1
Complete Metric Spaces

> MONSIEUR JOURDAIN: Well, what do you know about that! These forty years now I've been speaking in prose without knowing it!
>
> –Molière, The Bourgeois Gentleman, 1670

1.1 Maps

Imagine that a map of the county, city or village you are living in is placed on the ground somewhere within the country's, city's or village's borders. It can be proved, and our intuition confirms this conjecture, that there is a point (precisely one!) on this map that lies directly above the point it describes. This statement is true regardless of the map scale. And of course it does not matter which country, city or village we have in mind. What is important is only that the map describes the entire area in which it is placed.

Moreover, there is nothing special about two dimensions in this example. If a one-dimensional 'map' of a road from town A to town B is prepared and placed somewhere on this road, then a point on this map can be found that lies directly above the place on the road it describes. The same is true in three dimensions: if a three-dimensional map of a lecture hall is placed in that lecture hall, there is a point in this map that is placed precisely at the point it describes.

We have thus found a common denominator for a number of 'spaces': these spaces are distinguished by the fact that their maps, when placed in the corresponding spaces, have one point lying precisely in the place it describes.

There are also, of course, spaces that do not posses this property. Think, for example, of a punched ball B, that is, of a ball with removed center, call this center O. Any smaller punched ball B', with the same center, that is contained

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in B, can be thought of as a map of B. Namely, if $k > 1$ is the ratio of radii of B and B', a point $P' \in B'$ can be thought of as an image of $P \in B$ if and only if $\overrightarrow{OP} = k\,\overrightarrow{OP'}$. Certainly on the map B' there is no point P' that is an image of itself. The reason for this situation is that O, the removed point, is the only candidate for having this property.

There are thus two types of spaces: those with holes and those without holes. The latter are professionally termed *complete* and the holey spaces are said to be incomplete (see further down for a more precise definition).

1.2 Roots

We have encountered complete metric spaces in mathematics a number of times before without perhaps knowing it (like Monsieur Jourdain from our Molière quote). To present an example of such an encounter, we start from the Bernoulli inequality

$$(x+1)^n \geq 1 + nx, \text{ where } x \geq -1, n \geq 1, \tag{1.1}$$

which is easy to prove by induction. We will show first, following Lech Maligranda (see [28] and the papers cited there; an almost identical proof was given even earlier by Bengt Åkerberg [3]), that (1.1) implies the following relation:[1]

$$x_1 \cdot x_2 \cdots x_n \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n, \quad x_1, x_2, \ldots, x_n > 0, n \geq 1. \tag{1.2}$$

Let $A_n = \frac{x_1+x_2+\cdots+x_n}{n}$. Since $\frac{A_n}{A_{n-1}} > 0$, taking $x := \frac{A_n}{A_{n-1}} - 1 > -1$ in (1.1), we have

$$\left(\frac{A_n}{A_{n-1}}\right)^n \geq 1 + n\left(\frac{A_n}{A_{n-1}} - 1\right) = \frac{nA_n - (n-1)A_{n-1}}{A_{n-1}} = \frac{x_n}{A_{n-1}}.$$

It follows that $A_n^n \geq x_n A_{n-1}^{n-1}$. This allows proving (1.2) by induction: for $n = 1$ the inequality is obvious, and assuming it holds for $n-1$ we obtain

$$A_n^n \geq x_n A_{n-1}^{n-1} \geq x_n(x_1 \cdots x_{n-1}) = x_1 \cdots x_n,$$

as claimed.

Using the obtained inequality, in turn, we will show[2] that for every positive number a and every integer n, there is a number b_n, denoted $\sqrt[n]{a}$ and termed the nth root of a, such that $b_n^n = a$ (how could you check that such a

[1] Written as $\sqrt[n]{x_1 \cdot x_2 \cdots x_n} \leq \frac{x_1+x_2+\cdots+x_n}{n}$, this becomes the well-known inequality between the arithmetic and geometric means. However, we do not want to use the notion of root at this point.

[2] Following Daniel Daners, Ulmer Seminare 2013, Notebook 18, Three Line Proofs.

number is uniquely determined?). To this end, let's consider the sequence given recursively by

$$x_1 = a, \qquad x_{k+1} = \frac{1}{n}\left((n-1)x_k + \frac{a}{x_k^{n-1}}\right). \tag{1.3}$$

This sequence is bounded from below by 0: all its members are positive, which is easy to check by induction. Also, because of (1.2), we have

$$x_{k+1}^n = \left(\frac{\overbrace{x_k + \cdots + x_k}^{(n-1)\text{terms}} + \frac{a}{x_k^{n-1}}}{n}\right)^n \geq a\,,$$

and this proves that

$$nx_{k+1} = \left((n-1) + \frac{a}{x_k^n}\right)x_k \leq nx_k,$$

that is, that the sequence is non-increasing. Hence, it has the limit $b_n := \lim_{k\to\infty} x_k$. Letting k tend to infinity in (1.3), we obtain

$$b_n = \frac{1}{n}\left((n-1)b_n + \frac{a}{b_n^{n-1}}\right).$$

Simple algebra now shows that $b_n^n = a$, completing the proof.

Let's have a closer look at this argument. Besides somewhat straightforward (though ingenious) calculations, it involves the following important step:

any non-increasing sequence that is bounded from below has a limit.

As we shall see later (see Exercise 1.5), this sentence is a disguised statement that the set of real numbers is complete, without holes, full.

Is this completeness completely obvious? It seems to be: from our childhood we became accustomed to the fact that real numbers can be identified with points on a line (this was not at all obvious before R. Descartes, though), and the line does not have holes. We were also taught that real numbers are limits of sequences of rational numbers, and we think of π, for example, in a similar way. Therefore, we tend to think of the set of real numbers as a completion of the set of rational numbers: if there is any hole in the latter set, a real number fills this place.[3]

[3] A formal proof that real numbers fill the gaps in the set of rational numbers can be found in [34].

By the way, in the reasoning presented above we take it for granted that $\mathbb{Q}$, the set of rational numbers, is holey. Are there any grounds for such prejudice? Of course, there are. To explain, we know that some computations do not make sense in $\mathbb{Q}$. For example, in $\mathbb{Q}$ $\sqrt{2}$ is meaningless, that is, $\sqrt{2}$ is not a rational number,[4] and this has a bearing on our previous argument on the existence of $\sqrt[n]{a}$.

For, if a is rational number then, by induction, all elements of the sequence obtained from the recurrence (1.3) are rational also. The algebra remains the same, proving that the sequence does not increase and is bounded from below. As we have just recalled, the limit cannot be a rational number for $a = n = 2$ (and a great many other cases). Thus we have found a sequence of rational numbers that converges to an irrational number. If we were unaware of the existence of irrational numbers (for some, something that cannot be expressed as a fraction $\frac{m}{n}$ where m and n are integers is as strange as a pink elephant and does not resemble a number at all), we would be forced to say that

> not all sequences of rational numbers that are non-increasing and bounded from below converge.

This, however, means that $\mathbb{Q}$ is not a 'full' set; this set is not complete, for it has holes. A closer look at $\mathbb{Q}$ reveals that between any two distinct rational numbers there are infinitely many non-rational numbers. One could even say that there are more holes than there are non-holes ($\mathbb{Q}$ is countable, $\mathbb{R}$ and $\mathbb{R} \setminus \mathbb{Q}$ are uncountable). The most significant difference between $\mathbb{Q}$ and $\mathbb{R}$ is that the former has holes whereas the latter is complete.

[4] For, supposing that

$$\sqrt{2} = \frac{l}{m} = \frac{2^{l_2} 3^{l_3} \cdots p^{l_p}}{2^{m_2} 3^{m_3} \cdots q^{m_q}},$$

where l_2 is the number of times 2 shows up in the prime factorization of l, and so on, then by taking squares and multiplying both sides by m^2 we obtain

$$2^{2m_2+1} 3^{2m_3} \cdots q^{2m_q} = 2^{2l_2} 3^{2l_3} \cdots p^{2l_p}.$$

Note that on the left-hand side 2 is raised to an odd power, but on the right-hand side it is raised to an even power. Since this contradicts uniqueness of prime factorization, $\sqrt{2}$ cannot be rational.

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1.3 Achilles

Some readers may dislike the previous section. Not all of us think taking roots is something they fancy doing. It is hard to argue against such an attitude, for it is possible to have a good living (in fact, be a billionaire) while being illiterate. But, in fact, completeness lies at the heart of something that for ages fascinated philosophers.

To see what I mean, let's recall Zeno of Elea, one of the most prominent students of Parmenides, who lived around 490 to 430 BCE. He is mostly known for his paradoxes, which were to substantiate his teacher's beliefs that plurality, change and motion in particular are but an illusion. Let's look at the apparently most famous and representative of these paradoxes: Achilles and the tortoise (see Figure 1.1). We all know that nobody is able to outrun the swift Achilles – and this is definitely impossible for a tortoise. But is the latter truly in a hopeless position? Suppose that initially the tortoise is at a distance $d > 0$ away from its pursuer, and that Achilles runs $\frac{1}{k}$ times faster than the little animal (where $k < 1$). The ill-matched competition begins – the tortoise tries to escape, and Achilles chases it. However, before Achilles catches the tortoise, he needs to come to the place where the tortoise had been at the beginning, and by that time the tortoise has moved a little (by a distance kd). Thus, Achilles faces a similar situation to the one he had initially: he needs to chase the tortoise who is at a distance kd away. Again, by the time Achilles comes to the place where tortoise had been this time, the tortoise has moved slightly away. This cycle will repeat infinitely, without end! And so, Achilles will never catch the tortoise. Quite a paradox!

Some may see this argument as pure sophistry. Many others (those who see that this argument cannot be easily refuted) may in fact start to doubt the world they see with their eyes is real. Such cases are known in history – for example, Georgias of Leontinoi, one of the philosophical followers of Zeno, became famous for his three nihilistic statements that can roughly be expressed as follows (see [40], p. 23): 1) there is nothing, 2) even if there were something, that something could not be apprehended, and 3) even if something were apprehended, this knowledge could neither be communicated nor understood by others. This leads to the following bold hypothesis:

Ignorance (of mathematics) is harmful.

Let's take neither of these roads, for both are disastrous. Zeno's paradoxes cannot be taken lightly or disregarded, because, as stated by W. Tatarkiewicz

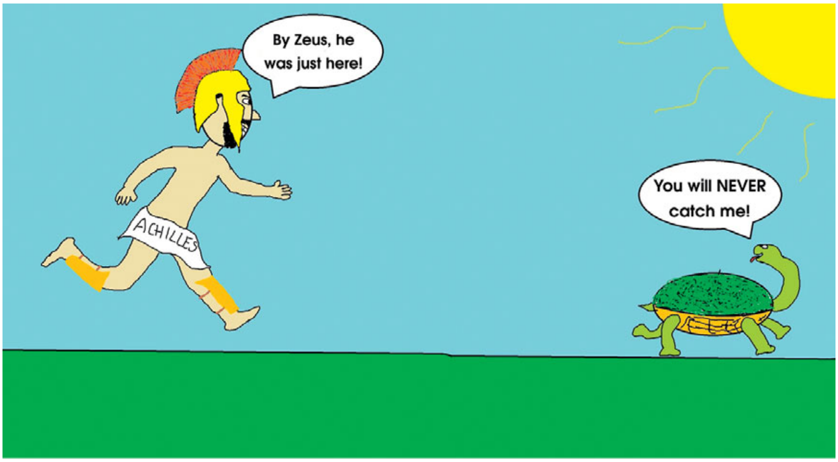

Figure 1.1 Way to go, tortoise!

(see [40] p. 25), 'Zeno's paradoxes (...) were inspiring for and discussed by outstanding philosophers, including Bayle, Descartes, Leibniz, Kant, Hegel, Herbart, Hamilton, Mili, Renouvier, Bergson, and Russell.' Great minds have contemplated these matters, and we should appreciate the solution that came with the development of modern mathematical analysis and with the theory of convergence of infinite series in particular in the nineteenth century (after over two thousand years!).

Here is an explanation of the paradox. First of all, we note a gap in Zeno's reasoning: the fact that something takes place infinitely many times need not imply that it will take place for ever. More specifically, the sum of infinitely many terms need not be infinite. And that's the crux of the matter.

Let us take an even closer look at the paradox. Let t_0 be the time Achilles needs to reach the place where the tortoise was initially. As we have noted before, by that time the tortoise moves away by the distance dk. Thus the time needed for Achilles to cover the latter distance is kt_0. In that time, the tortoise moves $k^2 d$ away, and the time needed for Achilles to cover this distance is $k^2 t_0$, etcetera. Notice that 'etcetera' is not a scary word anymore,[5] because

$$t_\infty := \sum_{n=0}^{\infty} k^n t_0 < \infty;$$

it is at t_∞ that Achilles catches the tortoise.

[5] Unless you are afraid of Latin.

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Let's work out the details: for any natural N, we have

$$\sum_{n=0}^{N} k^n t_0 = \frac{1-k^{N+1}}{1-k} t_0$$

(to see this, it suffices to multiply both sides by $1-k$ and do a little canceling), and the last expression converges to $\frac{t_0}{1-k}$ as $N \to \infty$. It does, we hasten to add, because k is smaller than 1, Achilles being faster than the tortoise. For $k > 1$, the sum on the right converges to infinity and Achilles turns out to be not so swift after all. By the way, for $k = 1$ he is not so swift either, but the formula above is different (can you provide it?).

It is perhaps worth looking at the yet more specific case of $k = \frac{1}{2}$ (Achilles is twice as fast as the tortoise) and $t_0 = 1$. Here, we are dealing with the sum of the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots .$$

Some might argue that this sum cannot be equal to 2, because it never 'reaches' 2. But even such notorious doubters can be somewhat convinced: beyond reasonable doubt, the finite sums

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^N}, \qquad N \geq 1,$$

increase with N, and are bounded from above by 2. Zeno's paradox disappears if we agree that a non-decreasing sequence that is bounded from above has a limit; it is immaterial whether the limit here is 2; what is important is that the limit exists – the limit is the time when Achilles catches the tortoise. In other words, Zeno's paradox disappears if we agree that (see Exercise 1.4)

time is a complete space, a space without holes.

If we reject this assumption, there need not be a time when Achilles catches the tortoise, and we will need to admit that reality does not agree with reason. As we see, completeness influences our living with no quarter. It is simply impossible to live without it.

1.4 Metric Spaces, Cauchy Sequences, Completeness

Let us come back to the subject of Section 1.1. What lies behind the fact that on a map of a county there is a point that lies directly above the point it describes?

The reason seems to be related to the notion of distance: there is a constant $q \in (0,1)$ (termed the scale of a map) equal to the ratio of distances between points on a map and corresponding points in the area the map describes (this ratio is independent of the choice of these points). It is thus reasonable to start with the notion of distance.

To recall, a set X equipped with a function d (called a metric), mapping $X \times X$ to $\mathbb{R}^+$ and satisfying the following three conditions, is said to be a metric space:

(a) for all $x, y \in X$, equality $d(x,y) = 0$ holds if and only if $x = y$,
(b) for all $x, y \in X$, we have $d(x,y) = d(y,x)$,
(c) for all $x, y, z \in X$, we have $d(x,z) \le d(x,y) + d(y,z)$.

Of course, $d(x,y)$ is interpreted as a distance between points x and y. With this interpretation, the conditions given above are plausible and agree with our intuition nicely: a distance between two points is zero if and only if these points coincide, distance to y measured from x is the same as the distance to x measured from y, and the way from x to z that leads through an intermediate point y cannot be shorter than the way that leads directly from x to z. Because of the last intuition, condition (c) is termed the triangle inequality and is best visualized if x, y and z are thought of as vertices of a triangle.

If we think now of the county we live in as a metric space X (with distance measured with a measuring ruler – even if this ruler is really long), then by placing a map of the county on a ground we define a transformation of X. In this transformation, to a point $x \in X$ we assign the x' lying directly below the point describing x on the map. Then

$$d(x', y') = q d(x,y), \tag{1.4}$$

where, as before, $q < 1$ is the map's scale.

As we shall see in the next chapter, this property, together with completeness of X, is a key to the property discussed in Section 1.1. For now, having the notion of distance defined, let us think of how this notion can be used to define complete spaces. To this end, let us come back to the example of Section 1.2. We have seen there that the set of rational numbers is not complete, and the argument for that was that there exists a non-increasing sequence of rational numbers that is bounded from below, and yet does not converge (to a rational number).

This idea is promising: why not, following Augustin Louis Cauchy, define complete spaces with the help of sequences? Why not detect holes by examining appropriate sequences? In an abstract metric space, however, we cannot work with non-decreasing or non-increasing sequences because in an abstract metric space more often than not there is no natural order. Hence, we need to have a good (gut) feeling what is an 'appropriate' sequence. Cauchy's brilliant idea was to define them as follows:

1.1 Definition A sequence $(x_n)_{n\geq 1}$ of elements of a metric space X is said to be a Cauchy sequence (or a fundamental sequence), if for any $\epsilon > 0$ there is an $n_0 \geq 0$ such that $d(x_n, x_m) < \epsilon$, as long as $n, m \geq n_0$.

In other words, for any ϵ one can throw away a finite number of elements of a Cauchy sequence in such a way that distances between any two of the remaining elements will be smaller than ϵ. Readers should convince themselves (at least intuitively) that non-increasing sequences of real numbers that are bounded from below are fundamental.

Let's see how fundamental sequences are related to convergent sequences. The latter are, to recall, defined as follows.

1.2 Definition A sequence $(x_n)_{n\geq 1}$ of elements of a metric space X is said to converge if there is an $x \in X$, said to be its limit, such that $\lim_{n\to\infty} d(x_n, x) = 0$; that is, for all $\epsilon > 0$ there is an n_0 such that $d(x_n, x) < \epsilon$ for $n \geq n_0$.

It is easy to check that any sequence that converges is fundamental. For, if x is its limit then, given $\epsilon > 0$ we can find n_0 such that $d(x_n, x) < \frac{\epsilon}{2}$ for all $n \geq n_0$. By the triangle inequality this implies, however, that $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$, proving the claim.

Nevertheless, the converse is not true: there are Cauchy sequences that do not converge, and it is precisely the existence of such sequences that indicates the existence of holes in a metric space. The basic idea is that Cauchy sequences behave as if they were convergent. If we cannot find a limit of a Cauchy sequence, we suspect that the space we examine is holey.

To gain some more insight and to see a connection between fundamental sequences and holes in a metric space, let us think about a being that is living in the punched ball of Section 1.1. He/she knows our Euclidean metric but does not have a way of looking at the ball he/she lives in 'from outside' and thus discovering a hole. He/she only thinks that a point with all three coordinates equal to zero is not a point at all; for him/her it is a no-point.[6] We arrange things this way because we want him/her to detect the existence of the hole

[6] Ancient Greeks asked 'how can nothing be something?', and there are still many who cannot accept the existence of zero as a number. Or a person with zero morale. My dear referee of the Polish edition hastened, however, to recall S. J. Lec's aphorism which roughly translated goes as follows: 'When I reached the bottom, I heard knocking from below.'

from the inside, without looking from the outside. If he/she is a mathematician he/she can try to use a fundamental sequence. For instance, he/she can think of $(x_n)_{n\geq 1}$ given by $x_n = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n})$, and argue as follows: 'given $\epsilon > 0$, I can throw away a finite number of elements of this sequence in such a way that for any x_n, x_m of the remaining elements the distances

$$d(x_n, x_m) = \sqrt{3}\left|\frac{1}{n} - \frac{1}{m}\right| < \frac{\sqrt{3}}{\min(m, n)}$$

between them are smaller than ϵ: it suffices to throw away all x_n with $n \leq \sqrt{3}\epsilon^{-1}$. Hence, this sequence seems to converge. However,' – he/she continues – 'I cannot think of a point that could be the limit of this sequence. For, any point, say x, in my decent space – the best space one can live in – has three coordinates of which at least one, say a, is non-zero (how ugly it would be for a point in my space to have all coordinates equal zero!). This shows, then, that the distance between x_n and x is at least $|\frac{1}{n} - a|$, and the latter quantity cannot converge to 0. I have thus found a Cauchy sequence that cannot converge. There must be something wrong with my space. I wonder what is it? Does it have a hole?'.

These considerations lead us to the following definition.

1.3 Definition A metric space is said to be complete if all fundamental (Cauchy) sequences of elements of this space converge.

As already discussed, the space of real numbers with distance $d(x, y) = |x - y|$ is a basic example of a complete space. On the other hand, the space of rational numbers, with the same distance, is full of holes. We will not give a formal proof of the fact that reals form a complete space – this is done in any decent course of real analysis (see e.g., [34]). Instead, we will show how completeness of $\mathbb{R}$ implies completeness of $\mathbb{R}^k, k \in \mathbb{N}$ when equipped with a Euclidean metric; more examples of complete spaces will be presented later in the book.

The argument that $\mathbb{R}^k$ is complete is in fact quite simple. For, let $(x_n)_{n\geq 1}$, where

$$x_n = (\xi_{n,1}, \xi_{n,2}, \dots, \xi_{n,k}) \in \mathbb{R}^k$$

is a Cauchy sequence in $\mathbb{R}^k$. An easy-to-establish inequality

$$|\xi_{n,i} - \xi_{m,i}| \leq d(x_n, x_m) \tag{1.5}$$

(which holds for all $i = 1, \dots, k$) implies that the numerical sequences $\left(\xi_{n,i}\right)_{n\geq 1}, i = 1, \dots, k$ are fundamental in $\mathbb{R}$. Indeed, this inequality shows that $|\xi_{n,i} - \xi_{m,i}|$ is smaller than a given ϵ whenever $d(x_n, x_m)$ is smaller

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than this ϵ, and we know by assumption that $d(x_n, x_m)$ is smaller than an ϵ for all sufficiently large n, m. Since $\mathbb{R}$ is complete, there exist the limits $\xi_i = \lim_{n\to\infty} \xi_{n,i}, i = 1, \ldots, k$. Vector $x = (\xi_1, \xi_2, \ldots, \xi_k)$ is a member of $\mathbb{R}^k$. We are left with showing that x is a limit of $(x_n)_{n\geq 1}$. By assumption, for all $\epsilon > 0$ there is $n_0(\epsilon)$ such that

$$d(x_n, x_m) = \sqrt{\sum_{i=1}^{k} (\xi_{n,i} - \xi_{m,i})^2} < \epsilon$$

as long as $n, m \geq n_0(\epsilon)$. Letting $m \to \infty$, we obtain

$$d(x_n, x) = \sqrt{\sum_{i=1}^{k} (\xi_{n,i} - \xi_i)^2} \leq \epsilon$$

for $n \geq n_0(\epsilon)$. Hence, for $n \geq n_1(\epsilon) := n_0(\epsilon/2)$, we have $d(x_n, x) < \epsilon$, completing the proof.

1.5 Yet Another Encounter

As I have already mentioned above, completeness of metric spaces is a key to a number of theorems in pure and applied mathematics. Here is another, slightly more advanced, example: Dirichlet's test for convergence of functional series. The test says that a series of the form

$$\sum_{i=1}^{\infty} a_i x_i(s), \qquad s \in S,$$

where S is a set, a_i's are positive numbers and $x_i : S \to \mathbb{R}$ are functions, converges uniformly with respect to s, provided that the following two conditions are satisfied:

(a) $a_{i+1} \leq a_i$ for all $i \geq 1$ and $\lim_{i\to\infty} a_i = 0$,
(b) there is an $M > 0$ such that $|\sum_{i=1}^{n} x_i(s)| \leq M$ for all $s \in S$ and $n \geq 1$.

All one needs to know to prove validity of this test, besides a bit of algebra, is that the space of *bounded*[7] functions on S is a complete metric space when equipped with the distance

$$d(x, y) = \sup_{s\in S} |x(s) - y(s)|,$$

[7] Note that, were either x or y not bounded, $d(x, y)$ could be infinite.

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and after reading a couple of chapters that follow, the reader will be able to check this completeness with ease, see Exercise 3.10.

As for the algebra, we first let

$$y_n(s) := \sum_{i=1}^{n} a_i x_i(s),$$
$$z_n(s) := \sum_{i=1}^{n} x_i(s), \qquad s \in S, n \geq 1.$$

By assumption (b), we have $|z_n(s)| \leq M$ for all $s \in S$ and $n \geq 1$. Then, as long as $m > n \geq 1$,

$$\begin{aligned} y_m(s) - y_n(s) &= \sum_{i=n+1}^{m} a_i[z_i(s) - z_{i-1}(s)] = \sum_{i=n+1}^{m} a_i z_i(s) - \sum_{i=n}^{m-1} a_{i+1} z_i(s) \\ &= \sum_{i=n+1}^{m-1} (a_i - a_{i+1}) z_i(s) + a_m z_m(s) - a_{n+1} z_n(s). \end{aligned}$$

Thus, since $a_i \geq a_{i+1}$, we see that $|y_m(s) - y_n(s)|$ does not exceed

$$M[a_m + \sum_{i=n+1}^{m-1} (a_i - a_{i+1}) + a_{n+1}] = 2Ma_{n+1},$$

yielding

$$d(y_m, y_n) \leq 2Ma_{n+1}.$$

Now, the second part of assumption (a) tells us that $(y_n)_{n\geq 1}$ is a Cauchy sequence. There is thus a bounded function y on S such that $\lim_{n\to\infty} d(y_n, y) = 0$, that is,

$$\sup_{s\in S} |y(s) - \sum_{i=1}^{n} a_i x_i(s)|$$

converges to 0, as $n \to \infty$. But this is precisely the uniform convergence of the series (to y).

1.6 Exercises

Exercise 1.1. Prove the Bernoulli inequality.

Exercise 1.2. Let $(x_n)_{n\geq 1}$ be a sequence of elements of a metric space, and suppose that $(x_n)_{n\geq 1}$ converges to an x in this space. Use the triangle inequality

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to show that then, for any y in this space, the numerical sequence $(d(x_n, y))_{n\geq 1}$ converges to $d(x, y)$.

Exercise 1.3. Check that $\sqrt{3}$ and $\sqrt{5}$ are not rational.

Exercise 1.4. Show that if any non-increasing sequence of reals that is bounded from below converges, then so does any any non-decreasing sequence of reals that is bounded from above.

Exercise 1.5. Prove that completeness of the space of reals implies that any non-increasing sequence of reals that is bounded from below converges. Show also that the fact that any non-increasing sequence of reals that is bounded from below converges, implies completeness of the space of reals.

Exercise 1.6. Let $X := \{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\} \subset \mathbb{R}$ be equipped with the metric $d(x, y) = |x - y|$. Is X complete? What about $X \cup \{0\}$?

Exercise 1.7. Prove (1.5). **Hint:** A sum of non-negative terms is no smaller than any of these terms.

Exercise 1.8. Let X be the space of sequences $(\xi_i)_{i\geq 1}$ such that ξ_i is either $+1$ or -1 for each $i \geq 1$. Check to see that this is a complete metric space with metric defined as follows:

$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} |\xi_i - \eta_i|,$$

where $x = (\xi_i)_{i\geq 1}$ and $y = (\eta_i)_{i\geq 1}$.

Exercise 1.9. If you are already convinced that Achilles will catch the tortoise, the time needed for him to do that can be found without summing the infinite series of Section 1.3, but simply by calculating t_∞ from the following relations:

$$k(d + x) = x, \qquad t_\infty = \frac{x}{v_{\text{tortoise}}}, \qquad t_0 = \frac{d}{v_{\text{Achilles}}} = \frac{dk}{v_{\text{tortoise}}}.$$

Provide the details.

Exercise 1.10. Argue as in Section 1.5 to prove the following test for convergence of series, attributed to Weierstrass (and known as the M-test). Suppose $a_n, n \geq 1$ are positive numbers such that $\sum_{n=1}^{\infty} a_n < \infty$ and $x_n, n \geq 1$ are functions on a set S. Assume that

$$|x_n(s)| \leq a_n, \qquad n \geq 1, s \in S.$$

Then the series $\sum_{n=1}^{\infty} x_n(s)$ converges absolutely and uniformly.

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☞ CHAPTER SUMMARY

Guided by basic intuitions, we introduce the notion of a complete metric space and discover that we have in fact encountered it before in our study of mathematics. In particular, we learn that if the set of real numbers were not complete, bounded increasing (or decreasing) sequences would not have limits. Similarly, we realize that if time were not complete, Achilles would never catch the tortoise. In a slightly more advanced part, we show that the criteria for convergence of functional series involve the notion of completeness of the space of continuous functions.