ORDER IN FINITE AFFINE PLANES

Mary Beattie, J.E. Marsden and R.W. Sharpe

(received February 14, 1966)

Definition of an Ordered Plane. An ordered plane [1, p.177 ff.] is a set of undefined entities called points, together with an undefined relation of intermediacy, satisfying the following axioms: (The symbol [ABC] means "B is between A and C".)

O1. There exist at least two points.

O2. If A and B are distinct points, then there is at least one point C such that [ABC].

O3. If [ABC], then A, B and C are distinct points.

O4. If [ABC], then not [BCA].

(If A and B are distinct points, the line AB is the set consisting of A and B together with every point P such that [PAB] or [ABP] or [APB].)

O5. If C and D are distinct points of line AB, then A is a point of line CD.

O6. If AB is a line, there exists a point C not on this line.

(The triangle ABC, where A, B and C are non-collinear points, is the set of the three points A, B and C and the lines AB, BC and AC.)

O7. If ABC is a triangle and if [BCD] and [AEC], then there is a point F on line DE such that [AFB].

From axioms O1 - O6 it follows that if [ABC] then [CBA] and no other such relation holds among A, B and C. Axiom O5 makes the notion of collinearity meaningful, and it is easily shown that if C and D are distinct points on the line AB,

https://doi.org/10.4153/CMB-1966-047-6 Published online by Cambridge University Press

then line AB = line CD. It follows that two distinct lines meet in at most one point.

The addition of O7 enables us to prove the expected results about the order of four points on a line, e.g. that [ABC] and [BCD] imply [ABD] and [ACD], and hence to show that any line in an ordered plane contains an infinite number of points.

Finite Ordered Planes. The purpose of this note is to investigate the structure of certain finite planes which obey axioms O1 - O6 (and, of course, fail to satisfy O7). Let us call an intermediacy relation satisfying O1 - O6 a finite order.

For any two points A and B, there must be a third point C such that [ABC], a fourth point D such that [ACD] and a fifth point E such that [CAE]. A, B, C, D and E are all distinct by O3, and all on the line AB. Hence any line must contain at least five points.

With the absence of axiom O7, the definition of a finite order on one line is independent of its definition on any other line. It is therefore desirable to impose some other condition on the order; we have chosen to have it interact with an algebraic structure on the plane.

Dilatations on $Z_p \times Z_p$. We now turn our attention to the finite plane $Z_p \times Z_p$, the Cartesian product of Z_p with itself, where Z_p is the field of residue classes modulo p, a prime integer [2, p.54]. Lines are sets of points satisfying non-trivial linear equations with coefficients in Z_p.

Definition. Consider an affine plane P; a dilatation is a transformation of P onto itself which leaves invariant each class of parallel lines [1, p.194].

It is easily shown that all dilatations D acting on the points X of $Z_p \times Z_p$ have the form $D(X) = aX + B$ where $a \neq 0$ and B are fixed elements of Z_p and $Z_p \times Z_p$ respectively.

A natural finite order is a finite order which is invariant under dilatations. Our main result gives the prime integers p

https://doi.org/10.4153/CMB-1966-047-6 Published online by Cambridge University Press

for which a natural finite order is possible on $Z_p \times Z_p$.

THEOREM. A natural finite order can be defined on $Z_p \times Z_p$ $(p \geq 5)$ if and only if $p \equiv -1$ (modulo 3).

Proof. Axioms O1 and O6 are automatically satisfied; axioms O3, O4 and O5 will be satisfied if and only if for any three distinct collinear points A, B and C of $Z_p \times Z_p$ we can distinguish exactly one as being between the other two, and for non distinct or non collinear points no order relation is defined.

The dilatations form a group. Hence, to obtain a natural finite order relation for the plane, we may confine our attention to one member of each parallel class, say the line through (0, 0). Since these representatives are additively isomorphic to Z_p, it is sufficient to consider possible orders on Z_p itself. Our dilatations become the automorphisms $T: Z_p \to Z_p$, defined by $T(x) = ax + b$, $a \neq 0$ and b fixed in Z_p, for all $x \in Z_p$. These transformations T form a group Γ of order $p(p-1)$. A transformation belonging to Γ is determined by the images of two distinct points of Z.

If there is an intermediacy relation on each set of three distinct points, or triplet, of Z_p satisfying axioms O2 and O4, then there is a natural finite order on Z_p.

Consider any triplet of points (a, b, c). We distinguish two cases.

(i) Suppose it is possible to label the three points a, b and c so that $b - a = c - b$. Then for any T in Γ, $T(b) - T(a) = T(c) - T(b)$. Now the transformation $G \in \Gamma$ defined by $G(x) = 2b - x$ interchanges $T(a)$ and $T(c)$ and leaves $T(b)$ fixed for all T in Γ. Further, no S in Γ can interchange $T(a)$ and $T(b)$, leaving $T(c)$ fixed. For suppose it did; then $T(a) - T(b) = T(c) - T(a)$ and hence $a - b = c - a$ or $3(b - a) = 0$. Since $p > 3$, $b = a$, contradicting of our assumption of distinctness. By a simple argument, no other permutation of a, b and c can occur among the $(T(a), T(b), T(c))$. Hence in this case we can and must say that for any natural finite order $[abc]$ and $[T(a)\ T(b)\ T(c)]$.

https://doi.org/10.4153/CMB-1966-047-6 Published online by Cambridge University Press

If for a triplet (d, e, f) there is a $T \in \Gamma$ such that $T(d) = f$, $T(f) = d$ and $T(e) = e$, then $e - d = f - e$, and there is an H in Γ such that $H(d) = a$, $H(e) = b$ and $H(f) = c$. For, let H be the member of Γ such that $H(d) = a$ and $H(f) = c$. Then $HTH^{-1}(a) = c$ and $HTH^{-1}(c) = a$. Hence $HTH^{-1} = G$, $(G(x) = 2b - x)$ and $HTH^{-1}(b) = b$; T therefore leaves $H^{-1}(b) = e$ invariant. (A member of Γ leaving two points fixed would be the identity.) Axiom O2 is now guaranteed. For, given a and b, let $c = 2b - a$; then $[abc]$.

(ii) Suppose that case (i) is not true, i.e., that no T interchanges a and c while leaving b fixed. Suppose we try to define an intermediacy arbitrarily on a, b and c, requiring, for example, that $[abc]$. Then for every $T \in \Gamma$ we must have $[T(a)\ T(b)\ T(c)]$. Clearly, if there exists a T such that $T(a) = b$, $T(b) = c$, $T(c) = a$, axiom O4 will not be satisfied and we will be unable to define a natural finite order on $Z_p \times Z_p$. If there does not exist such a T, then because Γ is a group, a consistent order has been defined on the class of triplets $\{(T(a), T(b), T(c)); T \in \Gamma\}$. We then choose a triplet outside this class, (d, e, f) say, and define an arbitrary order $[def]$ which will determine an order on the class of triplets $\{(T(d), T(e), T(f)); T \in \Gamma\}$, provided no T permutes d, e and f cyclically. We continue until we find a triplet so permuted under a T in Γ, or until an intermediacy relation has been defined for every triplet. In fact, if the number of classes of triplets of type (ii) is m, there are 3^m possible definitions of order on $Z_p \times Z_p$.

Hence a natural finite order exists if there is no T in Γ which permutes some three points a, b and c of Z_p cyclically and which is therefore of period three. (For such a T, T^3 leaves a, b and c fixed and is therefore the identity.)

If such a T exists, then since Γ is of order $p(p-1)$, 3 must divide $p - 1$, and hence $p \equiv 1 \pmod 3$. Suppose 3 divides $p-1$. Then there exists an $\omega \neq 1$, $\omega \in Z_p$ such that $\omega^3 = 1$, [2, p.112]. Consider the triplet $(1, \omega, \omega^2)$ and the transformation S in Γ defined by $S(x) = \omega x$, which takes 1 into ω, ω into ω^2, and ω^2 into 1. This proves the result.

https://doi.org/10.4153/CMB-1966-047-6 Published online by Cambridge University Press

REFERENCES

1. H.S.M. Coxeter, Introduction to Geometry. Wiley, (1961).

2. B.L. Van der Waerden, Modern Algebra. Vol. 1, Ungar, (1953).

University of Toronto