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GK DIMENSION AND LOCALLY NILPOTENT SKEW DERIVATIONS

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Abstract. Let A be a domain over an algebraically closed field with Gelfand–Kirillov dimension in the interval $[2, 3)$. We prove that if A has two locally nilpotent skew derivations satisfying some natural conditions, then A must be one of five algebras. All five algebras are Noetherian, finitely generated, and have Gelfand–Kirillov dimension equal to 2. We also obtain some results comparing the Gelfand–Kirillov dimension of an algebra to its subring of invariants under a locally nilpotent skew derivation.

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1. Introduction. There have been several recent papers analysing the structure and Gelfand–Kirillov dimension of noncommutative domains with a locally nilpotent derivation [**1**, **2**]. The main result of this paper, Theorem 7, classifies domains with Gelfand–Kirillov dimension in the interval $[2, 3)$ having two locally nilpotent skew derivations satisfying certain compatibility conditions. We now state Theorem 7 without listing all the compatibility conditions.

THEOREM. *Let A be a domain over an algebraically closed field F with $2 \leq \mathrm{GKdim}(A) < 3$. Suppose, for $i = 1, 2$, $\delta_i \neq 0$ are locally nilpotent q_i-skew σ_i-derivations satisfying the compatibility conditions in Theorem 7. If either A is finitely generated or both σ_1 and σ_2 are locally algebraic, then A must be one of the following five algebras:*

1. *the commutative polynomial ring $F[x, y]$,*
2. *the Weyl algebra $F[x, y \mid xy - yx = 1]$,*
3. *the enveloping algebra $F[x, y \mid xy - yx = x]$,*
4. *the quantum plane $F[x, y \mid xy = qyx]$ with $q \neq 1$,*
5. *the quantum Weyl algebra $F[x, y \mid xy - qyx = 1]$ with $q \neq 1$.*

In particular, all five algebras are Noetherian, finitely generated, and have GK dimension equal to 2.

In order to prove Theorem 7, we will first need to prove

THEOREM 2. *Let A be a domain with* $\mathrm{GKdim}(A) < \infty$ *and let* $\delta \neq 0$ *be a locally nilpotent q-skew σ-derivation where* $char_q = 0$. *If either A is finitely generated or σ is locally algebraic, then* $\mathrm{GKdim}(A) - \mathrm{GKdim}(A^\delta) \geq 1$.

We then conclude this paper with two results on ordinary derivations. The first, Theorem 10, extends a well-known result on commutative domains to finitely generated semiprime algebras satisfying a polynomial identity.

THEOREM 10. *Let δ be a locally nilpotent derivation of a finitely generated semiprime algebra A of characteristic* 0 *satisfying a polynomial identity and having finite GK dimension.*

(1) *If δ is nilpotent on an ideal with the same GK dimension as A, then* $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^\delta)$.
(2) *If δ is not nilpotent on every ideal with the same GK dimension as A, then* $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^\delta) + 1$.

Finally, in Theorem 11, we show that the characteristic 0 assumption in Theorem 10 is necessary by proving that, for every derivation of a finitely generated semiprime algebra of characteristic $p > 0$, the invariants have the same GK dimension as the entire algebra.

We will now introduce the results from the literature and the terminology that will be used throughout this paper. Many of these results, as well as the definition of Gelfand–Kirillov dimension, can be found in the excellent book of Krause–Lenagan [**8**].

Throughout, we will let A be an algebra over a field F. We will let $\mathrm{GKdim}(A)$ denote the Gelfand–Kirillov dimension of A as an algebra over F. Observe that $\mathrm{GKdim}(A) = 0$ is equivalent to A being locally finite dimensional over F, meaning that every finitely generated subalgebra of A is finite dimensional. At various points, we may refer to $\mathrm{GKdim}(B)$, where B is an ideal of A. Since the definition of Gelfand–Kirillov dimension requires the algebra have a unit, if necessary, we can think of $\mathrm{GKdim}(B)$ as being a shorthand for $\mathrm{GKdim}(F + B)$.

For any set subset S of a ring, we will let S^* denote its nonzero elements. If h is a function such that $h(S) \subseteq S$, we say that S is h-stable. When $q \in F^*$ and σ is an F-linear automorphism of A, then an F-linear function $\delta : A \to A$ is called a q-skew derivation if

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s) \quad \text{and} \quad \delta(\sigma(r)) = q\sigma(\delta(r)),$$

for all $r, s \in A$. Sometimes δ may simply be referred to as a skew derivation. If σ is the identity map, then δ is an ordinary derivation. We say that σ is locally algebraic if every finite dimensional F-subspace is contained in a finite dimensional σ-stable F-subspace.

For a skew derivation δ, we let

$$A^\delta = \{r \in A \mid \delta(r) = 0\},$$

denote the rings of invariants of A under δ. Observe that since δ is q-skew, σ restricts to an automorphism of A^δ. We say that δ is locally nilpotent if, for every $r \in A$, there exists $n = n(r) \geq 1$ such that $\delta^n(r) = 0$. Another important algebra in examining A, δ and σ will be the skew polynomial ring $A[x; \sigma, \delta]$.

If T is a right Ore set of regular elements of A, we let AT^{-1} denote the right Ore localization obtained by inverting the elements of T. Every element of AT^{-1} can be written in the form at^{-1}, where $a \in A$ and $t \in T$. When A is an Ore domain, we let $Q(A)$ denote the division ring obtained from A be inverting all the nonzero elements in A^*. We will $Z(A)$ denote the centre of A.

If $q \in F^*$, we say that the $char_q = 0$, if $1 + q + \cdots + q^{n-1} \neq 0$, for all $n \in \mathbb{N}$. When $q = 1$, $char_q = 0$ is the same as the ordinary characteristic of F being 0. When $q \neq 1$, $char_q = 0$ is the same as q not being a root of 1. If a, b belong to some F-algebra, then $[a, b]_q = ab - qba$ is called the q-commutator of a and b. When $[a, b]_q = 0$, we say that a and b q-commute.

The following facts will be used frequently throughout this paper.

(1) There is no algebra A such that $0 < \text{GKdim}(A) < 1$ or $1 < \text{GKdim}(A) < 2$.
(2) If B is a subalgebra or homomorphic image of A, then $\text{GKdim}(B) \leq \text{GKdim}(A)$.
(3) If $A = A_1 \oplus \cdots \oplus A_n$, then $\text{GKdim}(A) = \max_i \text{GKdim}(A_i)$.
(4) If $I_1, \ldots, I_n$ are ideals of A and $I = \bigcap_{i=1}^n I_i$, then $\text{GKdim}(A/I) = \max_i \text{GKdim}(A/I_i)$.
(5) If T is a multiplicatively closed set of regular elements in $Z(A)$, then $\text{GKdim}(AT^{-1}) = \text{GKdim}(A)$.
(6) If B is a subalgebra of a finitely generated domain A such that $\text{GKdim}(A) < \text{GKdim}(B) + 1 < \infty$, then B^* is a left and right Ore set in A such that $Q(B)$ is both the left and right Ore localization of B at B^*, $Q(A)$ is both the left and right Ore localization of A at B^*, and $Q(A)$ is finite dimensional as both a right and left vector space over $Q(B)$.
(7) If B is a subalgebra of A such that A is finitely generated as a left or right B-module, then $\text{GKdim}(A) = \text{GKdim}(B)$.
(8) If $\text{GKdim}(A) = 1$, then every finitely generated subalgebra of A satisfies a polynomial identity. In addition, if A is a domain and the base field is algebraically closed, then A is commutative.
(9) If $\text{GKdim}(A) < \infty$ then $\text{GKdim}(A[x]) = \text{GKdim}(A) + 1$.
(10) If δ is a σ-derivation of A, then $\text{GKdim}(A[x; \sigma, \delta]) \geq \text{GKdim}(A) + 1$.

Item (1) is frequently referred to as the Bergman gap theorem (G. M. Bergman, A note on growth functions of algebras and semigroups, Research Note, University of California, Berkeley, 1978, unpublished mimeographed notes (unpublished data).). Item (6) is due to Borho–Kraft [**4**]. The first part of item (8) is due to Small–Warfield [**10**]. Tsen's theorem asserts that if F is algebraically closed and $F(x)$ is a rational function field over F, then any division ring which has centre $F(x)$ and is finite dimensional over $F(x)$ is equal to $F(x)$. The second part of (8) follows by applying the first part of (8) along with Tsen's theorem. Item (10) is due to Huh and Kim [**6**].

2. Locally nilpotent skew derivations. We will use the following lemma several times to guarantee that various locally nilpotent q-skew derivations are not nilpotent.

LEMMA 1. *Let A be a domain and $\delta \neq 0$ a q-skew derivation where $char_q = 0$. Then δ is not nilpotent*

Proof. By way of contradiction, suppose δ is nilpotent and let n be the index of nilpotence of δ. If $n > 1$ and $r, s \in A$ consider

$$0 = \delta^n(r\delta^{n-2}(s)) = (1 + q + \cdots + q^{n-1})\sigma(\delta^{n-1}(r))\delta^{n-1}(s).$$

Since $char_q = 0$, we have $1 + q + \cdots + q^{n-1} \neq 0$, and it follows that

$$\sigma(\delta^{n-1}(A))\delta^{n-1}(A) = \delta^{n-1}(\sigma(A))\delta^{n-1}(A) = \delta^{n-1}(A)\delta^{n-1}(A) = 0.$$

Therefore $\delta^{n-1} = 0$ which immediately leads to the contradiction $\delta = 0$. □

Our first main result relies heavily on item (6) which is a result of Borho–Kraft [**4**].

THEOREM 2. *Let A be a domain with* $\mathrm{GKdim}(A) < \infty$ *and let $\delta \neq 0$ be a locally nilpotent q-skew σ-derivation where $char_q = 0$. If either A is finitely generated or σ is locally algebraic, then* $\mathrm{GKdim}(A) - \mathrm{GKdim}(A^\delta) \geq 1$.

Proof. Suppose not; then $\mathrm{GKdim}(A) - \mathrm{GKdim}(A^\delta) < 1$. Since $\delta \neq 0$, there exists $a \in A$ such that $\delta(a) \neq 0$ and $\delta^2(a) = 0$. We first consider the case where A is not finitely generated. Observe that $\mathrm{GKdim}(A) - \mathrm{GKdim}(A^\delta) = 1 - \epsilon$, for some $\epsilon > 0$. Since σ is locally algebraic, there exists a finitely generated σ-stable subalgebra B contained in A^δ such that $\mathrm{GKdim}(A^\delta) - \mathrm{GKdim}(B) < \frac{\epsilon}{2}$. Hence, $\mathrm{GKdim}(A) - \mathrm{GKdim}(B) < 1$. Furthermore, B and a are contained in a finitely generated subalgebra C of A which is stable under both σ and δ. Thus $\mathrm{GKdim}(C) - \mathrm{GKdim}(B) < 1$.

By the result of Borho–Kraft [**4**], $Q(C)$ is obtained from C by inverting the nonzero elements of B and $Q(C)$ is finite dimensional as both a left and right vector space over $Q(B)$. Since B is σ-stable, we can extend δ and σ uniquely to $Q(C)$ by letting $\sigma(t^{-1}) = \sigma(t)^{-1}$ and $\delta(t^{-1}) = -\sigma(t)^{-1}\delta(t)t^{-1}$, for all $0 \neq t \in B$. The elements of B belong to the kernel of δ, therefore the extension of δ to $Q(C)$ remains locally nilpotent.

For any $i \geq 1$, δ^i is a right $Q(B)$-module map of $Q(C)$. However, since $Q(C)$ is finite dimensional over $Q(B)$, the powers of δ are linearly dependent over $Q(B)$ on the left. Therefore, there exists $a_s, \ldots, a_t \in Q(B)$, with $s \leq t$ and $a_s \neq 0$, such that $a_s\delta^s + \cdots + a_t\delta^t = 0$.

Since $\delta(a) \neq 0$, we know that $\delta \neq 0$ on $Q(C)$. Although δ is locally nilpotent, Lemma 1 asserts that δ is not nilpotent, Therefore there exists $c \in Q(C)$ such that $\delta^s(c) \neq 0$ but $\delta^{s+1}(c) = 0$. As a result,

$$0 = (a_s\delta^s + \cdots + a_t\delta^t)(c) = a_s\delta^s(c).$$

But this is a contradiction as both a_s and $\delta^s(c)$ are nonzero.

In the argument above, since A was not necessarily finitely generated, it was necessary to construct C and B in order to apply the Borho–Kraft result. If A is finitely generated, we can immediately apply the Borho–Kraft result and localize A and A^δ at $(A^\delta)^*$ to obtain $Q(A)$ and $Q(A^\delta)$. The powers of δ are now linearly dependent over $Q(A^\delta)$ and this leads to the same contradiction as above. □

We now begin the work necessary to prove Theorem 7.

LEMMA 3. *Let A be a domain and let $\delta_i \neq 0$ be q_i-skew σ_i-skew derivations, for $i = 1, 2$ such that $\sigma_1\sigma_2 = \sigma_2\sigma_1$ and $A^{\delta_1} \neq A^{\delta_2}$. If $q \in F^*$ and $\sigma = \sigma_1\sigma_2$, then $[\delta_1, \delta_2]_q$ is a σ-derivation if and only if $[\delta_1, \sigma_2]_q = [\sigma_1, \delta_2]_q = 0$.*

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Proof. If $r, s \in A$, we have

$$(\delta_1\delta_2)(rs) = \delta_1(\delta_2(r)s + \sigma_2(r)\delta_2(s))$$
$$= \delta_1(\delta_2(r))s + \sigma_1(\delta_2(r))\delta_1(s) + \delta_1(\sigma_2(r))\delta_2(s) + \sigma_1(\sigma_2(r))\delta_1(\delta_2(s)) \quad (1)$$

and

$$(\delta_2\delta_1)(rs) = \delta_2(\delta_1(r)s + \sigma_1(r)\delta_1(s))$$
$$= \delta_2(\delta_1(r))s + \sigma_2(\delta_1(r))\delta_2(s) + \delta_2(\sigma_1(r))\delta_1(s) + \sigma_2(\sigma_1(r))\delta_2(\delta_1(s)). \quad (2)$$

If we let σ replace $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$ and then subtract q times the second equation from the first, we obtain

$$[\delta_1, \delta_2]_q(rs) = [\delta_1, \delta_2]_q(r)s + \sigma(r)[\delta_1, \delta_2]_q(s) + [\sigma_1, \delta_2]_q(r)\delta_1(s) + [\delta_1, \sigma_2]_q(r)\delta_2(s). \quad (3)$$

In one direction, if $[\delta_1, \sigma_2]_q = [\sigma_1, \delta_2]_q = 0$, then equation (1) reduces to

$$[\delta_1, \delta_2]_q(rs) = [\delta_1, \delta_2]_q(r)s + \sigma(r)[\delta_1, \delta_2]_q(s).$$

Thus $[\delta_1, \delta_2]_q$ is a σ-derivation.

In the other direction, if $[\delta_1, \delta_2]_q$ is a σ-derivation, then equation (1) simplifies to

$$0 = [\sigma_1, \delta_2]_q(r)\delta_1(s) + [\delta_1, \sigma_2]_q(r)\delta_2(s). \quad (4)$$

Since $A^{\delta_1} \neq A^{\delta_2}$, either $A^{\delta_1} \not\subset A^{\delta_2}$ or $A^{\delta_2} \not\subset A^{\delta_1}$. If $A^{\delta_1} \not\subset A^{\delta_2}$, let $a \in A$ such that $\delta_1(a) = 0$ and $\delta_2(a) \neq 0$. Replacing s by a in equation (2) give us

$$0 = [\delta_1, \sigma_2]_q(r)\delta_2(a).$$

Since $\delta_2(a) \neq 0$ and A is a domain, it follows that $[\delta_1, \sigma_2]_q = 0$. Therefore equation (2) becomes

$$0 = [\sigma_1, \delta_2]_q(r)\delta_2(s).$$

By choosing some $s \notin A^{\delta_2}$, we see that $[\sigma_1, \delta_2]_q = 0$.

An identical argument holds if we instead assume that $A^{\delta_2} \not\subset A^{\delta_1}$, concluding the proof. □

The following result and its proof appear as part of Theorem 1 in [**3**].

LEMMA 4. *Let R be an algebra with a q-skew σ-derivation δ which is locally nilpotent such that $\delta(x) = 1$, for some $x \in R$, and $char_q = 0$. Then δ is surjective and R is the skew polynomial ring $R^\delta[x; \sigma^{-1}, d]$, where d is a σ^{-1}-derivation of R^δ.*

We continue with

LEMMA 5. *Let A be a domain over an algebraically closed field F such that $2 \leq$ GKdim$(A) < 3$. For $i = 1, 2$, let $\delta_i \neq 0$ be locally nilpotent q_i-skew σ_i-derivations such that*

(1) *δ_1 and δ_2 q-commute, for some $q \in F^*$,*
(2) *$char_{q_i} = 0$, for $i = 1, 2$,*

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(3) $\sigma_1\sigma_2 = \sigma_2\sigma_1$, *and*
(4) $A^{\delta_1} \neq A^{\delta_2}$.

If either A is finitely generated or both σ_i are locally algebraic, then

(1) $\mathrm{GKdim}(A^{\delta_1}) = \mathrm{GKdim}(A^{\delta_2}) = 1$,
(2) A^{δ_1} *and* A^{δ_2} *are commutative, and*
(3) $A^{\delta_1} \cap A^{\delta_2} = F$.

Proof. Let δ be either δ_1 or δ_2 and let σ be the automorphism associated to δ. Recall that both δ and σ are F-linear. By Theorem 2, $\mathrm{GKdim}(A) - \mathrm{GKdim}(A^{\delta}) \geq 1$, therefore by the Bergman Gap Theorem, $\mathrm{GKdim}(A^{\delta}) = 1$ or $\mathrm{GKdim}(A^{\delta}) = 0$.

To show that $\mathrm{GKdim}(A^{\delta}) = 1$, by way of contradiction, let us suppose that $\mathrm{GKdim}(A^{\delta}) = 0$; thus $A^{\delta} = F$. Since F is a field, there exists $x \in A$ such that $\delta(x) = 1$. We can now apply Lemma 4, to assert that A is the skew polynomial ring $F[x; \sigma^{-1}, d]$, where d is a σ^{-1}-derivation of F. However, $F \subseteq Z(A)$, therefore x commutes with F and we have $A = F[x; \sigma^{-1}, d] = F[x]$. Item (9) now implies that $\mathrm{GKdim}(A) = \mathrm{GKdim}(F[x]) = \mathrm{GKdim}(F) + 1 = 1$, a contradiction. As a result, $\mathrm{GKdim}(A^{\delta_1}) = \mathrm{GKdim}(A^{\delta_2}) = 1$. In addition, since F is algebraically closed, item (8) implies that A^{δ_1} and A^{δ_2} are commutative.

To conclude the proof and show that $A^{\delta_1} \cap A^{\delta_2} = F$, it suffices to show that $\mathrm{GKdim}(A^{\delta_1} \cap A^{\delta_2}) = 0$. Since $A^{\delta_1} \neq A^{\delta_2}$, either $A^{\delta_1} \not\subset A^{\delta_2}$ or $A^{\delta_1} \not\subset A^{\delta_2}$. Without loss of generality, we may assume that $A^{\delta_1} \not\subset A^{\delta_2}$. Since δ_1 and δ_2 q-commute, A^{δ_1} is stable under δ_2. Therefore there exists $a \in A^{\delta_1}$ such that $0 \neq \delta_2(a) \in A^{\delta_1} \cap A^{\delta_2}$. If $b = \delta_2(a)$, we can let B and C be localizations of A^{δ_1} and $A^{\delta_1} \cap A^{\delta_2}$, respectively, obtained by inverting the powers of b. Since $b \in A^{\delta_2}$, the extension of δ_2 to B remains locally nilpotent. However $ab^{-1} \in B$ and $\delta_2(ab^{-1}) = 1$. If we let $x = ab^{-1}$, we can once again apply Lemma 4 to now assert that B is the skew polynomial ring $B^{\delta_2}[x; \sigma_2^{-1}, d]$, where d is a σ_2^{-1}-derivation of B^{δ_2}. As a result,

$$\mathrm{GKdim}(B) = \mathrm{GKdim}(B^{\delta_2}[x; \sigma_2^{-1}, d]) \geq \mathrm{GKdim}(B^{\delta_2}) + 1.$$

However $C = B^{\delta_2}$ and both B and C are obtained by inverting central elements, therefore

$$1 = \mathrm{GKdim}(A^{\delta_1}) = \mathrm{GKdim}(B) = \mathrm{GKdim}(C[x; \sigma_2^{-1}, d])$$

$$\geq \mathrm{GKdim}(C) + 1 = \mathrm{GKdim}(A^{\delta_1} \cap A^{\delta_2}) + 1.$$

Thus $\mathrm{GKdim}(A^{\delta_1} \cap A^{\delta_2}) = 0$. □

The next lemma is the final piece needed to complete the classification in Theorem 7.

LEMMA 6. *Let A be an algebra over a field F generated by x, y with the relation $xy - qyx = \alpha x + \beta y + \gamma$, where $q, \alpha, \beta, \gamma \in F$ and $q \neq 0$. Then A must be a homomorphic image one of the following five algebras:*

(1) *the commutative polynomial ring* $F[x, y]$,
(2) *the Weyl algebra* $F[x, y \mid xy - yx = 1]$,
(3) *the enveloping algebra* $F[x, y \mid xy - yx = x]$,
(4) *the quantum plane* $F[x, y \mid xy = qyx]$ *with* $q \neq 1$,
(5) *the quantum Weyl algebra* $F[x, y \mid xy - qyx = 1]$ *with* $q \neq 1$.

Proof. We begin with the special case where $\alpha = \beta = 0$; thus $xy - qyx = \gamma$. If $\gamma = 0$, then $xy - qyx = 0$. A may have additional relations, therefore A is the homomorphic image of either the commutative polynomial ring in two variables or the quantum plane depending upon whether $q = 1$ or $q \neq 1$. If $\gamma \neq 0$, then if we let $y' = y\gamma^{-1}$, we see that A is generated by x, y' with the relation $xy' - qy'x = 1$. Next, if $q \neq 1$ then, since A may have other relations, A is a homomorphic image of the quantum Weyl algebra. However, if $q = 1$ then, since x and y' generate a simple algebra, A must be isomorphic to the Weyl algebra.

Next, we consider the case where $q \neq 1$. If we let $x' = x - \beta(1-q)^{-1}$ and $y' = y - \alpha(1-q)^{-1}$, then A is generated by x', y' with the relation $x'y' - qy'x' = \gamma' \in F$. Therefore we have reduced to the cases covered in the previous paragraph.

Finally, we consider the case where $q = 1$ and at least one of α, β is nonzero. Observe that we can rewrite $xy - yx = \alpha x + \beta y + \gamma$ as $yx - xy = -\beta y - \alpha x - \gamma$, essentially switching the roles of x and y. Therefore, in order to determine the structure of A, it suffices to consider the case where $\alpha \neq 0$. In this case, if we let $x' = x + y\beta\alpha^{-1} + \gamma\alpha^{-1}$ and $y' = y\alpha^{-1}$, then A is generated by x', y' with the relation $x'y' - y'x' = x'$. Since A may have other relations, A must be the homomorphic image of the enveloping algebra of the two-dimensional non-nilpotent Lie algebra. □

We can now prove the main result of this paper.

THEOREM 7. *Let A be a domain over an algebraically closed field F with $2 \leq \mathrm{GKdim}(A) < 3$. Suppose, for $i = 1, 2$, $\delta_i \neq 0$ are locally nilpotent q_i-skew σ_i-derivations with the following properties:*

(1) *δ_1 and δ_2 q-commute, for some $q \in F^*$,*
(2) *$char_{q_i} = 0$, for $i = 1, 2$,*
(3) *$\sigma_1\sigma_2 = \sigma_2\sigma_1$, and*
(4) *$A^{\delta_1} \neq A^{\delta_2}$.*

If either A is finitely generated or both σ_1 and σ_2 are locally algebraic, then A must be one of the following five algebras:

(1) *the commutative polynomial ring $F[x, y]$,*
(2) *the Weyl algebra $F[x, y \mid xy - yx = 1]$,*
(3) *the enveloping algebra $F[x, y \mid xy - yx = x]$,*
(4) *the quantum plane $F[x, y \mid xy = qyx]$ with $q \neq 1$,*
(5) *the quantum Weyl algebra $F[x, y \mid xy - qyx = 1]$ with $q \neq 1$.*

In particular, all five algebras are Noetherian, finitely generated, with GK dimension equal to 2.

Proof. We will begin by showing that A is a homomorphic image of one of the algebras (1)–(5) above. In light of Lemma 6, it suffices to show that A is generated over F by some x, y with the relation $xy - qyx = \alpha x + \beta y + \gamma$, where $\alpha, \beta, \gamma \in F$. By Lemma 5, A^{δ_1} and A^{δ_2} are commutative domains with GK dimension 1, whose intersection is the field F. Furthermore, by Lemma 3, $[\delta_1, \sigma_2]_q = [\sigma_1, \delta_2]_q = 0$, hence A^{δ_1} and A^{δ_2} are both stable under $\delta_1, \delta_2, \sigma_1, \sigma_2$. By Lemma 5(3), the restrictions of δ_1 and δ_2 to A^{δ_2} and A^{δ_1}, respectively, are nonzero locally nilpotent skew derivations.

Since, the field F is the invariants of the locally nilpotent actions of δ_1 on A^{δ_2} and of δ_2 on A^{δ_1}, there exists $x \in A^{\delta_2}, y \in A^{\delta_1}$ such that $\delta_1(x) = 1, \delta_2(y) = 1$. By Lemma 4, A^{δ_1} is a skew polynomial ring over F generated by y and A^{δ_2} is a skew polynomial

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ring over F generated by x. However, both A^{δ_1} and A^{δ_2} are commutative rings and are therefore ordinary polynomial rings. Thus $A^{\delta_1} = F[y]$ and $A^{\delta_2} = F[x]$.

We can also apply Lemma 4 to the actions of δ_1 and δ_2 on A to represent A in two different ways as skew polynomials rings. In particular, we have $A = A^{\delta_1}[x; \sigma_1^{-1}, d_1]$ and $A = A^{\delta_2}[y; \sigma_2^{-1}, d_2]$, where d_1 is a σ_1^{-1}-derivation of A^{δ_1} and d_2 is a σ_2^{-1}-derivation of A^{δ_2}. Combining this with that facts that $A^{\delta_1} = F[y]$ and $A^{\delta_2} = F[x]$, we now have $A = F[y][x; \sigma_1^{-1}, d_1] = F[x][y; \sigma_2^{-1}, d_2]$. Looking at our two representations of A as skew polynomials rings, we that A is generated over F by x, y subject to the relations

$$xy = \sigma_1^{-1}(y)x + f(y) \quad \text{and} \quad yx = \sigma_2^{-1}(x)y + g(x), \tag{5}$$

where $f(y) \in F[y]$ and $g(x) \in F[x]$.

Next, we need to compute $\sigma_1^{-1}(y)$ and $\sigma_2^{-1}(x)$. By Lemma 3, $[\delta_1, \sigma_2]_q = [\sigma_1, \delta_2]_q = 0$, which implies that

$$\sigma_2^{-1}(\delta_1(x)) = q\delta_1(\sigma_2^{-1}(x)) \quad \text{and} \quad \sigma_1^{-1}(\delta_2(y)) = q^{-1}\delta_2(\sigma_1^{-1}(y)).$$

Since $\delta_1(x) = 1$, $\delta_2(y) = 1$, the previous equations imply that

$$\delta_1(\sigma_2^{-1}(x)) = q^{-1} = \delta_1(q^{-1}x) \quad \text{and} \quad \delta_2(\sigma_1^{-1}(y)) = q = \delta_2(qy).$$

As a result,

$$\sigma_2^{-1}(x) - q^{-1}x \in A^{\delta_1} = F[y] \quad \text{and} \quad \sigma_1^{-1}(y) - qy \in A^{\delta_2} = F[x].$$

However, $A^{\delta_1} = F[y]$ is σ_2-stable and $A^{\delta_2} = F[x]$ is σ_1-stable. Therefore, we now have

$$\sigma_2^{-1}(x) - q^{-1}x \in F[x] \cap F[y] = F \quad \text{and} \quad \sigma_1^{-1}(y) - qy \in F[x] \cap F[y] = F.$$

Thus

$$\sigma_2^{-1}(x) = q^{-1}x + a \quad \text{and} \quad \sigma_1^{-1}(y) = qy + b,$$

for some $a, b \in F$.

Substituting the values of $\sigma_2^{-1}(x)$ and $\sigma_1^{-1}(y)$ into equation (3), we now have

$$xy = (qy + b)x + f(y) \quad \text{and} \quad yx = (q^{-1}x + a)y + g(x).$$

These equations can be rewritten as

$$xy - qyx = bx + f(y) \quad \text{and} \quad xy - qyx = -qay - qg(x).$$

As a result,

$$bx + f(y) = -qay - qg(x),$$

hence

$$bx + qg(x) = -qay - f(y) \in F[x] \cap F[y] = F.$$

Therefore $bx + qg(x) = -qay - f(y) = c \in F$ and both earlier equations now reduce to

$$xy - qyx = bx - qay - c.$$

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Thus A is indeed generated over F by x, y with a relation of the form $xy - qyx = \alpha x + \beta y + \gamma$, where $\alpha, \beta, \gamma \in F$.

We now know that A is isomorphic to B/P, where B is one of the algebras (1)–(5) and P is a prime ideal of B. Observe that B is a domain with $\mathrm{GKdim}(B) = 2$. If $P \neq 0$, then P contains a regular element of B and Proposition 3.15 of [**8**] shows that $\mathrm{GKdim}(A) + 1 \leq \mathrm{GKdim}(B)$. Thus $\mathrm{GKdim}(A) \leq 1$, contradicting our assumption that $2 \leq \mathrm{GKdim}(A) < 3$. As a result, $P = 0$ and A must be one of the algebras (1)–(5). Observe that all five algebras obtained in this classification are easily seen to be Noetherian, finitely generated, and have GK dimension equal to 2. □

The following examples illiustrate that all five algebras described in Theorem 7 do have two locally nilpotent skew derivations with the properties described in the theorem.

EXAMPLE 8. The five algebras described in Theorem 7 all have locally nilpotent skew derivations δ_1, δ_2 with the properties described in Theorem 7.

Case 1: $A = F[x, y \mid xy - qyx = 0]$. Since we are allowing $q = 1$, algebras (1) and (4) from Theorem 7 are both described in this manner. Next, suppose $q_1, q_2 \in F$ such that $char_{q_i} = 0$, for $i = 1, 2$. We can now define q_i-skew σ_i-derivations δ_i as follows:

$$\delta_1(x) = 1, \quad \delta_1(y) = 0, \quad \sigma_1(x) = q_1 x, \quad \sigma_1(y) = q^{-1} y$$

$$\delta_2(x) = 0, \quad \delta_2(y) = 1, \quad \sigma_2(x) = qx, \quad \sigma_2(y) = q_2 y.$$

In this case, $\delta_1, \delta_2, \sigma_1, \sigma_2$ extend uniquely to all of A such that δ_1, δ_2 are locally nilpotent and satisfy all the properties listed in Theorem 7.

Case 2: $A = F[x, y \mid xy - qyx = 1]$. Since we are allowing $q = 1$, algebras (2) and (5) from Theorem 7 are both described in this manner. In this case, we let δ_1, δ_2 be defined as in Case 1. However in order for σ_1 and σ_2 to preserve the relation $xy - qyx = 1$, we need $q = q_1 = {q_2}^{-1}$. Therefore, in the example, δ_1 is q-skew and δ_2 is q^{-1}-skew.

Case 3: $A = F[x, y \mid xy - yx = x]$. This case covers algebra (3) from Theorem 7. We once again define δ_1, δ_2 as in Case 1. However, in this case, we define σ_1, σ_2 as

$$\sigma_1(x) = q_1 x, \quad \sigma_1(y) = y - 1, \quad \sigma_2(x) = x, \quad \sigma_2(y) = y + a,$$

where $a \in F$. In this example, $q_2 = 1$, hence δ_2 commutes with both δ_1 and σ_2. However, δ_1 is q_1-skew for any q_1 with $char_{q_i} = 0$. Observe that δ_2 is an ordinary derivation if and only if $a = 0$.

3. Derivations. Let A be an algebra of characteristic 0 with $\mathrm{GKdim}(A) < \infty$ and let δ be an ordinary derivation of A which is locally nilpotent. The question of whether $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^{\delta}) + 1$ is open, even if A is a domain. It is well-known that the answer is yes if A is a commutative domain and the goal of Theorems 9 and 10 is to extend this to prime rings satisfying a polynomial identity and then to finitely generated semiprime rings satisfying a polynomial identity.

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THEOREM 9. *Let δ be a locally nilpotent derivation of a prime algebra A of characteristic* 0 *satisfying a polynomial identity and having finite GK dimension.*

(1) *If δ vanishes on $Z(A)$, then* $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^\delta)$.
(2) *If δ does not vanish on $Z(A)$, then* $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^\delta) + 1$.

Proof. Since A is prime and satisfies a polynomial identity, $A(Z(A)^*)^{-1}$ is a simple algebra which is finite dimensional over the field $Z(A)(Z(A)^*)^{-1}$. Therefore

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(A(Z(A)^*)^{-1}) = \mathrm{GKdim}(Z(A)(Z(A)^*)^{-1}) = \mathrm{GKdim}(Z(A)).$$

If δ vanishes on $Z(A)$, we have $Z(A) \subseteq A^\delta$. Therefore $\mathrm{GKdim}(Z(A)) \leq \mathrm{GKdim}(A^\delta)$, hence

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(Z(A)) \leq \mathrm{GKdim}(A^\delta) \leq \mathrm{GKdim}(A),$$

proving (1).

If δ does not vanish on $Z(A)$, it restricts to a locally nilpotent derivation of $Z(A)$ which, by Lemma 1, is not nilpotent. Therefore, there exists $y \in Z(A)$ such that $\delta(y) \neq 0$ and $\delta^2(y) = 0$. Let $\alpha = \delta(y)$ and then let S and T be the localizations of A and A^δ, respectively, obtained by inverting the powers of α. Then δ remains locally nilpotent on S and $S^\delta = T$. Since α is central, $\mathrm{GKdim}(S) = \mathrm{GKdim}(A)$ and $\mathrm{GKdim}(T) = \mathrm{GKdim}(A^\delta)$.

Next, let $x = y\alpha^{-1}$; then $x \in Z(S)$ and $d(x) = 1$. By Lemma 4, S is the skew polynomial ring $S^\delta[x; d]$, for some derivation d of S^δ. However, x is central, therefore $d = 0$ and $S = S^\delta[x]$. Therefore, by item (9), $\mathrm{GKdim}(S) = \mathrm{GKdim}(S^\delta) + 1$ and we now have

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(S) = \mathrm{GKdim}(S^\delta) + 1 = \mathrm{GKdim}(T) + 1 = \mathrm{GKdim}(A^\delta) + 1,$$

proving (2). □

We now extend the previous result from prime rings to finitely generated semiprime rings.

THEOREM 10. *Let δ be a locally nilpotent derivation of a finitely generated semiprime algebra A of characteristic* 0 *satisfying a polynomial identity and having finite GK dimension.*

(1) *If δ is nilpotent on an ideal with the same GK dimension as A, then* $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^\delta)$.
(2) *If δ is not nilpotent on every ideal with the same GK dimension as A, then* $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^\delta) + 1$.

Proof. Since A is semiprime, finitely generated, and satisfies a polynomial identity, it has a finite number of minimal primes ideals and is therefore Goldie, see 13.4.4 and 13.6.9 in **[9]**. Let $P_1, \ldots, P_n$ be the minimal primes of A and, for each i, let A_i be the right annihilator of P_i of A. Since A has characteristic 0, by **[7]**, each P_i is δ-stable. Observe that each A_i, when viewed as a ring, is a prime ring and is also δ-stable.

Let $B = A_1 + \cdots + A_n$; we claim that this sum of ideals is actually a direct sum. If not, without loss of generality, we may assume that there exists $a_i \in A_i$ such that $0 \neq a_1 = a_2 + \cdots + a_n$. Since $a_iP_i = 0$, it follows that

$$a_1P_2 \cdots P_n = (a_2 + \cdots + a_n)P_2 \cdots P_n = 0.$$

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Since $A_1 \cap P_1 = 0$, we know that $a_1 \notin P_1$. Therefore, since P_1 is prime, $P_j \subseteq P_1$, for some $j > 1$. However, $P_j \neq P_1$, thus P_j is properly contained in P_1 which contradicts that P_1 is a minimal prime.

Next, we claim that $B = A_1 \oplus \cdots \oplus A_n$ is essential as both a left and right ideal of A. If $aB = 0$, then $aA_i = 0$, for all i. Each P_i is prime and $A_i \not\subseteq P_i$, therefore $a \in \bigcap_{i=1}^{n} P_i = 0$. Thus B is a two-sided ideal with zero left annihilator in a semiprime ring and it immediately follows that B is essential as both a left and right ideal.

To prove part (1), suppose δ is nilpotent on an ideal I of A such that $\operatorname{GKdim}(A) = \operatorname{GKdim}(I)$. Since $\bigcap_{i=1}^{n} P_i = 0$, we have

$$\operatorname{GKdim}(A) = \operatorname{GKdim}(I) = \max_i \operatorname{GKdim}(I + P_i/P_i).$$

Therefore, there is a minimal prime P_m such that $\operatorname{GKdim}(A) = \operatorname{GKdim}(I + P_m/P_m)$. Derivations of prime rings which are nilpotent on a nonzero ideal are nilpotent on the entire ring [**5**]. Since δ is nilpotent on $I + P_m/P_m$, it now follows that δ is also nilpotent on A/P_m, hence must be nilpotent on A_m/P_m. Therefore, for some t, $\delta^t(A_m) \subseteq A_m \cap P_m = 0$. Thus δ is nilpotent on A_m and Lemma 1 now implies that δ vanishes on the domain $Z(A_m)$. Applying Theorem 9, we have

$$\operatorname{GKdim}(A_m) = \operatorname{GKdim}((A_m)^\delta).$$

All nonzero ideals of a prime ring satisfying a polynomial identity produce the same quotient ring when localized at their centre, therefore they all have the same GK dimension. Therefore $\operatorname{GKdim}(A_m/P_m) = \operatorname{GKdim}(I + P_m/P_m)$ and we have

$$\operatorname{GKdim}(A^\delta) \geq \operatorname{GKdim}((A_m)^\delta) = \operatorname{GKdim}(A_m) \geq \operatorname{GKdim}(A_m/P_m)$$

$$= \operatorname{GKdim}(I + P_m/P_m) = \operatorname{GKdim}(A).$$

The above inequalities immediately imply that $\operatorname{GKdim}(A) = \operatorname{GKdim}(A^\delta)$, proving (1).

To prove part (2), for any i, $Z(A_i)$ is δ-stable. Therefore there exists some nonzero $t_i \in Z(A_i) \cap A^\delta$. Since t_i is in the centre of the prime ring A_i, t_i is regular in A_i. Thus if we let $t = t_1 + \cdots + t_n$, then t is central and regular in B. However, B is essential, thus t is also central and regular in A. Next, we let AT^{-1} $(A^\delta)T^{-1}$, BT^{-1} and $(B^\delta)T^{-1}$ denote, respectively, the localizations of A, A^δ, B, and B^δ at the powers of t. If $a \in A$, we have $a = (at)t^{-1} \in BT^{-1}$, thus $AT^{-1} = BT^{-1}$. Furthermore, since t is a regular element of A belonging to B^δ, it also follows that

$$(A^\delta)T^{-1} = (AT^{-1})^\delta = (BT^{-1})^\delta = (B^\delta)T^{-1}.$$

Since we are localizing at central elements, we have

$$\operatorname{GKdim}(A^\delta) = \operatorname{GKdim}((A^\delta)T^{-1}) = \operatorname{GKdim}((B^\delta)T^{-1}) = \operatorname{GKdim}(B^\delta).$$

Observe that

$$B^\delta = (A_1)^\delta \oplus \cdots \oplus (A_n)^\delta.$$

Therefore, $\operatorname{GKdim}(B^\delta) = \max_i \operatorname{GKdim}((A_i)^\delta)$. Given i, one possibility is that $\operatorname{GKdim}(A_i) < \operatorname{GKdim}(A)$. The GK dimension of any semiprime ring satisfying a polynomial identity is an integer. Therefore, in this case, we have $\mathrm{GKdim}(A_i) \leq \mathrm{GKdim}(A) - 1$. Thus

$$\mathrm{GKdim}((A_i)^\delta) \leq \mathrm{GKdim}(A_i) \leq \mathrm{GKdim}(A) - 1.$$

The other possibility is that $\mathrm{GKdim}(A_i) = \mathrm{GKdim}(A)$. Since we are trying to prove (2), we know that δ is not nilpotent on A_i. Since A_i is prime and satisfies a polynomial identity, if we let $S = A(Z(A)^*)^{-1}$, then S is simple and finite dimensional over its centre. If δ vanished on $Z(A_i)$, the extension of δ to S would be both algebraic and locally nilpotent. contradicting that it is not nilpotent on A_i. Thus δ cannot vanish on $Z(A_i)$ and Theorem 9 now asserts that $\mathrm{GKdim}((A_i)^\delta) = \mathrm{GKdim}(A_i) - 1$, hence $\mathrm{GKdim}((A_i)^\delta) = \mathrm{GKdim}(A) - 1$.

Since there exists some j where $\mathrm{GKdim}(A_j) = \mathrm{GKdim}(A)$, the arguments above combine to show that $\max_i \mathrm{GKdim}((A_i)^\delta) = \mathrm{GKdim}(A) - 1$. Hence,

$$\mathrm{GKdim}(B^\delta) = \max_i \mathrm{GKdim}((A_i)^\delta) = \mathrm{GKdim}(A) - 1.$$

However, $\mathrm{GKdim}(A^\delta) = \mathrm{GKdim}(B^\delta)$, therefore whenever δ is not nilpotent on every ideal with the same GK dimension as A, we have $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^\delta) + 1$, proving (2). □

The result in Theorem 10 cannot be extended to the characteristic $p > 0$ case. In fact, we conclude this paper by showing that in the characteristic p case, A and A^δ have the same GK dimension for every derivation δ.

THEOREM 11. *Let δ be a derivation of a finitely generated semiprime algebra A of characteristic $p > 0$ satisfying a polynomial identity and having finite GK dimension. Then* $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^\delta)$.

Proof. We will first consider the case where A is prime. As in the proof of Theorem 9, since A satisfies a polynomial identity, A and $Z(A)$ have the same GK dimension. If L and K are the quotient fields of $Z(A)$ and $Z(A)^\delta$, respectively, then $\mathrm{GKdim}(L) = \mathrm{GKdim}(Z(A))$ and $\mathrm{GKdim}(K) = \mathrm{GKdim}(Z(A)^\delta)$. Since A has characteristic p and L is commutative, for every $\alpha \in L$, we have $\alpha^p \in K$. Therefore L is algebraic over K and since every algebraic extension of a field has the same GK dimension as the smaller field, we have $\mathrm{GKdim}(L) = \mathrm{GKdim}(K)$. Combining all of the above, we have

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(Z(A)) = \mathrm{GKdim}(L) = \mathrm{GKdim}(K) = \mathrm{GKdim}(Z(A)^\delta)$$

$$\leq \mathrm{GKdim}(A^\delta) \leq \mathrm{GKdim}(A),$$

proving the prime case.

For the semiprime case, as in the proof of Theorem 10, we can let $P_1, \ldots, P_n$ be the minimal primes of A and let A_i be the annihilator of P_i. Since A has characteristic p, the minimal primes need not be δ-stable. However, for each i, $\delta({P_i}^2) \subseteq P_i$ and

$$0 = \delta(A_i(P_i)^2) \subseteq \delta(A_i)(P_i)^2 + A_i\delta({P_i}^2) = \delta(A_i)(P_i)^2.$$

Therefore

$$(\delta(A_i)P_i)^2 \subseteq \delta(A_i)(P_i)^2 = 0,$$

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hence $\delta(A_i)P_i$ is nilpotent and must therefore be zero. Since $\delta(A_i)P_i = 0$, $\delta(A_i)$ annihilates P_i, thus A_i is δ-stable.

Therefore, again as in the proof of Theorem 10, each A_i is a prime ring and $B = A_1 \oplus \cdots \oplus A_n$ is a direct sum of δ-stable ideals of A which is essential as both a left and right ideal of A. We can again localize A and B at the powers of some central, regular element in B^δ to conclude that for some $j \leq n$, $\mathrm{GKdim}(A) = \mathrm{GKdim}(A_j)$. We can now apply the prime case to conclude that $\mathrm{GKdim}(A_j) = \mathrm{GKdim}({A_j}^\delta)$. As a result,

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(A_j) = \mathrm{GKdim}({A_j}^\delta) \leq \mathrm{GKdim}(A^\delta) \leq \mathrm{GKdim}(A),$$

thereby proving (2). □

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