau(\chi)_p, \tag{3.8}$$

where

$$\tau(\chi)_p = \chi_p\left(\frac{D}{p^{D_p}}\right)\tau(\chi_p) \tag{3.9}$$

for $\tau(\chi_p) = \sum_{m\in(\mathbf{Z}/p^{D_p}\mathbf{Z})^*} \chi_p(m)e^{2\pi i m/p^{D_p}}$, the Gauss sum of the character χ_p. Then, by (3.8) and (2.5),

$$\tau(\chi) = \prod_{p|D} \tau(\chi)_p. \tag{3.10}$$

Given a prime p, we define a local function $f_p^\chi : G(\mathbf{Q}_p) \longrightarrow \mathbf{C}$ as follows. If $p|D$, we take

$$\mathrm{Supp}(f_p^\chi) = \bigcup_{\substack{m \bmod D\mathbf{Z}_p \\ p\nmid m}} \begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} Z_p K_p \tag{3.11}$$

(a disjoint union), and define $f_p^\chi = \sum_m f_{p,m}^\chi$, where $f_{p,m}^\chi$ is supported on the coset indexed by m in (3.11), and is given by

$$f_{p,m}^\chi\left(\begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} zk\right) = \frac{\overline{\chi_p(m)}\psi_p(z)}{\tau(\overline{\chi})_p} \tag{3.12}$$

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for $\tau(\overline{\chi})_p$ as in (3.9). The value is independent of the choice of representative for $m \in (\mathbf{Z}_p/p^{D_p}\mathbf{Z}_p)^*$, since χ_p has conductor p^{D_p}. If $p \nmid D$, then f_p^χ is supported on $Z_pK_1(N)_p$, and we define it by

$$f_p^\chi(zk) = \nu_p(N)\psi_p(z),$$

where $\nu_p(N) = [K_p : K_0(N)_p] = \nu(p^{N_p})$. It is easily verified using (2.2) (applied to χ^*) and (3.10) that $f^\chi = \prod_p f_p^\chi$.

4. The Hecke operator

We refer to [KL1, Section 13] for a more detailed account of the adelic Hecke operator defined here. Fix a positive integer n with $(n, DN) = 1$. Define

$$M(n, N) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\widehat{\mathbf{Z}}) \,\middle|\, \det g \in n\widehat{\mathbf{Z}}^*, c \in N\widehat{\mathbf{Z}} \right\}.$$

Define a function $f^n : G(\mathbf{A}_{\mathrm{fin}}) \longrightarrow \mathbf{C}$ with

$$\mathrm{Supp}(f^n) = Z(\mathbf{A}_{\mathrm{fin}})M(n, N) = Z(\mathbf{Q}^+)M(n, N)$$

by

$$f^n(z_{\mathbf{Q}}m) = \nu(N)\psi^*(d_N) \quad \left(z_{\mathbf{Q}} \in Z(\mathbf{Q}^+), m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(n, N) \right).$$

One shows easily that f^n is bi-$K_1(N)$-invariant.

For any prime p, let

$$M(n, N)_p = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_p) \,\middle|\, \det g \in n\mathbf{Z}_p^*, c \in N\mathbf{Z}_p \right\}.$$

Notice that if $p \nmid n$, then $M(n, N)_p = K_0(N)_p$. Define a function $f_p^n : G(\mathbf{Q}_p) \to \mathbf{C}$, supported on $Z(\mathbf{Q}_p)M(n, N)_p$, by

$$f_p^n(zm) = \nu_p(N)\psi_p(z)\psi_p((d_N)_p) \quad \left(z \in Z(\mathbf{Q}_p), m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(n, N)_p \right). \tag{4.1}$$

Then it is straightforward to check that $f^n(g) = \prod_p f_p^n(g_p)$ for $g \in G(\mathbf{A}_{\mathrm{fin}})$.

Proposition 4.1. *For $h \in S_k(N, \psi)$,*

$$R(f^n)\phi_h = n^{1-k/2}\phi_{T_n h}. \tag{4.2}$$

Proof. See [KL1, Proposition 13.6]. □

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5. The global test function

We take $f_\infty(g) = d_k\overline{\langle \pi_k(g)v_0, v_0\rangle}$, where π_k is the weight k discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ with formal degree $d_k = (k-1)/4\pi$, central character $\left(\begin{smallmatrix} x & \\ & x\end{smallmatrix}\right) \mapsto \operatorname{sgn}(x)^k$, and lowest weight unit vector v_0. Explicitly,

$$f_\infty\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \frac{(k-1)}{4\pi}\,\frac{\det(g)^{k/2}(2i)^k}{(-b+c+(a+d)i)^k}$$

if $ad - bc > 0$, and it vanishes otherwise (see [KL1, Theorem 14.5]). This function is self-adjoint, meaning that

$$f_\infty(g) = \overline{f_\infty(g^{-1})}. \tag{5.1}$$

It is integrable if and only if $k > 2$ [KL1, Proposition 14.3].

Given two functions $f_1, f_2 \in L^1(\psi^*)$, we define their convolution

$$f_1 * f_2(x) = \int_{\overline{G}(\mathbf{A})} f_1(g)f_2(g^{-1}x)\,dg = \int_{\overline{G}(\mathbf{A})} f_1(xg^{-1})f_2(g)\,dg.$$

It is straightforward to show that

$$R(f_1 * f_2) = R(f_1)\circ R(f_2) \tag{5.2}$$

as operators on $L^2(\overline{\psi^*})$, where

$$R(f)\phi(x) = \int_{\overline{G}(\mathbf{A})} f(g)\phi(xg)\,dg.$$

Fix an integer $n > 0$ relatively prime to DN, and set

$$f = (f_\infty \times f^\chi) * (f_\infty \times f^n). \tag{5.3}$$

Local components for f can be defined as follows.

Proposition 5.1. *With notation as in the previous two sections, define*

$$f_p = \begin{cases} f_p^\chi = f_p^n & \text{if } p \nmid nD,\\ f_p^\chi & \text{if } p|D,\\ f_p^n & \text{if } p|n.\end{cases}$$

Then $f = f_\infty \prod_p f_p$.

Proof. Because $f_\infty \times f^\chi$ and $f_\infty \times f^n$ are both factorizable and identically 1 on K_p for almost every p, the integral defining their convolution is factorizable and hence

$$f = (f_\infty \times f^\chi) * (f_\infty \times f^n) = (f_\infty * f_\infty)\prod_p (f_p^\chi * f_p^n).$$

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It follows directly from the orthogonality relations for discrete series that $f_\infty * f_\infty = f_\infty$. Indeed,

$$\begin{aligned} f_\infty * f_\infty(x) &= d_k^2 \int_{\overline{G}(\mathbf{R})} \overline{\langle \pi_k(g)v_0, v_0\rangle\langle \pi_k(g^{-1}x)v_0, v_0\rangle}\, dg \\ &= d_k^2 \int_{\overline{G}(\mathbf{R})} \langle \pi_k(g)v_0, \pi_k(x)v_0\rangle \overline{\langle \pi_k(g)v_0, v_0\rangle}\, dg \\ &= d_k^2 \frac{\langle v_0, v_0\rangle \overline{\langle \pi_k(x)v_0, v_0\rangle}}{d_k} \\ &= f_\infty(x). \end{aligned}$$

Likewise, simple direct computation shows that for finite primes p, $f_p^\chi * f_p^n = f_p$, as given. □

Globally, the support of f is

$$\mathrm{Supp}(f) = G(\mathbf{R})^+ \times \bigcup_{\substack{m \bmod D,\\ (m,D)=1}} \begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} Z(\mathbf{Q}^+)M(n,N). \tag{5.4}$$

The union over m is easily seen to be disjoint, using the fact that $(n, D) = 1$. Accordingly, we can write

$$f_{\mathrm{fin}} = \sum_{m \in (\mathbf{Z}/D\mathbf{Z})^*} f_m,$$

where f_m is supported on the coset indexed by m in (5.4), and

$$f_m\left(\begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} zk\right) = \frac{\nu(N)\overline{\chi(m)}\psi^*(d_N)}{\tau(\overline{\chi})} \tag{5.5}$$

for $z \in Z(\mathbf{Q}^+)$ and $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M(n, N)$.

Under the condition $k > 2$ (which will be in force throughout), $f \in L^1(\psi^*)$, so the operator $R(f)$ on $L^2(\overline{\psi^*})$ is defined.

Proposition 5.2. *The operator $R(f)$ factors through the orthogonal projection of $L^2(\overline{\psi^*})$ onto the finite-dimensional subspace $S_k(N, \psi)$ (embedded in $L^2(\overline{\psi^*})$ via (2.8)).*

Proof. The operator $R(f_\infty \times f^n)$ factors through the orthogonal projection onto $S_k(N, \psi)$ [KL1, Corollary 13.13]. Therefore, by (5.2) and (5.3), $R(f)$ has the same property. □

Remark 5.3. The image of $R(f)$ is *not* contained in any classical space of cusp forms $S_k(\Gamma_1(M))$. Indeed, it is immediate from (3.7) that the image of $R(f)$ is not left $K_1(M)$-invariant for any M, since a congruence condition on a is needed.

The operator $R(f)$ is an integral operator given by the continuous kernel

$$K(g_1, g_2) = \sum_{h \in \mathcal{F}} \frac{R(f)\phi_h(g_1)\overline{\phi_h(g_2)}}{\|h\|^2} = \sum_{\gamma \in \overline{G}(\mathbf{Q})} f(g_1^{-1}\gamma g_2). \tag{5.6}$$

Here, the spectral sum is taken over any orthogonal basis $\mathcal{F}$ for $S_k(N, \psi)$, as a consequence of Proposition 5.2, and both sums are absolutely convergent.

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6. Spectral side

Fix a positive integer r relatively prime to D. We prove Theorem 1.1 by computing the following integral:

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}\int_{\mathbf{Q}\backslash\mathbf{A}} K\left(\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)\theta_r(x)\overline{\chi^*(y)}|y|^{s-k/2}\,dx\,d^*y \tag{6.1}$$

using the two expressions for the kernel (5.6). We note that the double integral is absolutely convergent for all $s\in\mathbf{C}$ (see below).

On the spectral side, (6.1) becomes

$$\sum_{h\in\mathcal{F}}\frac{1}{\|h\|^2}\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}R(f)\phi_h\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\overline{\chi^*(y)}|y|^{s-k/2}\,d^*y\int_{\mathbf{Q}\backslash\mathbf{A}}\overline{\phi_h\left(\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)}\theta_r(x)\,dx. \tag{6.2}$$

Choose $\mathcal{F}$ in (5.6) to consist of eigenvectors of T_n. Then, for $h\in\mathcal{F}$, we write $T_nh=\lambda_n(h)h$, so that, by (3.7) and (4.2),

$$R(f)\phi_h\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)=n^{1-k/2}\lambda_n(h)\chi^*(y)\phi_{h_\chi}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)$$

for all $y\in\mathbf{R}^+\times\widehat{\mathbf{Z}}^*\cong\mathbf{Q}^*\backslash\mathbf{A}^*$. Consequently, the factor $\overline{\chi^*(y)}$ in (6.2) is cancelled out, and (6.2) becomes

$$\sum_{h\in\mathcal{F}}\frac{n^{1-k/2}\lambda_n(h)}{\|h\|^2}\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}\phi_{h_\chi}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)|y|^{s-k/2}\,d^*y\int_{\mathbf{Q}\backslash\mathbf{A}}\overline{\phi_h\left(\begin{pmatrix} 1 & x\\ & 1\end{pmatrix}\right)}\theta_r(x)\,dx \tag{6.3}$$

$$=\frac{n^{1-k/2}}{e^{2\pi r}}\sum_{h\in\mathcal{F}}\frac{\lambda_n(h)\overline{a_r(h)}}{\|h\|^2}\Lambda(s,h,\chi) \tag{6.4}$$

by Lemma 2.2. The two integrals in (6.4) are absolutely convergent for all s, so we have the following proposition.

Proposition 6.1. *The double integral (6.1) is absolutely convergent for all $s\in\mathbf{C}$.*

7. Geometric side

For the moment, let $H(\mathbf{A})=\overline{M}(\mathbf{A})\times N(\mathbf{A})\cong\mathbf{A}^*\times\mathbf{A}$. Inserting the geometric expression $K(g_1,g_2)=\sum_\gamma f(g_1^{-1}\gamma g_2)$ into (6.1),

$$\int_{H(\mathbf{Q})\backslash H(\mathbf{A})}\sum_{\gamma\in\overline{G}(\mathbf{Q})} f\left(\begin{pmatrix} y^{-1} & 0\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)\theta_r(x)\overline{\chi^*(y)}|y|^{s-k/2}\,dx\,d^*y$$

$$=\int_{H(\mathbf{Q})\backslash H(\mathbf{A})}\sum_{\delta}\sum_{\gamma\in[\delta]} f\left(\begin{pmatrix} y^{-1} & 0\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)\theta_r(x)\overline{\chi^*(y)}|y|^{s-k/2}\,dx\,d^*y,$$

where δ ranges over a set of representatives for the $H(\mathbf{Q})$-orbits in $\overline{G}(\mathbf{Q})$ relative to the action $(m,n)\cdot\gamma=m^{-1}\gamma n$, and $[\delta]=\{m^{-1}\delta n\mid(m,n)\in H_\delta(\mathbf{Q})\backslash H(\mathbf{Q})\}$ is the orbit. It is

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not hard to check that in fact for all δ, the stabilizer $H_\delta(\mathbf{Q}) = \{1\}$. Therefore, (6.1) is formally equal to

$$\sum_\delta \int_{\mathbf{A}^*} \int_{\mathbf{A}} f\left(\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \theta_r(x)\overline{\chi^*(y)}|y|^{s-k/2}\,dx\,d^*y, \tag{7.1}$$

where δ runs through a set of representatives for $\overline{M}(\mathbf{Q})\backslash\overline{G}(\mathbf{Q})/N(\mathbf{Q})$. By the Bruhat decomposition

$$G(\mathbf{Q}) = M(\mathbf{Q})N(\mathbf{Q}) \cup M(\mathbf{Q})N(\mathbf{Q})\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} N(\mathbf{Q}),$$

a set of representatives δ is given by

$$\{1\} \cup \left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix} \,\middle|\, t \in \mathbf{Q}^*\right\}. \tag{7.2}$$

The equality between (6.1) and (7.1) is valid on the strip $1 < \mathrm{Re}(s) < k-1$. This is a consequence of the following proposition.

Proposition 7.1. *Suppose that* $1 < \mathrm{Re}(s) < k-1$. *Then*

$$\sum_\delta \int_{\mathbf{A}^*} \int_{\mathbf{A}} \left| f\left(\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \theta_r(x)\overline{\chi^*(y)}|y|^{s-k/2} \right| dx\,d^*y < \infty.$$

Proof. This is proven in just the same way as the analogous result in [KL2, Proposition 3.3]. We outline the steps. Because f_{fin} is bounded and compactly supported as a function of y, x, the argument hinges on bounding the infinite part

$$I_\delta^{abs}(f)_\infty = \int_0^\infty \int_{-\infty}^\infty \left| f_\infty\left(\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\right| dx\, y^{\sigma-k/2-1}\,dy,$$

where $\sigma = \mathrm{Re}(s)$. However, by (5.1), the above is

$$= \int_0^\infty \int_{-\infty}^\infty \left| f_\infty\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \delta^{-1} \begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)\right| dx\, y^{\sigma-k/2-1}\,dy.$$

Noting that the set of the δ in (7.2) is exactly the set of inverses of the δ in [KL2], the above coincides with $I_{\delta^{-1}}^{abs}(f)_\infty$ considered in Section 3.3 there. Hence, those results give

$$I_\delta^{abs}(f) < \infty \quad \text{for} \begin{cases} \delta = 1 & \text{and } 0 < \sigma < k-1, \\ \delta = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} & \text{and } 1 < \sigma < k, \\ \delta = \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix} & \text{and } 0 < \sigma < k. \end{cases}$$

Furthermore, by [KL2, Proposition 3.3], for $\delta_t = \left(\begin{smallmatrix} t & -1 \\ 1 & 0 \end{smallmatrix}\right)$,

$$I_{\delta_t}^{abs}(f)_\infty \ll |t|^{\sigma-k} \quad \text{if } 0 < \sigma < k. \tag{7.3}$$

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Thus, to complete the proof, it remains to show that for $1 < \sigma < k-1$,

$$\sum_{t \in \mathbf{Q}^*} I^{abs}_{\delta_t}(f) < \infty.$$

We will prove in Proposition 8.1 below that the finite part $I^{abs}_{\delta_t}(f)_{\mathrm{fin}}$ vanishes unless $t = Nb/nD$ for some $b \in \mathbf{Z} - \{0\}$. We will show in (8.14) that for such t,

$$|I^{abs}_{\delta_t}(f)_{\mathrm{fin}}| \leq \frac{n^{\sigma-k/2}\nu(N)\varphi(D)\gcd(r,n)}{N^{2\sigma-k}D^{1/2}} \sum_{d|b} d^{k-2\sigma}.$$

Together with (7.3), the fact that $\#\{d : d|b\} \ll |b|^{\varepsilon}$ for $\varepsilon > 0$, and using $d^{k-2\sigma} \leq 1$ when $\sigma > k/2$, this gives the global estimate

$$\sum_{t \in \mathbf{Q}^*} I^{abs}_{\delta_t}(f) \ll_{N,D,n,\varepsilon} \begin{cases} \displaystyle\sum_{b \in \mathbf{Z}-\{0\}} |b|^{-\sigma+\varepsilon} & \text{if } \sigma \leq k/2, \\ \displaystyle\sum_{b \in \mathbf{Z}-\{0\}} |b|^{\sigma-k+\varepsilon} & \text{if } \sigma > k/2. \end{cases}$$

This is finite when $1 < \sigma < k-1$ and ε is sufficiently small. □

Let $I_\delta(s)$ denote the double integral attached to δ in (7.1). For $1 < \mathrm{Re}(s) < k-1$ and each δ in (7.2), we need to compute $I_\delta(s)$. It factorizes as

$$I_\delta(s) = I_\delta(s)_\infty I_\delta(s)_{\mathrm{fin}} = I_\delta(s)_\infty \prod_p I_\delta(s)_p,$$

where

$$I_\delta(s)_\infty = \int_{\mathbf{R}^*} \int_{\mathbf{R}} f_\infty\left(\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \overline{\theta_\infty(rx)\chi_\infty(y)}|y|^{s-k/2}\, dx\, \frac{dy}{|y|}$$

and likewise

$$I_\delta(s)_p = \int_{\mathbf{Q}_p^*} \int_{\mathbf{Q}_p} f_p\left(\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \overline{\theta_p(rx)\chi_p(y)}|y|_p^{s-k/2}\, dx\, d^*y.$$

From the definition of f_∞, the integrand for $I_\delta(s)_\infty$ vanishes unless $y > 0$. Because χ_∞ is trivial on $\mathbf{R}^+$, it has no effect on $I_\delta(s)_\infty$ and can be removed. Using (5.1),

$$I_\delta(s)_\infty = \overline{I'_{\delta^{-1}}(\overline{s})_\infty}, \tag{7.4}$$

where

$$I'_{\delta^{-1}}(s)_\infty = \int_0^\infty \int_{-\infty}^\infty f_\infty\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \delta^{-1} \begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \theta_\infty(rx) y^{s-k/2}\, dx\, \frac{dy}{y}$$

is the archimedean factor computed in [KL2].

For convenience, when computing the finite part $I_\delta(s)_{\mathrm{fin}}$, we will replace y by y^{-1}. (It is a property of unimodular (for example abelian) groups that this does not affect the value of the integral.) Thus,

$$I_\delta(s)_{\mathrm{fin}} = \int_{\mathbf{A}^*_{\mathrm{fin}}} \int_{\mathbf{A}_{\mathrm{fin}}} f_{\mathrm{fin}}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \overline{\theta_{\mathrm{fin}}(rx)}\, dx\, \chi^*(y)|y|_{\mathrm{fin}}^{k/2-s}\, d^*y.$$

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Looking at determinants, by (5.4), the integrand is nonzero only if $y = nu/\ell^2$ for some $\ell \in \mathbf{Q}^+$ and $u \in \widehat{\mathbf{Z}}^*$. Thus, since $\mathbf{Q}^+ \cap \widehat{\mathbf{Z}}^* = \{1\}$, the above is

$$= \sum_{\ell \in \mathbf{Q}^+} \Big(\frac{n}{\ell^2}\Big)^{s-k/2} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\mathrm{fin}}} f_{\mathrm{fin}}\bigg(\begin{pmatrix} \ell & \\ & \ell \end{pmatrix}^{-1} \begin{pmatrix} \frac{nu}{\ell} & \\ & \ell \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\bigg) \overline{\theta_{\mathrm{fin}}(rx)}\, dx\, \chi^*\Big(\frac{nu}{\ell^2}\Big)\, d^*u.$$

Note that $\chi^*_{\mathrm{fin}}(n/\ell^2) = \chi^*(n/\ell^2) = 1$ since $n/\ell^2 \in \mathbf{Q}^+$. Likewise, the scalar factor of ℓ_{fin}^{-1} pulls out of f_{fin} as $\psi^*_{\mathrm{fin}}(\ell^{-1}) = 1$. Hence,

$$I_\delta(s)_{\mathrm{fin}} = \sum_{\ell \in \mathbf{Q}^+} \Big(\frac{n}{\ell^2}\Big)^{s-k/2} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\mathrm{fin}}} f_{\mathrm{fin}}\bigg(\begin{pmatrix} \frac{nu}{\ell} & \\ & \ell \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\bigg) \overline{\theta_{\mathrm{fin}}(rx)}\, dx\, \chi^*(u)\, d^*u. \tag{7.5}$$

Proposition 7.2. *When $\delta = 1$, the integral*

$$I_1(s) = \int_{\mathbf{A}^*} \int_{\mathbf{A}} f\bigg(\begin{pmatrix} y & xy \\ 0 & 1 \end{pmatrix}\bigg) \overline{\theta(rx)}\, dx\, \chi^*(y)|y|^{k/2-s}\, d^*y$$

converges absolutely on $0 < \mathrm{Re}(s) < k-1$, and for such s it is equal to

$$\frac{n^{1-k/2}}{e^{2\pi r}} \frac{2^{k-1}(2\pi rn)^{k-s-1}}{(k-2)!} \Gamma(s)\nu(N) \sum_{d|(n,r)} d^{2s-k+1} \psi\Big(\frac{n}{d}\Big)\chi\Big(\frac{rn}{d^2}\Big). \tag{7.6}$$

Proof. The absolute convergence was proven in Proposition 7.1. We factorize the integral as $I_1(s) = I_1(s)_\infty I_1(s)_{\mathrm{fin}}$. By (7.4) and the proof of [KL2, Proposition 3.4],

$$I_1(s)_\infty = \frac{2^{k-1}(2\pi r)^{k-s-1}}{(k-2)!\, e^{2\pi r}} \Gamma(s). \tag{7.7}$$

Now consider the finite part, which by (7.5) is

$$I_1(s)_{\mathrm{fin}} = \sum_{\ell \in \mathbf{Q}^+} \Big(\frac{n}{\ell^2}\Big)^{s-k/2} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\mathrm{fin}}} f_{\mathrm{fin}}\bigg(\begin{pmatrix} \frac{nu}{\ell} & \frac{xnu}{\ell} \\ 0 & \ell \end{pmatrix}\bigg) \overline{\theta_{\mathrm{fin}}(rx)}\, dx\, \chi^*(u)\, d^*u.$$

Replacing x by $\ell x/nu$, the above is

$$\begin{aligned} &= \sum_{\ell \in \mathbf{Q}^+} \frac{n^{s-k/2+1}}{\ell^{2s-k+1}} \sum_{m \in (\mathbf{Z}/D\mathbf{Z})^*} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\mathrm{fin}}} f_m\bigg(\begin{pmatrix} \frac{nu}{\ell} & x \\ 0 & \ell \end{pmatrix}\bigg) \overline{\theta_{\mathrm{fin}}\Big(\frac{r\ell x}{nu}\Big)}\, dx\, \chi^*(u)\, d^*u \\ &= \sum_{\ell \in \mathbf{Q}^+} \frac{n^{s-k/2+1}}{\ell^{2s-k+1}} \sum_m \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\mathrm{fin}}} f_m\bigg(\begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{nu}{\ell} & x + \frac{\ell m}{D} \\ 0 & \ell \end{pmatrix}\bigg) \overline{\theta_{\mathrm{fin}}\Big(\frac{r\ell x}{nu}\Big)}\, dx\, \chi^*(u)\, d^*u. \end{aligned}$$

By (5.4), the integrand is nonzero if and only if $nu/\ell, \ell, x + \ell m/D \in \widehat{\mathbf{Z}}$. Replacing x by $x - \ell m/D$,

$$\begin{aligned} I_1(s)_{\mathrm{fin}} &= \frac{\nu(N)}{\tau(\overline{\chi})} \sum_{\ell | n} \frac{n^{s-k/2+1}}{\ell^{2s-k+1}} \psi^*(\ell_N) \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \int_{\widehat{\mathbf{Z}}} \overline{\theta_{\mathrm{fin}}\Big(\frac{r\ell(x - \frac{\ell m}{D})}{nu}\Big)}\, dx\, \chi^*(u)\, d^*u \\ &= \frac{\nu(N)}{\tau(\overline{\chi})} \sum_{\ell | n} \frac{n^{s-k/2+1}}{\ell^{2s-k+1}} \psi(\ell) \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \theta_{\mathrm{fin}}\Big(\frac{r\ell^2 m}{nDu}\Big) \int_{\widehat{\mathbf{Z}}} \overline{\theta_{\mathrm{fin}}\Big(\frac{r\ell x}{nu}\Big)}\, dx\, \chi^*(u)\, d^*u. \end{aligned}$$

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The integral over $\widehat{\mathbf{Z}}$ evaluates to 1 if $\frac{n}{\ell}|r$, and 0 otherwise. Setting $d = n/\ell$, the above is

$$\begin{aligned}
&= \frac{\nu(N)}{\tau(\overline{\chi})} \sum_{d|(n,r)} \frac{n^{s-k/2+1}}{(\frac{n}{d})^{2s-k+1}} \psi\Big(\frac{n}{d}\Big) \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \theta_{\mathrm{fin}}\Big(\frac{rnm}{d^2 Du}\Big) \chi^*(u)\, d^*u \\
&= \frac{\nu(N) n^{k/2-s}}{\tau(\overline{\chi})} \sum_{d|(n,r)} d^{2s-k+1} \psi\Big(\frac{n}{d}\Big) \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \theta_{\mathrm{fin}}\Big(\frac{rnm}{d^2 D} u\Big) \overline{\chi^*(u)}\, d^*u \\
&= \frac{\nu(N) n^{k/2-s}}{\tau(\overline{\chi})} \sum_{d|(n,r)} d^{2s-k+1} \psi\Big(\frac{n}{d}\Big) \sum_m \overline{\chi(m)} \chi\Big(\frac{rnm}{d^2}\Big) \frac{\tau(\overline{\chi})}{\varphi(D)} \\
&= \nu(N) n^{k/2-s} \sum_{d|(n,r)} d^{2s-k+1} \psi\Big(\frac{n}{d}\Big) \chi\Big(\frac{rn}{d^2}\Big).
\end{aligned}$$

Passing to the third line, we applied (2.5). Multiplying by (7.7) gives the result. □

Although we computed the orbital integral globally, it may be of interest to know the value of the local orbital integrals, for example if one wishes to compute the analogous trace formula over a number field, or use a test function which differs from ours at a finite number of places. Letting

$$I_1(s)_p = \int_{\mathbf{Q}_p^*} \int_{\mathbf{Q}_p} f_p\left(\begin{pmatrix} y & xy \\ 0 & 1 \end{pmatrix}\right) \overline{\theta_p(rx)} \chi_p(y) |y|_p^{k/2-s}\, dx\, d^*y,$$

by calculations very similar to the above, we find, for $r \in \mathbf{Z}_p$, that

$$I_1(s)_p = \begin{cases} \chi_p(r) & (p|D), \\ \nu(p^{N_p}) & (p|N), \\ (p^{n_p})^{k/2-s} \displaystyle\sum_{d_p=0}^{\min(r_p,n_p)} (p^{d_p})^{2s-k+1} \psi_p\Big(\frac{p^{d_p}}{p^{n_p}}\Big) \chi_p\Big(\frac{p^{2d_p}}{p^{n_p}}\Big) & (p|n), \\ 1 & (p \nmid nND). \end{cases} \tag{7.8}$$

Proposition 7.3. *When $\delta = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$, the integral*

$$I_\delta(s) = \int_{\mathbf{A}^*} \int_{\mathbf{A}} f\left(\begin{pmatrix} 0 & -y \\ 1 & x \end{pmatrix}\right) \overline{\theta(rx)}\, dx\, \chi^*(y) |y|^{k/2-s}\, d^*y$$

converges absolutely on $1 < \mathrm{Re}(s) < k$, and for such s it vanishes unless $N = 1$. When $N = 1$ (so k is even by (2.6)),

$$I_\delta(s) = \frac{n^{1-k/2}}{e^{2\pi r}} \frac{2^{k-1}(2\pi rn)^{s-1}}{(k-2)!} \Gamma(k-s) \frac{i^k}{D^{2s-k}} \frac{\tau(\chi)^2}{D} \sum_{d|(r,n)} d^{k-2s+1} \overline{\chi\Big(\frac{rn}{d^2}\Big)}. \tag{7.9}$$

Remark 7.4. Comparing with the identity term (7.6) when $N = 1$,

$$\frac{i^k}{D^{2s-k}} \frac{\tau(\chi)^2}{D} I_1(k-s, \overline{\chi}) = I_\delta(s, \chi),$$

mirroring the functional equation (1.2) on the spectral side.

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Proof. The absolute convergence was proven in Proposition 7.1. By (7.4) and the proof of [KL2, Proposition 3.5],

$$I_\delta(s)_\infty = \frac{2^{k-1}(2\pi r)^{s-1}}{e^{2\pi r}(k-2)!\, i^k}\Gamma(k-s). \tag{7.10}$$

Now consider the finite part (7.5):

$$I_\delta(s)_{\mathrm{fin}} = \sum_{\ell\in\mathbf{Q}^+}\Big(\frac{n}{\ell^2}\Big)^{s-k/2}\int_{\widehat{\mathbf{Z}}^*}\int_{\mathbf{A}_{\mathrm{fin}}} f_{\mathrm{fin}}\left(\begin{pmatrix}0 & -\frac{nu}{\ell}\\ \ell & x\ell\end{pmatrix}\right)\overline{\theta_{\mathrm{fin}}(rx)}\,dx\,\chi^*(u)\,d^*u.$$

Because the above matrix has determinant $nu\in n\widehat{\mathbf{Z}}^*$, we see from (5.4) that

$$\begin{pmatrix}0 & -\frac{nu_p}{\ell}\\ \ell & \ell x_p\end{pmatrix}\in M(n,N)_p \quad \text{for all } p\nmid D \tag{7.11}$$

when $\ell x_p\in\mathbf{Z}_p$. Likewise, we can assume that for some $m\in(\mathbf{Z}/D\mathbf{Z})^*$,

$$\begin{pmatrix}1 & \frac{m}{D}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -\frac{nu}{\ell}\\ \ell & x\ell\end{pmatrix} = \begin{pmatrix}\frac{\ell m}{D} & -\frac{nu}{\ell}+\frac{m\ell x}{D}\\ \ell & \ell x\end{pmatrix}\in M(n,N). \tag{7.12}$$

The latter implies that $x\ell\in\widehat{\mathbf{Z}}$ and $\ell\in N\mathbf{Z}^+$. If $p|N$, then, by considering the upper right entry of (7.11), we have $\mathrm{ord}_p(n)\ge\mathrm{ord}_p(\ell)\ge\mathrm{ord}_p(N)>0$. This contradicts $(n,N)=1$, and therefore we may assume that $N=1$. From the upper left entry of (7.12), we see that $D|\ell$. Write $\ell=Dd$ for $d\in\mathbf{Z}^+$. Replacing x by $x/\ell=x/Dd$, the measure is scaled by $|Dd|_{\mathrm{fin}}^{-1}=Dd$, so $I_\delta(s)_{\mathrm{fin}}$ is equal to

$$\sum_{d\in\mathbf{Z}^+}\frac{n^{s-k/2}}{(Dd)^{2s-k-1}}$$
$$\times\sum_{m\in(\mathbf{Z}/D\mathbf{Z})^*}\int_{\widehat{\mathbf{Z}}^*}\int_{\widehat{\mathbf{Z}}} f_m\left(\begin{pmatrix}1 & -\frac{m}{D}\\ 0 & 1\end{pmatrix}\begin{pmatrix}md & \frac{-nu}{Dd}+\frac{mx}{D}\\ Dd & x\end{pmatrix}\right)\overline{\theta_{\mathrm{fin}}\Big(\frac{rx}{Dd}\Big)}\,dx\,\chi^*(u)\,d^*u.$$

From the upper right entry of (7.12), $-nu/Dd+mx/D\in\widehat{\mathbf{Z}}$. Since $x\in\widehat{\mathbf{Z}}$, this means that

$$mdx\in md\widehat{\mathbf{Z}}\cap(nu+Dd\widehat{\mathbf{Z}}). \tag{7.13}$$

Generally, it is not hard to show that for any $h,j,k\in\widehat{\mathbf{Z}}$,

$$h\widehat{\mathbf{Z}}\cap(j+k\widehat{\mathbf{Z}}) = \begin{cases} hc+\dfrac{hk}{\gcd(h,k)}\widehat{\mathbf{Z}} & \text{if } \gcd(h,k)|j,\\ \emptyset & \text{if } \gcd(h,k)\nmid j,\end{cases} \tag{7.14}$$

where $c\in\mathbf{Z}$ is any solution to $hc\equiv j \bmod k\widehat{\mathbf{Z}}$. Applying this to (7.13), we have $\gcd(h,k)=\gcd(md,Dd)=d$, so the set in (7.13) is nonempty if and only if $d|n$. Assuming that this holds, the range of x is determined by

$$x\in c_m+D\widehat{\mathbf{Z}},$$

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where c_m is any positive integer satisfying

$$mdc_m \equiv nu \bmod dD\widehat{\mathbf{Z}}. \tag{7.15}$$

For such x, the value of f_m is identically equal to $\overline{\chi(m)}/\tau(\overline{\chi})$, since $N = 1$ (and so $\psi = 1$). Replace x by $c_m + Dx$. This changes the measure by a factor of $|D|_{\mathrm{fin}} = D^{-1}$, and

$$I_\delta(s)_{\mathrm{fin}} = \frac{n^{s-k/2}}{\tau(\overline{\chi})D^{2s-k}} \sum_{d|n} d^{k-2s+1} \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \int_{\widehat{\mathbf{Z}}} \overline{\theta_{\mathrm{fin}}\Big(\frac{r(c_m + Dx)}{Dd}\Big)}\, dx\, \chi^*(u)\, d^*u.$$

The inner integral $\int_{\widehat{\mathbf{Z}}} \overline{\theta_{\mathrm{fin}}(rx/d)}\, dx$ is equal to 1 if $d|r$, and 0 otherwise. Therefore, the above is

$$= \frac{n^{s-k/2}}{\tau(\overline{\chi})D^{2s-k}} \sum_{d|(r,n)} d^{k-2s+1} \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \overline{\theta_{\mathrm{fin}}\Big(\frac{rc_m}{Dd}\Big)}\chi^*(u)\, d^*u.$$

Since $d|(r,n)$, (7.15) is equivalent to $c_m \equiv \overline{m}(n/d)u \bmod D\widehat{\mathbf{Z}}$, where $m\overline{m} \equiv 1 \bmod D$. Using this along with (2.5),

$$\begin{aligned}
I_\delta(s)_{\mathrm{fin}} &= \frac{n^{s-k/2}}{\tau(\overline{\chi})D^{2s-k}} \sum_{d|(r,n)} d^{k-2s+1} \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \theta_{\mathrm{fin}}\Big(\frac{-(\frac{r}{d})(\frac{n}{d})\overline{m}u}{D}\Big)\chi^*(u)\, d^*u \\
&= \frac{n^{s-k/2}}{\tau(\overline{\chi})D^{2s-k}} \sum_{d|(r,n)} d^{k-2s+1} \sum_m \overline{\chi(m)}\chi(-1)\overline{\chi\Big(\frac{rn}{d^2}\Big)}\chi(m)\frac{\tau(\chi)}{\varphi(D)} \\
&= \frac{n^{s-k/2}}{D^{2s-k}} \frac{\tau(\chi)}{\chi(-1)\tau(\overline{\chi})} \sum_{d|(r,n)} d^{k-2s+1}\overline{\chi\Big(\frac{rn}{d^2}\Big)}.
\end{aligned} \tag{7.16}$$

Since χ is primitive, $\tau(\chi)\chi(-1)\tau(\overline{\chi}) = \tau(\chi)\overline{\tau(\chi)} = D$. Therefore, $\tau(\chi)/\chi(-1)\tau(\overline{\chi}) = \tau(\chi)^2/D$. Using this and multiplying the above by (7.10), equation (7.9) follows. □

We state here the value of the local orbital integrals

$$I_\delta(s)_p = \int_{\mathbf{Q}_p^*} \int_{\mathbf{Q}_p} f_p\Big(\begin{pmatrix} 0 & -y \\ 1 & x \end{pmatrix}\Big) \overline{\theta_p(rx)}\chi_p(y)|y|_p^{k/2-s}\, dx\, d^*y.$$

By local calculations similar to the above, we find, for $\delta = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, that

$$I_\delta(s)_p = \begin{cases} (p^{D_p})^{k-2s}\overline{\psi_p(D)}\,\overline{\chi_p(D^2)}\,\overline{\chi_p(-r)}\dfrac{\tau(\chi)_p}{\tau(\overline{\chi})_p} & (p|D), \\ 0 & (p|N), \\ (p^{n_p})^{s-k/2} \displaystyle\sum_{d_p=0}^{\min(r_p,n_p)} (p^{d_p})^{k-2s+1}\overline{\psi_p(p^{d_p})\chi_p\Big(\frac{p^{n_p}}{p^{2d_p}}\Big)} & (p|n), \\ 1 & (p \nmid nND). \end{cases} \tag{7.17}$$

Here, recall that $\tau(\chi)_p = \chi_p(D/p^{D_p})\tau(\chi_p)$ as in (3.9). Using (3.10), it is straightforward to show that the product of the above over all p agrees with (7.16) when $N = 1$.

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8. Computation of $I_{\delta_t}(s)$

In this section, we will prove the following proposition.

PROPOSITION 8.1. *When* $\delta_t = \left(\begin{smallmatrix} t & -1 \\ 1 & 0 \end{smallmatrix}\right)$, *the integral*

$$I_{\delta_t}(s) = \int_{\mathbf{A}^*} \int_{\mathbf{A}} f\left(\begin{pmatrix} yt & yxt - y \\ 1 & x \end{pmatrix}\right) \overline{\theta(rx)}\, dx\, \chi^*(y)|y|^{k/2-s}\, d^*y$$

converges absolutely on the strip $0 < \sigma < k$, *where* $\sigma = \mathrm{Re}(s)$. *The integrand vanishes unless* $t = Nb/nD$ *for* $b \in \mathbf{Z} - \{0\}$. *When* $1 < \sigma < k - 1$, *the sum* $\sum_{t \in \mathbf{Q}^*} I_{\delta_t}(s)$ *is absolutely convergent, and* $E := (e^{2\pi r}/\nu(N) n^{1-k/2}) \sum I_{\delta_t}(s)$ *is equal to*

$$\frac{(4\pi rn)^{k-1} \varphi(D) \psi(nD) e^{i\pi s/2}}{N^s D^{s-k} (k-2)!\, \tau(\overline{\chi})} \times \sum_{\substack{a \neq 0, d > 0 \text{ sat. } (8.2), \\ \gcd(a, Nd^{(D)}) | \gcd(r,n)}} \frac{a^{s-k} \gcd(a, Nd^{(D)})}{d^s \psi(a) e^{(2\pi i r \ell / a d_D)}} J_\chi(a,d)\, {}_1f_1\left(s; k; \frac{-2\pi i r n D}{N a d}\right),$$

where $a^s = e^{-i\pi s}|a|^s$ *if* $a < 0$, $d^{(D)} = \prod_{p \nmid D} p^{d_p}$ *is the prime-to-D part of d, and similarly for* $d = d^{(D)} d_D$, ℓ *is any integer satisfying* $Nd^{(D)}\ell \equiv -nD \bmod ad_D$, J_χ *is a product of certain explicit local factors of absolute value* ≤ 1 *given in* (8.13),

$${}_1f_1(s,k;w) = \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} {}_1\mathrm{F}_1(s;k;w) = \int_0^1 e^{wx} x^{s-1} (1-x)^{k-s-1}\, dx \tag{8.1}$$

for $\mathrm{Re}(k) > \mathrm{Re}(s) > 0$, *and, writing* a_p, d_p *for the p-adic valuations of* a, d,

$$p | D \implies \begin{cases} a_p = D_p & \text{if } d_p > D_p, \\ a_p \ge D_p & \text{if } d_p = D_p, \\ a_p = d_p & \text{if } 0 \le d_p < D_p. \end{cases} \tag{8.2}$$

REMARK 8.2. We give an expression for $I_{\delta_t}(s)$ in (8.16) below. As in [KL2], this can be used in principle to compute the sum over t to any level of precision.

The absolute convergence was proven in Proposition 7.1. To begin the computation, write $\delta = \delta_t$, and consider the finite part given by (7.5):

$$I_\delta(s)_{\mathrm{fin}} = \sum_{\ell \in \mathbf{Q}^+} \left(\frac{n}{\ell^2}\right)^{s-k/2} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\mathrm{fin}}} f_{\mathrm{fin}}\left(\begin{pmatrix} \frac{nut}{\ell} & \frac{nutx}{\ell} - \frac{nu}{\ell} \\ \ell & \ell x \end{pmatrix}\right) \overline{\theta_{\mathrm{fin}}(rx)}\, dx\, \chi^*(u)\, d^*u.$$

We will show that this vanishes unless $t \in (N/nD)\mathbf{Z}$. In anticipation of this, write $t = Nb/nD$, where (for now) $b \in \mathbf{Q}^*$. Since the determinant of the matrix is $nu \in n\widehat{\mathbf{Z}}^*$, by (5.4) the integrand vanishes unless $\ell \in N\mathbf{Z}^+$ and $\ell x \in \widehat{\mathbf{Z}}$. Therefore, we shall set $\ell = Nd$, and replace x by $\ell^{-1}x = x/Nd$, so that dx becomes $d(x/Nd) = |Nd|_{\mathrm{fin}}^{-1} dx = Nd \cdot dx$. The above then becomes

$$\sum_{d \in \mathbf{Z}^+} \frac{n^{s-k/2} Nd}{(Nd)^{2s-k}} \int_{\widehat{\mathbf{Z}}^*} \int_{\widehat{\mathbf{Z}}} f_{\mathrm{fin}}\left(\begin{pmatrix} \frac{ub}{dD} & \frac{ubx}{Nd^2 D} - \frac{nu}{Nd} \\ Nd & x \end{pmatrix}\right) \overline{\theta_{\mathrm{fin}}\left(\frac{rx}{Nd}\right)}\, dx\, \chi^*(u)\, d^*u. \tag{8.3}$$

We will show that the integrand vanishes unless $b \in \mathbf{Z}$ and $d|b$. We now work locally. The pth factor of the double integral is

$$\int_{\mathbf{Z}_p^*}\int_{\mathbf{Z}_p} f_p\left(\begin{pmatrix} \frac{ub}{dD} & \frac{ubx}{Nd^2D} - \frac{nu}{Nd} \\ Nd & x \end{pmatrix}\right)\overline{\theta_p\left(\frac{rx}{Nd}\right)}\,dx\,\chi_p(u)\,d^*u. \tag{8.4}$$

8.1. Local computation at $p|D$. Suppose first that $p|D$. Then N is a unit, so, replacing u by Nu, (8.4) becomes

$$\int_{\mathbf{Z}_p^*}\int_{\mathbf{Z}_p} f_p\left(\begin{pmatrix} \frac{Nub}{dD} & \frac{ubx}{d^2D} - \frac{nu}{d} \\ Nd & x \end{pmatrix}\right)\overline{\theta_p\left(\frac{rx}{Nd}\right)}\,dx\,\chi_p(Nu)\,d^*u. \tag{8.5}$$

Recall that $f_p = \sum_{m\in(\mathbf{Z}_p/D\mathbf{Z}_p)^*} f_{p,m}$, where $f_{p,m} = f_{p,m}^{\chi}$ is given in (3.12). Fix m and consider

$$f_{p,m}\left(\begin{pmatrix} \frac{Nub}{dD} & \frac{ubx}{d^2D} - \frac{nu}{d} \\ Nd & x \end{pmatrix}\right) = f_{p,m}\left(\begin{pmatrix} 1 & -\frac{m}{D} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{Nub}{dD} + \frac{Ndm}{D} & \frac{ubx}{d^2D} - \frac{nu}{d} + \frac{mx}{D} \\ Nd & x \end{pmatrix}\right).$$

By (3.11) and the fact that the determinant of the rightmost matrix is $nNu \in \mathbf{Z}_p^*$, this is nonzero if and only if the rightmost matrix belongs to K_p or, equivalently,

$$\text{(i)}\quad \frac{ub}{dD} + \frac{dm}{D} \in \mathbf{Z}_p,$$
$$\text{(ii)}\quad \frac{x}{d}\left(\frac{ub}{dD} + \frac{dm}{D}\right) - \frac{nu}{d} \in \mathbf{Z}_p.$$

We assume henceforth that these conditions hold. Notice that if $p \nmid d$, the first condition already implies the second. On the other hand, since $x \in \mathbf{Z}_p$,

$$p|d, \text{(i)}, \text{(ii)} \implies \begin{array}{ll} \text{(iii)} & x \in \mathbf{Z}_p^*, \\ \text{(iv)} & \left(\frac{ub}{dD} + \frac{dm}{D}\right) \in \mathbf{Z}_p^*. \end{array}$$

Letting $D_p = \mathrm{ord}_p(D)$, and similarly for b_p, d_p, we find by condition (i) (if $d_p = 0$) and condition (iv) (if $d_p > 0$) that

$$p|D \implies \begin{cases} b_p = d_p + D_p & \text{if } d_p > D_p, \\ b_p \geq 2D_p & \text{if } d_p = D_p, \\ b_p = 2d_p & \text{if } 0 \leq d_p < D_p. \end{cases} \tag{8.6}$$

Suppose first that $p \nmid d$, so that by (8.6), $d_p = b_p = 0$. Then condition (i) is equivalent to

$$m \equiv \frac{-bu}{d^2} \bmod D\mathbf{Z}_p,$$

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d, b, u, m being units. So, given u, there is exactly one m for which the above condition holds and, by (3.12), the inner integral of (8.5) is equal to

$$\int_{\mathbf{Z}_p} \frac{\overline{\chi_p(m)}}{\tau(\overline{\chi})_p} \overline{\theta_p\Big(\frac{rx}{Nd}\Big)}\, dx = \frac{\overline{\chi_p(\frac{-b}{d^2})}}{\tau(\overline{\chi}_p)} \overline{\chi_p(u)},$$

since $\theta_p(rx/Nd) = 1$. Therefore, the double integral (8.4) is equal to

$$\chi_p(N) \frac{\chi_p(\frac{-d}{b/d})}{\tau(\overline{\chi})_p} \int_{\mathbf{Z}_p^*} \overline{\chi_p(u)} \chi_p(u)\, d^*u = \frac{\chi_p(\frac{-Nd}{b/d})}{\tau(\overline{\chi})_p}. \tag{8.7}$$

Now suppose that $p|d$. In view of (ii) and (iv),

$$x \in \frac{Dnu}{u\frac{b}{d} + dm} + d\mathbf{Z}_p \subset \mathbf{Z}_p^*.$$

This is the only condition on x required for the $f_{p,m}$-term to be nonzero. Make the substitution $x = Dnu/(ub/d + dm) + d \cdot w$, so that $dx = |d|_p dw = |Nd|_p dw$. The value of $f_{p,m}$ is $\overline{\chi_p(m)}/\tau(\overline{\chi})_p$, so, assuming that (iv) holds, the inner integral in (8.5) is equal to

$$|Nd|_p \sum_m \frac{\overline{\chi_p(m)}}{\tau(\overline{\chi})_p} \overline{\theta_p\Big(\frac{rDnu}{Nub + Nd^2m}\Big)} \int_{\mathbf{Z}_p} \overline{\theta_p\Big(\frac{rw}{N}\Big)}\, dw.$$

The latter integral has value 1, since $r \in \mathbf{Z}^+$ and N is a unit. The variable u ranges through the set $U_{b,d,m} = (-d^2m/b + (dD/b)\mathbf{Z}_p^*) \cap \mathbf{Z}_p^*$ determined by condition (iv) above. By considering the possibilities for $d_p > 0$ listed in (8.6), we find easily that

$$U_{b,d,m} = \begin{cases} \mathbf{Z}_p^* & \text{if } d_p > D_p \text{ (so } b_p = d_p + D_p),\\ -\dfrac{d^2m}{b} + \dfrac{D}{d}\mathbf{Z}_p^* & \text{if } 0 < d_p < D_p \text{ (so } b_p = 2d_p),\\ \mathbf{Z}_p^* & \text{if } d_p = D_p \text{ and } b_p > 2D_p,\\ \displaystyle\bigcup_{\substack{a \in (\mathbf{Z}/p\mathbf{Z})^*,\\ a \not\equiv -(d^2m/b) \bmod p}} (a + p\mathbf{Z}_p) & \text{if } d_p = D_p \text{ and } b_p = 2D_p. \end{cases}$$

The double integral (8.4) is equal to

$$\frac{|Nd|_p}{\tau(\overline{\chi})_p} \sum_{m \in (\mathbf{Z}_p/D\mathbf{Z}_p)^*} \overline{\chi_p(m)} \int_{U_{b,d,m}} \chi_p(Nu) \overline{\theta_p\Big(\frac{rDnu}{Nub + Nd^2m}\Big)}\, d^*u.$$

Noting that $U_{b,d,m} = mU_{b,d,1}$, we can replace u by mu and integrate over $U_{b,d,1}$. This has the effect of cancelling every m, so that the above is

$$\frac{|Nd|_p \varphi_p(D)}{\tau(\overline{\chi})_p} \int_{U_{b,d,1}} \chi_p(Nu) \overline{\theta_p\Big(\frac{rDnu}{Nub + Nd^2}\Big)}\, d^*u. \tag{8.8}$$

We leave this as something that could be computed given χ_p, if desired. For our purposes, it will be enough to bound the integral trivially by 1. (We do not think that a more careful treatment of the integral can yield enough power saving in D to enable the type of hybrid subconvexity bound mentioned in the introduction.)

8.2. Local computation at $p \nmid D$. Now we suppose that $p \nmid D$. In this case, χ_p is unramified, so that (8.4) is equal to

$$\int_{\mathbf{Z}_p^*}\int_{\mathbf{Z}_p} f_p\left(\begin{pmatrix} \frac{ub}{dD} & \frac{ubx}{Nd^2D} - \frac{nu}{Nd} \\ Nd & x \end{pmatrix}\right)\overline{\theta_p\left(\frac{rx}{Nd}\right)}\,dx\,d^*u. \tag{8.9}$$

Since the support of f_p is $Z(\mathbf{Q}_p)M(n,N)_p$ and the determinant of the above matrix is $nu \in n\mathbf{Z}_p^*$, the integrand is nonzero precisely when

$$\text{(i)}\quad \frac{ub}{dD} \in \mathbf{Z}_p,$$
$$\text{(ii)}\quad \frac{x}{Nd}\left(\frac{ub}{dD}\right) - \frac{nu}{Nd} \in \mathbf{Z}_p.$$

Both conditions are in fact independent of u. By (i), we see that $0 \le d_p \le b_p$ since $p \nmid D$. Together with (8.6), this proves our assertion that $I_\delta(s)$ vanishes unless $b \in \mathbf{Z}$, and that the sum in (8.3) can be taken just over $d|b$. Since $u, D \in \mathbf{Z}_p^*$, condition (ii) is equivalent to

$$\frac{b}{d}x \in (Dn + Nd\mathbf{Z}_p) \cap \frac{b}{d}\mathbf{Z}_p. \tag{8.10}$$

(If $p|N$, this is possible only if $d_p = b_p$.) Applying the local analog of (7.14) to (8.10), and then dividing by b/d,

$$x \in \begin{cases} c + \dfrac{Nd}{\gcd(b/d, Nd)}\mathbf{Z}_p & \text{if } \gcd_p(b/d, Nd)|Dn, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\gcd_p$ denotes the p-part of the gcd, and $c \in \mathbf{Z}$ is given by

$$\frac{b}{d}c \equiv Dn \bmod Nd\mathbf{Z}_p.$$

We shall specify c further as follows, so that the above holds simultaneously for all $p \nmid D$. Write $d = d^{(D)}d_D$, where $(D, d^{(D)}) = 1$ and $d_D = \prod_{p|D} p^{d_p}$. Then we take $c \in \mathbf{Z}$, so that

$$\begin{cases} \dfrac{b}{d}c \equiv Dn \bmod Nd^{(D)}\mathbf{Z}, \\ c \equiv 0 \bmod d_D\mathbf{Z}. \end{cases} \tag{8.11}$$

It is not hard to see that such c exists under the hypothesis that $\gcd_p(b/d, Nd)|Dn$ for all $p \nmid D$. Indeed, $\prod_{p\nmid D}\gcd_p(b/d, Nd) = \gcd(b/d, Nd^{(D)})|Dn$, which implies the existence of an integer c satisfying the first congruence. If necessary, we can multiply c by $d_D\overline{d_D} \equiv 1 \bmod Nd^{(D)}$ to further ensure that $c \in d_D\mathbf{Z}$.

The first congruence in (8.11) implies that b/d is relatively prime to N. Therefore, $\psi_p(x) = \psi_p(c)$. Since $\mathrm{meas}(\mathbf{Z}_p^*) = 1$ and everything is independent of $u \in \mathbf{Z}_p^*$, the double integral (8.9) is equal to

$$\nu_p(N)\psi_p((c_N)_p)\int_{c+(Nd/\gcd(b/d,Nd))\mathbf{Z}_p}\overline{\theta_p\left(\frac{rx}{Nd}\right)}\,dx,$$

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where we used the formula (4.1) for f_p. Now let $x = c + (Nd/\gcd(b/d, Nd))w$. Then the above is

$$= \nu_p(N)\psi_p((c_N)_p)\left|\frac{Nd}{\gcd(b/d, Nd)}\right|_p \overline{\theta_p\Big(\frac{rc}{Nd}\Big)} \int_{\mathbf{Z}_p} \overline{\theta_p\Big(\frac{rw}{\gcd(b/d, Nd)}\Big)}\,dw. \tag{8.12}$$

The integral is nonzero (and hence equal to 1) if and only if $\gcd_p(b/d, Nd)|r$.

8.3. The finite part. Multiply the local factors (8.8) (respectively (8.7)) and (8.12), together with the coefficient of the double integral in (8.3). We set

$$J_\chi\Big(\frac{b}{d}, d\Big) = \prod_{p|D} J_p\Big(\frac{b}{d}, d\Big), \tag{8.13}$$

where $J_p(b/d, d)$ denotes the integral in (8.8) if $d_p > 0$ (respectively the quantity $\chi_p(-Nd/b/d)/\varphi_p(D)$ if $d_p = 0$). We find that

$$I_{\delta_t}(s)_{\mathrm{fin}} = \sum_{\substack{d|b \text{ satisfying } (8.6),\\ \gcd(b/d, Nd^{(D)})|(r,n)}} \frac{n^{s-k/2}Nd}{(Nd)^{2s-k}} J_\chi\Big(\frac{b}{d}, d\Big) \prod_{p|D} \frac{|Nd|_p\varphi_p(D)}{\tau(\overline{\chi})_p}$$
$$\times \prod_{p\nmid D} \nu_p(N)\psi_p((c_N)_p)\left|\frac{Nd}{\gcd(\frac{b}{d}, Nd)}\right|_p \overline{\theta_p\Big(\frac{rc}{Nd}\Big)}.$$

Here, $d^{(D)} = \prod_{p\nmid D} p^{d_p}$, as before. We can make a few simplifications. First,

$$\prod_{p\nmid D} \psi_p((c_N)_p) = \psi^*(c_N) = \psi(c) = \frac{\psi(nD)}{\psi(b/d)},$$

since $(b/d)c \equiv nD \bmod N$ and $(b/d, N) = 1$ by (8.10). By the second congruence of (8.11), namely $d_D|c$, we have $\theta_p(rc/Nd) = 1$ for all $p|D$. Hence,

$$\prod_{p\nmid D} \overline{\theta_p\Big(\frac{rc}{Nd}\Big)} = \overline{\theta_{\mathrm{fin}}\Big(\frac{rc}{Nd}\Big)} = \theta_\infty\Big(\frac{rc}{Nd}\Big) = e^{-2\pi irc/Nd}.$$

Therefore,

$$I_{\delta_t}(s)_{\mathrm{fin}} = \frac{n^{s-k/2}\varphi(D)\nu(N)}{N^{2s-k}\tau(\overline{\chi})} \sum_{\substack{d|b \text{ satisfying } (8.6),\\ \gcd(b/d, Nd^{(D)})|(r,n)}} \frac{\psi(nD)}{\psi(\frac{b}{d})} \frac{\gcd(\frac{b}{d}, Nd^{(D)})}{d^{2s-k}e^{2\pi irc/Nd}} J_\chi\Big(\frac{b}{d}, d\Big). \tag{8.14}$$

8.4. Archimedean integral and global expression. Finally, we consider the archimedean integral $I_{\delta_t}(s)_\infty$. By (7.4) and the proof of [KL2, Proposition 3.7],

$$I_{\delta_t}(s)_\infty = \overline{\frac{(4\pi r)^{k-1}t^{\overline{s}-k}}{(k-2)!\,e^{2\pi r}} e^{-i\pi\overline{s}/2}e^{-2\pi ir/t}\,{}_1f_1(\overline{s}; k; 2\pi ir/t)},$$

where $t^s = e^{i\pi s}|t|^s$ if $t < 0$. Therefore,

$$I_{\delta_t}(s)_\infty = \frac{(4\pi r)^{k-1} t^{s-k} e^{i\pi s/2} e^{2\pi ir/t}}{(k-2)!\, e^{2\pi r}} {}_1f_1(s;k;-2\pi ir/t),$$

where now $t^s = e^{-i\pi s}|t|^s$ if $t < 0$. By the discussion above, we can take $t = Nb/nD$, so that

$$I_{\delta_t}(s)_\infty = \frac{(4\pi r)^{k-1} N^{s-k} e^{i\pi s/2}}{(k-2)!\, e^{2\pi r} n^{s-k} D^{s-k}} b^{s-k} e^{2\pi irnD/Nb} {}_1f_1\Big(s;k;\frac{-2\pi irnD}{Nb}\Big). \tag{8.15}$$

When we multiply by the finite part (8.14), the terms $e^{2\pi irnD/Nb}$ and $e^{-2\pi irc/Nd}$ combine as follows. By (8.11), we can write $(b/d)c = nD + \ell Nd^{(D)}$, where $\ell \in \mathbf{Z}$ is an integer satisfying

$$\ell Nd^{(D)} \equiv -nD \bmod \Big(\frac{b}{d}\Big)d_D.$$

Conversely, any ℓ satisfying the above determines an integer c satisfying (8.11). Then

$$e^{2\pi irnD/Nb} e^{-2\pi irc/Nd} = e^{2\pi ir(nD-(b/d)c)/Nb} = e^{-2\pi ir\ell Nd^{(D)}/Nb} = e^{-2\pi ir\ell/(b/d)d_D}.$$

Multiplying (8.15) by the finite part (8.14), we find, for $t = Nb/nD$, that

$$\begin{aligned} I_{\delta_t}(s) = {} & \frac{(4\pi r)^{k-1} N^{s-k} e^{i\pi s/2}}{(k-2)!\, e^{2\pi r} n^{s-k} D^{s-k}} b^{s-k} {}_1f_1\Big(s;k;\frac{-2\pi irnD}{Nb}\Big) \\ & \times \frac{n^{s-k/2}\varphi(D)\nu(N)}{N^{2s-k}\tau(\overline{\chi})} \sum_{\substack{d\mid b \text{ sat. (8.6)}, \\ \gcd(b/d, Nd^{(D)}) \mid (r,n)}} \frac{\psi(nD)\gcd(\frac{b}{d}, Nd^{(D)})}{\psi(\frac{b}{d}) d^{2s-k} e^{2\pi ir\ell/(b/d)d_D}} J_\chi\Big(\frac{b}{d}, d\Big). \end{aligned} \tag{8.16}$$

Writing $b = ad$, the condition (8.6) becomes (8.2). Summing over t, we see that $e^{2\pi r}/\nu(N)n^{1-k/2}\sum_{t\in\mathbf{Q}^*} I_{\delta_t}(s)$ is equal to

$$\begin{aligned} & \frac{(4\pi rn)^{k-1}\varphi(D)\psi(nD)e^{i\pi s/2}}{N^s D^{s-k}(k-2)!\,\tau(\overline{\chi})} \\ & \quad \times \sum_{\substack{a\neq 0, d>0 \text{ sat. (8.2)}, \\ \gcd(a, Nd^{(D)})\mid\gcd(r,n)}} \frac{a^{s-k}\gcd(a, Nd^{(D)})}{d^s\psi(a)e^{2\pi ir\ell/ad_D}} J_\chi(a,d) {}_1f_1\Big(s;k;\frac{-2\pi irnD}{Nad}\Big), \end{aligned}$$

where $a^s = e^{-i\pi s}|a|^s$ if $a < 0$. This completes the proof of Proposition 8.1.

9. Asymptotics

Grouping a with $-a$, we rewrite the above sum as follows:

$$\begin{aligned} \sum_{\substack{a,d>0 \text{ sat. (8.2)}, \\ \gcd(a, Nd^{(D)})\mid\gcd(r,n)}} & \Big[\frac{a^{s-k}}{\psi(a)e^{2\pi ir\ell/ad_D}} J_\chi(a,d) {}_1f_1\Big(s;k;\frac{-2\pi irnD}{Nad}\Big) \\ & + \frac{e^{-i\pi s}(-1)^k a^{s-k}}{\psi(-1)\psi(a)e^{(-2\pi ir\ell)/ad_D}} J_\chi(-a,d) {}_1f_1\Big(s;k;\frac{2\pi irnD}{Nad}\Big)\Big] \frac{\gcd(a, Nd^{(D)})}{d^s}. \end{aligned}$$

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From the integral representation (8.1),

$$|{}_1f_1(s,k,2\pi iw)| \le B(\sigma,k-\sigma) \le 1 \tag{9.1}$$

when $1 \le \sigma \le k-1$, where $B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\,du = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function. Because $|J_\chi(a,d)| \le 1$, the absolute value of the above is

$$\le \gcd(r,n)B(\sigma,k-\sigma)(1+e^{\pi\tau})\sum_{a,d>0} a^{\sigma-k}d^{-\sigma} \quad (s=\sigma+i\tau).$$

Using $|e^{i\pi s/2}|(1+e^{\pi\tau}) = 2\cosh(\pi\tau/2)$, we obtain the following proposition.

Proposition 9.1. *Write $s=\sigma+i\tau$ for $1<\sigma<k-1$. Then the term E given in Proposition 8.1 satisfies the bound*

$$|E| \le \frac{(4\pi rn)^{k-1}D^{k-\sigma-1/2}\varphi(D)\gcd(r,n)B(\sigma,k-\sigma)}{N^\sigma(k-2)!}2\cosh\Big(\frac{\pi\tau}{2}\Big)\zeta(k-\sigma)\zeta(\sigma).$$

Theorem 1.1 now follows immediately. In order to prove Corollary 1.2, we must show that the quotient $Q=E/F$ has the limit 0 as $N+k\to\infty$, where F is the first geometric term of (1.4), and E is the error term of (1.4) discussed above. We take $k\ge 3$, $N>1$, and $\gcd(n,r)=1$, so, for $(k-1)/2<\sigma<(k+1)/2$, we have $|F| = 2^{k-1}(2\pi rn)^{k-\sigma-1}|\Gamma(s)|/(k-2)!$. Thus, by the above proposition and (9.1),

$$|Q| = \left|\frac{E}{F}\right| \ll_{D,\tau} \frac{D^{k-\sigma}(2\pi rn)^\sigma}{N^\sigma|\Gamma(s)|}\zeta(k-\sigma)\zeta(\sigma). \tag{9.2}$$

We write $\sigma = k/2+\delta$ for $|\delta|<\frac{1}{2}$. Then each zeta factor is bounded by the constant $\zeta(\frac{3}{2}-|\delta|)$. By Stirling's approximation [AS, 6.1.39],

$$\Gamma(s)^{-1} = \Gamma\Big(\frac{k}{2}+\delta+i\tau\Big)^{-1} \sim \frac{e^{k/2}}{\sqrt{2\pi}(k/2)^{k/2+\delta+i\tau-1/2}}$$

as $k\to\infty$. With (9.2), this shows that

$$|Q| \ll \frac{(4D\pi rne)^{k/2}}{N^{(k-1)/2}k^{k/2-1}},$$

where the implied constant depends on δ, D, r, n, τ. This clearly goes to 0 as $N+k\to\infty$.

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