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PRE-WEIGHTED HOMOGENEOUS MAP GERMS
FINITE DETERMINACY AND TOPOLOGICAL TRIVIALITY

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Abstract. In this paper we introduce the notion of G-pre-weighted homogeneous map germ, (G is one of Mather's groups $\mathcal{A}$ or $\mathcal{K}$) and show that any G-pre-weighted homogeneous map germ is G-finitely determined. We also give an explicit "order", based on the Newton polyhedron of a pre-weighted homogeneous germ of function, such that the topological structure is preserved after perturbations by terms of higher order.

The characterization of finite determinacy of map germs and of topological triviality of families of map germs are fundamental subjects in singularity theory.

An analytic map germ $g: K^n, 0 \to K^p, 0$, ($K = \mathbb{R}$ or $\mathbb{C}$), is G-semi-weighted homogeneous, (G is one of the Mather's groups $\mathcal{A}$ or $\mathcal{K}$), if it can be decomposed in a form $g = f + k$, where f is weighted homogeneous, G-finitely determined and k is a map germ with weighted filtration higher than the weighted degree of f.

It is well known that any G-semi-weighted homogeneous map germ g is G-finite and that any deformation of g by terms with weighted degree higher than the weighted degree of f is topologically trivial (see [3]).

In this article we shall investigate map germs $g: K^n, 0 \to K^p, 0$ which can be decomposed in a form $g = f + h$, where f is weighted homogeneous, G-finitely determined and $h = (h_1, h_2, \ldots, h_p)$ is a polynomial map germ such that any monomial h_{ij} of h_i has weighted degree lower than the weighted degree of f_i.

Looking for the properties of semi-weighted homogeneous map germs, we ask the questions:

1. If a map germ g can be decomposed in a form $g = f + h$ as above, is g G-finitely determined?

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2. Given a map germ of type $g = f + h$, when can we find a curve $\alpha(t)$, with $\alpha(0) = 0$, such that the family $f_t = f + \alpha(t)h$ is topologically trivial for $t > 0$?

3. Given a deformation g_t of g, by terms with weighted degree higher than the weighted degree of f, is the family g_t topologically trivial for all t?

In the Theorem 1.5 we show that the answer to Question **1** is affirmative. In the Lemma 1.6 we show how to find a curve $\alpha(t)$, such that the family f_t is topologically trivial for all $t > 0$, showing that Question **2** is affirmative as well. We also give a partial answer to Question **3**, *i.é,* for the case $p = 1$, we follow the approach of Yoshinaga in [9] to give sufficient conditions for the topological triviality of families of germs of functions $g_t: K^n, 0 \to K, 0$.

§1. Pre-weighted homogeneous map germs

We consider map germs $g : K^n, 0 \to K^p, 0$ which are either real analytic if $K = \mathbb{R}$ or holomorphic if $K = \mathbb{C}$. We fix a system of local coordinates x for K^n and y for K^p.

DEFINITION 1.1. Given $(r_1, \ldots, r_n; d_1, \ldots, d_p)$ with r_i, $d_j \in \mathbb{Q}_+$, a map germ $f: K^n, 0 \to K^p, 0$ is weighted homogeneous of type $(r_1, \ldots, r_n; d_1, \ldots, d_p)$ if for all $\lambda \in K - \{0\}$

$$f(\lambda^{r_1}x_1, \lambda^{r_2}x_2, \ldots, \lambda^{r_n}x_n) = (\lambda^{d_1}f_1(x), \lambda^{d_2}f_2(x), \ldots, \lambda^{d_p}f_p(x)).$$

DEFINITION 1.2. Given $(r_1, r_2, \ldots, r_n)$ with $r_i \in \mathbb{Q}_+$, the weighted degree of a monomial $x^m = x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}$ is defined by $\text{fil}(x^m) = \sum_{i=1}^n m_i r_i$.

For any germ $f: K^n, 0 \to K, 0$, we define $\text{fil}(f) = min_m\{\text{fil}(x^m)\}$ with $a_m \neq 0$ in the Taylor series $j^\infty f(x) = \sum a_m x^m$ of f.

For any map germ $f: K^n, 0 \to K^p, 0$, $f = (f_1, f_2, \ldots, f_p)$ we define $\text{fil}(f) = (d_1, d_2, \ldots, d_p)$, where $d_i = \text{fil}(f_i)$ for all $i = 1, \ldots, p$.

DEFINITION 1.3. A map germ $f: K^n, 0 \to K^p, 0$ is G-finitely determined if there exists an integer ℓ such that for each map germ g with $j^\ell g(0) = j^\ell f(0)$, g is G-equivalent to f.

DEFINITION 1.4. A map germ $g: K^n, 0 \to K^p, 0$ is G-pre-weighted homogeneous if it can be decomposed in the form $g = f + h$, where f is weighted homogeneous of type $(r_1, \ldots, r_n; d_1, \ldots, d_p)$, G-finitely determined and h is a polynomial map germ with $\mathrm{fil}(h_{ij}) < d_i$ for all monomials h_{ij} of each h_i.

THEOREM 1.5. *Let $g\colon K^n, 0 \to K^p, 0$ be a G-pre-weighted homogeneous map germ. Then g is G-finitely determined.*

EXAMPLES.

1. Let $g : \mathbb{C}^2 \to \mathbb{C}^2$, the map germ having the form

$$g(x,y) = \left(x^p + y^q + \sum_{aq+bp<pq} \alpha_{ab} x^a y^b,\ xy \right), \text{ with } \gcd(p,q) = 1.$$

As $f(x,y) = (x^p + y^q,\ xy)$ is $\mathcal{A}$-finitely determined [see 5, p. 466], g is $\mathcal{A}$-pre-weighted homogeneous and $\mathcal{A}$-finitely determined.

2. Let $g : \mathbb{C}^2 \to \mathbb{C}^2$, $g(x,y) = (x^9 - y^6 + xy(x^3 - y^2)^2, 3x^2y^5 - (x^3 - y^2)^2)$. The germ g is of type $g = f + h$, where $f(x,y) = \left(x^9 - y^6, 3x^2y^5\right)$ is weighted homogeneous of type $(2,3;\ 18,19)$ and $h(x,y) = (xy(x^3 - y^2)^2, -(x^3 - y^2)^2)$ is weighted homogeneous of type $(2,3;17,12)$. As f has isolated singularity at 0 it is $\mathcal{K}$-finitely determined, hence g is $\mathcal{K}$-pre-weighted homogeneous and $\mathcal{K}$-finitely determined.

In [2] we apply Theorem 1.5 in order to give a simple and direct proof that the set of map germs that are not contact-sufficient is of infinite codimension. Similar results are obtained for the group $\mathcal{A}$ for certain pairs of dimensions (n,p). Another application of Theorem 1.5 is given in [1].

The following lemmas are essential to prove Theorem 1.5.

LEMMA 1.6. *Let $f\colon K^n, 0 \to K^p, 0$ be a weighted homogeneous map germ of type $(r_1, \dots, r_n; d_1, \dots, d_p)$ and $\theta_{ij}\colon K^n, 0 \to K, 0$ weighted homogeneous polynomial germs of type $(r_1, \dots, r_n; d_{ij})$ with $d_{ij} \neq d_i$, for all $i = 1, 2, \dots, p$.*

Then the family of map germs $f_t\colon K^n \times \mathbb{R}, 0 \to K^p, 0$, for $t > 0$ defined by $f_t = \left(f_1 + \sum_{j=1}^{q_1} t^{(d_1 - d_{1j})}\, \theta_{1j}, \dots, f_p + \sum_{j=1}^{q_p} t^{(d_p - d_{pj})}\, \theta_{pj} \right)$ *is G-trivial.*

Proof. For all $t > 0$, we let $\psi_t\colon K^p, 0 \to K^p, 0$, $\psi_t(y) = \left(t^{d_1} y_1, \dots, t^{d_p} y_p\right)$ and $\phi_t\colon K^n, 0 \to K^n, 0$, $\phi_t(x) = (t^{-r_1} x_1, \dots, t^{-r_n} x_n)$.

Hence $f_t = \psi_t \circ g \circ \phi_t$, where $g = \left(f_1 + \sum_{j=1}^{q_1} \theta_{1j}, \dots, f_p + \sum_{j=1}^{q_p} \theta_{pj} \right)$ and the result follows. ∎

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Lemma 1.6 does not ensure the G-finite determinacy of the map germs g_t. Indeed, let $g_t(x,y) = (x^2-y^2)^2 + t^2xy(x^2-y^2)^2$. Applying Lemma 1.6, we see that the family g_t is trivial, but g_t is not finitely determined for any $t \in \mathbb{R}$.

LEMMA 1.7. *Let $f(x,t)\colon K^n \times \mathbb{R}, 0 \to K^p, 0$ be smooth with $f_0(x)\colon K^n, 0 \to K^p, 0$ G-finitely determined. Then $f_t\colon K^n, 0 \to K^p, 0$ is G-finitely determined for all t sufficiently small.*

Proof. See [6 p.306] for $G = \mathcal{A}$, the case $G = \mathcal{K}$ is easier. □

Proof of Theorem 1.5. Any pre-weighted homogeneous map germ g can be decomposed in the form

$$g = \left(f_1 + \sum_{j=1}^{q_1} h_{1j}, \ldots, f_p + \sum_{j=1}^{q_p} h_{pj} \right),$$

where f is weighted homogeneous of type $(r_1, \ldots, r_n; d_1, \ldots, d_p)$ and $\mathrm{fil}(h_{ij}) < d_i$ for all i, j. Then, according to Lemma 1.6, the family

$$g_t = \left(f_1 + \sum_{j=1}^{q_1} t^{(d_1-d_{1j})}\, h_{1j}\ , \ldots, f_p + \sum_{j=1}^{q_p} t^{(d_p-d_{pj})}\, h_{pj} \right)$$

is G-trivial. Now we apply the Lemma 1.7 to the family g_t, and conclude that for small values of t, g_t is G-finitely determined, therefore for all $t > 0$ g_t is G-finitely determined and $g = g_1$ is G-finitely determined. □

Remark. Theorem 1.5 holds only when $d_{ij} < d_i$ for all i, j (*i.e*, when the map germ is pre-weighted homogeneous). If $d_{ij} > d_i$ for all i, j, following the proof of Theorem 1.5, we have an easy way to prove that any G-semi-weighted homogeneous map germ is G-finite. But if there exist some degrees $d_{ij} > d_i$ and some degrees $d_{ij} < d_i$ in the family g_t, Theorem 1.5 does not hold.

Let $g\,(x,y) = \left(x^2 - y^2\right)^2 \left(x^2 - y\right)$. The function g can be decomposed in the form $g = f + h + k$, where $f(x,y) = x^6 + y^5$ is weighted homogeneous of type $(5, 6; 30)$, $h(x,y) = -2x^2y^3 + x^4y$ has only terms with weighted degree lower than $\mathrm{fil}(f)$ and $k(x,y) = -2x^4y^2 + x^2y^4$ has only terms with weighted degree higher than $\mathrm{fil}(f)$.

By Lemma 1.6, the family

$$g_t(x,y) = \left(x^6 + y^5\right) + t^{-2}\left(-2x^2y^3\right) + t^{-4}\left(x^4y\right) + t^2\left(-2x^4y^2\right) + t^4\left(x^2y^4\right)$$

is G-trivial, but Theorem 1.5 does not apply since $\lim_{t\to 0} g_t \neq \left(x^6 + y^5\right)$.

In this example f is finitely determined whereas g is not.

§2. Topological triviality of pre-weighted homogeneous germs of functions

A deformation $G : K^n \times K, 0 \to K, 0$ of a germ g is topologically trivial if there exists a germ of homeomorphism $\psi : K^n \times K, 0 \to K^n \times K, 0$, where $\psi(x,t) = (\overline{\psi}(x,t), t)$ such that $G \circ \psi(x,t) = g(x)$.

We are interested in the following question:
"When is a deformation of a pre-weighted homogeneous germ $g = f + h$ by terms with weighted degree higher than the weighted degree of f topologically trivial?"

In [7], following the method described by Damon and Gaffney in [4], we have an answer to this question in the case of commode and irreducible pre-weighted homogeneous germs $g\colon K^2, 0 \to K, 0$. In this section we provide an answer for any commode pre-weighted homogeneous germ $g\colon K^n, 0 \to K, 0$, following the method of constructing the toroidal embedding associated to the Newton polyhedron of the germ g, as described by Yoshinaga in [9].

Definition 2.1. The Newton polyhedron, $\Gamma_+(g)$ of a germ $g : K^n, 0 \to K, 0$, with $j^\infty g = \sum a_m x^m$ is the convex hull in $\mathbb{R}^n_+$ of the set

$$\bigcup \{m + v \,:\, v \in \mathbb{R}^n_+ \text{ and } a_m \neq 0\}.$$

We denote by $\Gamma(g)$ the union of the compact faces of $\Gamma_+(g)$.

We define a partition into convex cones of the positive octant in the dual $\mathbb{R}^{n*}$ of $\mathbb{R}^n$. Let $(a_1, \dots, a_n)$ be dual coordinates in $\mathbb{R}^{n*}$.

Definition 2.2. For each $a = (a_1, \dots, a_n) \in \mathbb{R}^{n*}_+$ we define:

(a) $\ell(a) = \min\{\langle a, k\rangle \,:\, k \in \Gamma_+(g)\}$, where $\langle a, k\rangle = \sum_{\imath=1}^{n} a_\imath k_\imath$.

(b) $\Delta(a) = \{k \in \Gamma_+(g) \,:\, \langle a, k\rangle = \ell(a)\}$.

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(c) Two vectors $a, a' \in \mathbb{R}^{n*}_{+}$ are said to be equivalent if $\Delta(a) = \Delta(a')$.

The vector a is called a primitive integer if it is the vector with minimum length in $C(a) \cap (\mathbb{Z}^n_+ - \{0\})$, where $C(a)$ is the half ray emanating from 0 passing through a.

It is easy to see that each $(n-1)$-dimensional face Δ of $\Gamma(g)$ is associated to a primitive integer $a \in \mathbb{R}^{n*}_{+}$ such that $\Delta = \Delta(a)$.

Given an $\ell \in \mathbb{R}_+$, we call $\Delta_\ell(a) = \{m \in \Gamma_+(g) \colon \langle m, a\rangle \leq \ell\}$.

For any subset $D \subset \Gamma_+(g)$ we call $g_D = \sum\limits_{m \in D} a_m x^m$.

DEFINITION 2.3. A finite subset $D \subset \Gamma_+(g)$ is non-degenerate if

$$\left\{x \in K^n : \frac{\partial g_D}{\partial x_1}(x) = \frac{\partial g_D}{\partial x_2}(x) = \ldots = \frac{\partial g_D}{\partial x_n}(x) = 0\right\} \subset \{x_1 x_2 \ldots x_n = 0\}$$

A germ g is Newton non-degenerate if all compact faces of $\Gamma(g)$ are non-degenerate.

DEFINITION 2.4. For each $(n-1)$-dimensional compact face $\Delta(a^j) \subset \Gamma(g)$, we let $Q_j = \min\left\{\ell \colon \Delta_\ell(a^j) \text{ is non-degenerate}\right\}$.

DEFINITION 2.5. A germ g is commode if for each $i = 1, \ldots, n$, there exists a pure term $x_i^{m_i}$ with nonzero coefficient in $j^\infty g$.

If $m = (0, \ldots, 0, m_i, 0, \ldots, 0) \in \Gamma(g)$, we call the monomial $x_i^{m_i}$ an initial pure term of g.

PROPOSITION 2.6. *Let $g \colon K^n, 0 \to K, 0$ be a commode pre-weighted homogeneous germ of type $g = f + h$, such that every monomial of h is a vertex of $\Gamma_+(g)$.*

Then deformations of g of type $G(x,t) = g(x) + t\theta(x)$ are topologically trivial for all $t \in [0,1]$ if $\Gamma_+(\theta) \subset$ interior of $\Gamma_+(g)$, and for small values of t if $\Gamma_+(\theta) \subset \Gamma_+(g)$.

PROPOSITION 2.7. *Let $g \colon K^n, 0 \to K, 0$ be a commode pre-weighted homogeneous germ of type $g = f + h$, such that:*

i. *The germ h has only mixed terms.*

ii. *For each initial pure term $x_i^{m_i}$ of g, and for each $(n-1)$-dimensional compact face $\Delta(a^j) \in \Gamma(g)$, $Q_j \leq m_i a_i^j$, where $a^j = (a_1^j, \ldots, a_n^j)$*

Then:

(a) *If* $\Gamma_+(\theta) \subset$ *interior of* $\Gamma_+(f)$, *the family* $G(x,t) = g(x) + t\theta(x)$ *is topologically trivial for all* $t \in [0,1]$.

(b) *If* $\Gamma_+(\theta) \subset \Gamma_+(f)$, $G(x,t) = g(x) + t\theta(x)$ *is topologically trivial for sufficiently small values of* t.

EXAMPLES.

1. Let $g(x,y) = x^9 + y^8 + xy(x^2 - y^2)^2$. The polyhedron $\Gamma_+(g)$ is composed by five 1-dimensional faces $\Delta(a^j)$, where $a^1 = (1,0)$, $a^2 = (0,1)$, $a^3 = (3,1)$, $a^4 = (1,1)$ and $a^5 = (1,4)$ are the corresponding primitive integer vectors. $\Delta(a^1)$ and $\Delta(a^2)$ are the non-compact faces in the semi-axis, with vertices $\{(0,8)\}$ and $\{(9,0)\}$. The compact faces are $\Delta(a^3)$, $\Delta(a^4)$ and $\Delta(a^5)$ with vertices $\{(0,8),\ (1,5)\}$, $\{(1,5),\ (5,1)\}$ and $\{(5,1),(9,0)\}$ respectively.

The faces $\Delta(a^3)$ and $\Delta(a^5)$ are non-degenerate, hence $Q_3 = 8$ and $Q_5 = 9$, but $\Delta(a^4)$ is a degenerate face of $\Gamma(g)$, because the curve $\psi(t) = (t,t)$ is a solution for the equations $\left\{\frac{\partial g_{\Delta(a^4)}}{\partial x}(x) = \frac{\partial g_{\Delta(a^4)}}{\partial y}(x) = 0\right\}$. As $\Delta_8(a^4)$ is the first non-degenerate set associated to a^4, we have $Q_4 = 8$. Since $\langle (9,0); a^j\rangle$ and $\langle (0,8); a^j\rangle$ are both $\geq Q_j$, for $j = 3,4,5$, we conclude that for any θ such that $\Gamma_+(\theta) \subset$ interior of $\Gamma_+(f)$, the family $G(x,t) = g(x) + t\theta(x)$ is topologically trivial for all $t \in [0,1]$.

2. Let $g_1(x,y) = x^9 + \left(y^4 + xy\right)(x^2 - y^2)^2$. In this example $\Gamma_+(g_1) = \Gamma_+(g)$ (Ex.1), the compact face $\Delta(a^4)$ is degenerate and the set $D_8(a^4)$ is degenerate as well. The first non-degenerate set associated to a^4 is $D_9(a^4)$, hence $Q_4 = 9$. Here $\langle (0,8); a^4\rangle = 8 < Q_4$. After the proof of the Proposition 2.7, we shall show that deformations by terms x^m with $m_1 + m_2 \geq 9$ are topologically trivial.

3. Let $k(x,y) = x^9 + y^8 + xy(x^2 + y^2)^2$. Here $\Gamma_+(k) = \Gamma_+(g)$. If k is a complex germ of function, the compact face $\Delta(a^4)$ is degenerate, as the complex curve $\varphi(t) = (t, it)$ is a solution for the equations $\left\{\frac{\partial k_{\Delta(a^4)}}{\partial x}(x) = \frac{\partial k_{\Delta(a^4)}}{\partial y}(x) = 0\right\}$. If k is a real germ of function, $\Gamma_+(k)$ is non-degenerate since there is no real solution for the equations $\left\{\frac{\partial k_{\Delta(a^j)}}{\partial x}(x) = \frac{\partial k_{\Delta(a^j)}}{\partial y}(x) = 0\right\}$, for $j = 3,4,5$.

4. Let $f(x,y) = x^{12} + y^{12} + z^{12} + xyz(x^2 - y^2 + z^2)^2$. The polyhedron $\Gamma_+(f)$ is composed by seven 2-dimensional faces $\Delta(a^j)$, where $a^1 = (1,0,0)$, $a^2 = (0,1,0)$, $a^3 = (0,0,1)$, $a^4 = (1,6,1)$, $a^5 = (6,1,1)$, $a^6 = (1,1,6)$ and $a^7 = (1,1,1)$ are the corresponding primitive integer vectors. $\Delta(a^7)$ is the unique degenerate face of $\Gamma_+(f)$ and it is easy to check that $Q_7 = 12$, hence deformations of f by terms of degree higher than 12 are topologically trivial.

2.1. Proofs of the Propositions 2.6 and 2.7

Yoshinaga defined in [9] the gradient polyhedron associated to a germ g, and applied it to show the topological triviality of deformations of g.

Let $|\nabla g|^2 = \sum_{i=1}^{n} \left| x_i \dfrac{\partial g}{\partial x_i} \right|^2$. For a fixed $m \in \mathbb{Z}_+^n$ consider the following condition:

$(\mathcal{G})$ there exists a positive $\mathcal{E} = \mathcal{E}(m)$ such that $|\nabla g| \geq \mathcal{E}|x^m|$ in a neighbourhood of the origin in K^n.

DEFINITION 2.8. [9, p.804] The gradient polyhedron of g, $\Lambda_+(g)$ is the convex hull of the set $\bigcup \{m+v,\ m \in \mathbb{Z}_+^n,\ v \in \mathbb{R}_+^n :$ condition $(\mathcal{G})$ holds$\}$.

THEOREM 2.9. [9, p.805] *Let $G : K^n \times K, 0 \to K, 0$,b e a deformation of g of type $G(x,t) = g(x) + t\theta(x)$, such that $V(g) = \{0\}$, where $V(g) = \{x \in K^n : |\nabla\ g| = 0\}$.*

(a) *If $\Gamma_+(\theta) \subset$* interior of *$\Lambda_+(g)$, then the family $g_t(x) = G(x,t)$ is topologically trivial for all $t \in [0,1]$.*

(b) *If $\Gamma_+(\theta) \subset \Lambda_+(g)$, then the family $g_t(x) = G(x,t)$ is topologically trivial for sufficiently small values of t.*

Remark. Yoshinaga also proves in [9, Theorem 1.7] that $\Lambda_+(g) = \Gamma_+(g)$ if and only if g is Newton non-degenerate.

Proof of Proposition 2.6. Since g is a commode pre-weighted homogeneous germ, we have $V(g) = \{0\}$, hence if we prove that all the compact faces of $\Gamma(g)$ are non-degenerate, the result follows from the Theorem 1.7 of Yoshinaga.

Let Δ be an r-dimensional compact face of $\Gamma(g)$, since every monomial of h is a vertex of the Newton polyhedron of g, the face Δ is composed only by their vertices, hence the set of solutions of the equations

$\left\{\frac{\partial g_\Delta}{\partial x_1}(x) = \ldots = \frac{\partial g_\Delta}{\partial x_n}(x) = 0\right\}$ is a subset of $\{x_1 x_2 \ldots x_n = 0\}$ if and only if the rank of the $(n \times r+1)$-matrix $B = [b_{ij}]$ is maximal, where $b_i = (b_{i1}, \ldots, b_{in})$ are the $r+1$ vectors in $\mathbb{R}^{n*}$ defining the vertices of Δ, therefore they are linearly independent in $\mathbb{R}^{n*}$ and the result follows. $\square$

The essential tool used in proving Proposition 2.7 is the construction of the toroidal embedding associated to the Newton polyhedron of a germ g. The notion of toroidal embeddings was developed by Kempf et all in [6], the procedure of constructing the toroidal embedding associated to $\Gamma_+(g)$ is a local modification of Khovanskii's method of assigning a compact complex nonsingular toroidal manifold to an integer-valued compact convex polyhedron in K^n. This construction is due to Varchenko [8, pp.183–184] and we now summarize it.

Considering the equivalence relation (c) of the Definition 2.2, any equivalence class is naturally identified with a convex cone with its vertex at zero that is specified by finitely many linear equations and strictly linear inequalities with rational coefficients.

We call Σ_0 the partition into closed convex cones of the positive cone $\mathbb{R}^{n*}_+$ specified by the closures of the equivalence classes.

Following the algorithm described in the proof of Theorem 11 of [6 p.32], we construct on the basis of Σ_0, a partition Σ of the cone $\mathbb{R}^{n*}_+$ into finitely many closed convex cones with their vertices at zero such that:

1. Any cone belonging to Σ lies in one of the cones in Σ_0 and is specified by finitely many linear equalities and linear inequalities with rational coefficients.

2. If σ_1 is a face of a cone σ in Σ, then $\sigma_1 \in \Sigma$.

3. For any cones σ_1 and σ_2 in Σ, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

4. Any cone σ in Σ is simplicial and unimodular, *i.e.*, if the dimension of σ is q, there exist a set of primitive integer vectors $a^1(\sigma), a^2(\sigma), \ldots, a^q(\sigma)$ in σ which are linearly independent over $\mathbb{R}$ and $n-q$ primitive integer vectors $a^{q+1}(\sigma), \ldots, a^n(\sigma)$ such that $\mathbb{Z}a^1(\sigma) + \ldots + \mathbb{Z}a^n(\sigma) = \mathbb{Z}^n$.

Let σ be an n-dimensional cone in Σ and $a^1(\sigma), a^2(\sigma), \ldots, a^n(\sigma)$ the corresponding set of primitive integer vectors of σ that has been ordered once and for all. We associate to each such σ a copy of K^n denoted by

$K^n(\sigma)$. Let us denote by $\pi_\sigma: K^n(\sigma) \to K^n$ the mapping given by the formulae

$$x_i = y_1^{a_i^1(\sigma)} \dots y_n^{a_i^n(\sigma)}$$

where $x_1, x_2, \dots, x_n$ are coordinates in K^n, $y_1, y_2, \dots, y_n$ are coordinates in $K^n(\sigma)$ and $a_1^j(\sigma), \dots, a_n^j(\sigma)$ denote the coordinates of the vector $a^j(\sigma)$.

We define the following equivalence relation on the disjoint union of the $K^n(\sigma)$, let $y_\sigma \in K^n(\sigma)$ and $y_\tau \in K^n(\tau)$, then $y_\sigma \sim y_\tau$ if and only if $\pi_\sigma(y_\sigma) = \pi_\tau(y_\tau)$. We denote by $X = X(\Gamma_+(g))$ the set obtained in this way.

It follows from the properties 1–4 of the partition Σ and Theorems 6, 7 and 8 of [6, pp. 24–26] that X is a nonsingular n-dimensional algebraic complex manifold and that $\pi: X \to K^n$ defined by $\pi(y) = \pi_\sigma(y_\sigma)$ is a proper analytic mapping onto K^n (where $y_\sigma \in K^n(\sigma)$ is a representative of the equivalence class $y \in X$). Therefore the following equations

1. $|\nabla g| \geq \mathcal{E}|x^m|$ for all x in a neighbourhood U of 0
2. $|\nabla g| \circ \pi_\sigma(y_\sigma) \geq \mathcal{E}|x^m| \circ \pi_\sigma(y_\sigma)$ for all $y_\sigma \in \pi_\sigma^{-1}(U)$

are equivalent for any monomial $x^m \in K^n$ and any n-dimensional cone $\sigma \in \Sigma$. We shall use the condition 2 to prove Proposition 2.7.

Proof of Proposition 2.7. If $\Gamma(f) \subset \Lambda_+(g)$, the statement of this proposition follows from the Theorem 2.8, hence we shall show that the vertices of $\Gamma(f)$ satisfy the $\mathcal{G}$ condition. As g is commode and h has only mixed terms, the vertices of $\Gamma(f)$ are the initial pure terms $x_i^{m_i}$ of g. Given an n-dimensional cone σ, with a set of generators $a^1, \dots, a^n$, for each point $y_\sigma^0 \in \pi_\sigma^{-1}(0)$ there exists an n-tuple of numbers $\left(Q_1(y_\sigma^0), \dots, Q_n(y_\sigma^0)\right)$ with $\ell(a^j) \leq Q_j(y_\sigma^0) \leq Q_j$ such that

$$|\operatorname{grad} g| \circ \pi_\sigma(y_\sigma^0) = y_1^{Q_1(y_\sigma^0)} y_2^{Q_2(y_\sigma^0)} \dots y_n^{Q_n(y_\sigma^0)} h_\sigma(y_\sigma^0) \text{ with } |h_\sigma(y_\sigma^0)| > 0.$$

We remark here that the first inequality $\ell(a^j) \leq Q_j(y_\sigma^0)$ is immediate since we are considering monomials on or above the Newton polyhedron $\Gamma(g)$. The second inequality $Q_j(y_\sigma^0) \leq Q_j$ follows from the fact that g has isolated singularity at 0 and that $\Delta_{Q_j}(a^j)$ is the first non-generate set of a^j.

Then, for each $y_\sigma^0 \in \pi_\sigma^{-1}(0)$ there exists a neighbourhood $V(y_\sigma^0)$ such that $|h(y_\sigma)| > 0$ for all $y_\sigma \in V$.

On the other side, for all y_σ, we have $x_i^{m_i} \circ \pi_\sigma(y_\sigma) = |y_1^{M_1} y_2^{M_2} \dots y_n^{M_n}|$, where $M_j = m_i a_i^j$.

Therefore, the equation 2 holds, since the inequality $Q_j \leq m_i a_i^j = M_j$ holds for all j and all n-dimensional cones $\sigma \in \Sigma$. □

EXAMPLES.

1. Let $g(x,y) = x^9 + y^8 + xy(x^2 - y^2)^2$. Making use of the vectors a^i associated to the compact faces of $\Gamma_+(g)$, we get the 2-dimensional cones σ and the corresponding mappings π_σ:

$$\begin{array}{ll} \sigma_{1,3} = (a^1, a^3) & \pi_{1,3}(y_1, y_3) = (y_1 y_3^3; y_3) \\ \sigma_{3,4} = (a^3, a^4) & \pi_{3,4}(y_3, y_4) = (y_3^3 y_4; y_3 y_4) \\ \sigma_{4,5} = (a^4, a^5) & \pi_{4,5}(y_4, y_5) = (y_4 y_5; y_4 y_5^4) \\ \sigma_{5,2} = (a^5, a^2) & \pi_{5,2}(y_5, y_2) = (y_5; y_5^4 y_2). \end{array}$$

As the compact faces $\Delta(a^3)$ and $\Delta(a^5)$ are non-degenerate, the initial pure terms x^9 and y^8 satisfy equation 2 for the cones $\sigma_{1,3}$ and $\sigma_{5,2}$ since $\langle(9,0);(3,1)\rangle = 27 > Q_3$; $\langle(9,0);(1,4)\rangle = 9 = Q_5$; $\langle(0,8);(3,1)\rangle = 8 = Q_3$ and $\langle(0,8);(1,4)\rangle = 32 > Q_5$.

Since $\Delta(a^4)$ is a degenerate face in $\Gamma(g)$, we need to check if the initial pure terms x^9 and y^8 satisfy equation 2 for the cones $\sigma_{3,4}$ and $\sigma_{4,5}$.

We have $|\text{grad } g_{\Delta_4}| \circ \pi_{\sigma_{3,4}}(y_3, y_4) = y_3^8 y_4^6 \cdot |(y_3^4 - 1)| \cdot \left(|5y_3^4 - 1| + |y_3^4 - 5|\right)$.

Then for any point (y_3^0, y_4) such that $(y_3^0)^4 = 1$, $|\text{grad } g_{\Delta_4}| \circ \pi_{\sigma_{3,4}}(y_3^0, y_4) = 0$ and $|\text{grad } g| \circ \pi_{\sigma_{3,4}}(y_3^0, y_4) = y_4^8 \cdot \left(|9(y_3^0)^{27}| + |(y_3^0)^{24}|\right)$. Hence $Q_4(y_3^0, y_4) = 8$.

We observe here that if (y_3, y_4) is a point with $(y_3)^4 \neq 1$,

$$|\text{grad } g_{\Delta_4}| \circ \pi_{\sigma_{3,4}}(y_3, y_4) \neq 0 \text{ and } |\text{grad } g| \circ \pi_{\sigma_{3,4}}(y_3, y_4) = y_3^8 y_4^6 h(y_3, y_4),$$

with $h(y_3, y_4) \neq 0$, then $Q_4(y_3, y_4) = 6$. It is easy to see that the pure terms x^9 and y^8 satisfy the $\mathcal{G}$ condition and $\Gamma_+(f) \in \Lambda_+(g)$.

2. Following the Example 1 above, it is easy to see that $m \in \Lambda_+(g_1)$ if and only if $\langle m, a^4 \rangle = m_1 + m_2 \geq Q_4 = 9$. Therefore $\Gamma_+(f) \notin \Lambda_+(g)$.

3. Let $h = x^9 + y^8 + xy(x^2 + y^2)^2$. In the complex case, the results are similar to example 1, *i.e*, $\Gamma_+(f) \in \Lambda_+(g)$, since the initial pure terms x^9 and y^8 satisfy the equation 2 for any cone σ. In the real case, as h is

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non-degenerate, we conclude from Theorem 1.7 of Yoshinaga that $\Gamma_+(g) = \Lambda_+(g)$.

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