J. Aust. Math. Soc. **106** (2019), 274–286
doi:10.1017/S1446788718000113

SHARP CONSTANTS FOR MULTIVARIATE HAUSDORFF q-INEQUALITIES

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(Received 15 August 2017; accepted 24 February 2018; first published online 7 June 2018)

Communicated by C. Meaney

Abstract

In this paper, we focus on the multivariate Hausdorff operator of the form

$$\mathbf{H}_\Phi(f)(x) = \int_{(0,+\infty)^n} \frac{\Phi(\frac{x_1}{t_1}, \frac{x_2}{t_2}, \ldots, \frac{x_n}{t_n})}{t_1 t_2 \cdots t_n} f(t_1, t_2, \ldots, t_n)\, \mathbf{dt},$$

where $\mathbf{dt} = dt_1\, dt_2 \cdots dt_n$ or $\mathbf{dt} = d_q t_1\, d_q t_2 \cdots d_q t_n$ is the discrete measure in q-analysis. The sharp bounds for the multivariate Hausdorff operator on spaces L^p with power weights are calculated, where $p \in \mathbb{R}\backslash\{0\}$.

2010 *Mathematics subject classification*: primary 26D10; secondary 26D15, 39A13.
Keywords and phrases: sharp constant, multivariate Hausdorff operator, q-inequalities.

1. Introduction

The aim of this paper is to study Hausdorff operators in the framework of quantum calculus (q-calculus). q-calculus, while in a sense dating back to Euler, Jacobi, and also Jackson more recently (see [10]), is now beginning to be more useful in quantum mechanics, having an intimate connection with commutativity relations and Lie algebra. The reader can investigate [2, 4, 6] and [7] to observe numerous applications in various fields of mathematics. One interesting topic, q-analogues of the many inequalities derived from classical analysis, has been established. Its use can be seen in works such as [3, 8, 13, 15, 17]. These integral inequalities can be used for the study of qualitative and quantitative properties of integrals, see [1, 14, 19].

Let $G = (0, +\infty)^n$ and let $\Phi(t_1, t_2, \ldots, t_n)$ be a locally integrable function on G. For any $x = (x_1, x_2, \ldots, x_n) \in G$, the multivariate Hausdorff operator is defined on G by

$$H_\Phi f(x) = \int_G \frac{\Phi(\frac{x_1}{t_1}, \frac{x_2}{t_2}, \ldots, \frac{x_n}{t_n})}{t_1 t_2 \cdots t_n} f(t_1, t_2, \ldots, t_n)\, dt_1\, dt_2 \cdots dt_n. \tag{1.1}$$

The research was supported by National Natural Science Foundation of China (Grant Nos. 11471288, 11601456).

Let χ_E be the characteristic function of a set E. If we take $\Phi(t_1,\ldots,t_n)=\prod_{j=1}^n \chi_{[1,+\infty)}(t_j)t_j^{-1}$, then the Hausdorff operator H_Φ is reduced to the multivariate Hardy operator H_n which can be found in [16]

$$H_n f(x)=\frac{1}{x_1x_2\cdots x_n}\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} f(t_1,t_2,\ldots,t_n)\,dt_1\,dt_2\cdots dt_n.$$

If we take $\Phi(t_1,t_2,\ldots,t_n)=\prod_{j=1}^n \chi_{(0,1]}(t_j)$, then the Hausdorff operator H_Φ is reduced to the adjoint of multivariate Hardy operator H_n^*

$$H_n^* f(x)=\int_{x_1}^{+\infty}\int_{x_2}^{+\infty}\cdots\int_{x_n}^{+\infty}\frac{f(t_1,t_2,\ldots,t_n)}{t_1t_2\cdots t_n}\,dt_1\,dt_2\cdots dt_n.$$

For any $x=(x_1,x_2,\ldots,x_n)\in G$, $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with $\alpha_i\in\mathbb{R}$ $(1\le i\le n)$, let $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ and $dx=dx_1\,dx_2\cdots dx_n$. We notice that the Hausdorff operator and its varieties have attracted a lot of research related to modern harmonic analysis in the last decade. One can find these facts in recent survey papers [5] and [12]. Among numerous research results in recent publications, one that interests us most is the work of Wu and Chen [18]. They showed that the operator H_Φ is bounded on power weighted Lebesgue spaces L^p $(1\le p\le+\infty)$, that is

$$\Big(\int_G |H_\Phi f(x)|^p x^\alpha\,dx\Big)^{1/p}\le C_0\Big(\int_G |f(x)|^p x^\alpha\,dx\Big)^{1/p},$$

provided that $\Phi(x)\ge 0$ and $C_0=\int_G \Phi(x)\prod_{i=1}^n x_i^{(1+\alpha_i)/p-1}\,dx<+\infty$. Moreover, they proved that the constant C_0 is the sharp one. On the other hand, in [13] Maligranda, Oinarov and Persson derived some q-analysis variants of the classical Hardy inequality and obtained their corresponding best constants. Motivated by their work, a natural question raised is whether the q-analogue of a multivariate Hausdorff operator enjoys the same properties as the classical multivariate Hausdorff operator defined in (1.1).

To this end, we first introduce some basic notations and definitions of q-calculus, which are necessary for understanding this paper. Fix a positive number $q\in(0,1)$. For a function $f:[0,b)\to\mathbb{R}$, $0<b\le+\infty$, the q-integral or the q-Jackson integral of f is defined by the formula:

$$\int_0^x f(t)\,d_qt=(1-q)x\sum_{k=0}^{+\infty} q^k f(q^kx),\ \text{for } x\in(0,b], \tag{1.2}$$

and the improper q-integral of a function $f:[0,+\infty)\to\mathbb{R}$ is defined by the series

$$\int_0^{+\infty} f(t)\,d_qt=(1-q)\sum_{k=-\infty}^{+\infty} q^k f(q^k), \tag{1.3}$$

provided that the series on the right-hand sides of (1.2) and (1.3) converge absolutely (see [9] and [11]). In the following, for simplicity of notation, for $\alpha\in\mathbb{R}$ and $p\in\mathbb{R}\backslash\{0\}$, we will write $f\in L^p(t^\alpha\,d_qt)$ if f satisfies

$$\int_0^{+\infty} |f(t)|^p t^\alpha\,d_qt<+\infty,$$

https://doi.org/10.1017/S1446788718000113 Published online by Cambridge University Press

and let

$$\|f\|_{L^p(t^\alpha\,d_qt)} = \Big(\int_0^{+\infty} |f(t)|^p t^\alpha\, d_qt\Big)^{1/p}.$$

Also, we write $f \in L^p(d_qt)$ if $\alpha = 0$, and write $d_qx = d_qx_1\, d_qx_2 \cdots d_qx_n$ for any $x = (x_1, x_2, \ldots, x_n) \in G$ and $0 < q < 1$. We now define the q-analogue of multivariate Hausdorff operator by

$$\mathbf{H}_\Phi(f)(x) = \int_G \frac{\Phi(\frac{x_1}{t_1}, \frac{x_2}{t_2}, \ldots, \frac{x_n}{t_n})}{t_1 t_2 \cdots t_n} f(t_1, t_2, \ldots, t_n)\, d_qt_1\, d_qt_2 \cdots d_qt_n,$$

the q-analogue of the multivariate Hardy operator by

$$\mathbf{H}_n f(x) = \frac{1}{x_1 x_2 \cdots x_n} \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} f(t_1, t_2, \ldots, t_n)\, d_qt_1\, d_qt_2 \cdots d_qt_n,$$

and the q-analogue of multivariate adjoint Hardy operator by

$$\mathbf{H}_n^* f(x) = \int_{x_1}^{+\infty}\int_{x_2}^{+\infty}\cdots\int_{x_n}^{+\infty} \frac{f(t_1, t_2, \ldots, t_n)}{t_1 t_2 \cdots t_n}\, d_qt_1\, d_qt_2 \cdots d_qt_n.$$

Now let us describe our main results. These results are new even if $\alpha = 0$.

Theorem 1.1. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Assume that Φ is a nonnegative function and $f \in L^p(x^\alpha\, d_qx)$. If $1 \le p < +\infty$, then the following inequality*

$$\|\mathbf{H}_\Phi f\|_{L^p(x^\alpha\, d_qx)} \le C_1 \|f\|_{L^p(x^\alpha\, d_qx)}, \tag{1.4}$$

holds, provided that

$$C_1 = \int_G \Phi(t_1, t_2, \ldots, t_n) \prod_{i=1}^n t_i^{(1+\alpha_i)/p-1}\, d_qt_1\, d_qt_2 \cdots d_qt_n < +\infty. \tag{1.5}$$

If $p < 1$ $(p \neq 0)$, then we have the reverse inequality

$$\|\mathbf{H}_\Phi f\|_{L^p(x^\alpha\, d_qx)} \ge C_1 \|f\|_{L^p(x^\alpha\, d_qx)},$$

provided that (1.5) holds. Here we assume $f \neq 0$ and $\Phi > 0$ if $p < 0$. Moreover, for $p \in \mathbb{R}\backslash\{0\}$, the constant C_1 is the best possible one.

When applied, we can easily obtain the following results.

Corollary 1.2. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(1 + \alpha_i)/p < 1$ for $i = 1, \ldots, n$. Assume that Φ is a nonnegative function and $f \in L^p(x^\alpha\, d_qx)$. If $1 \le p < +\infty$, then the following inequality*

$$\|\mathbf{H}_n f\|_{L^p(x^\alpha\, d_qx)} \le C_2 \|f\|_{L^p(x^\alpha\, d_qx)},$$

holds with

$$C_2 = (1-q)^n \prod_{i=1}^n \frac{1}{1 - q^{1-1/p-\alpha_i/p}}.$$

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If $p < 1$ $(p \neq 0)$, then the following inequality

$$\|\mathbf{H}_n f\|_{L^p(x^\alpha\, d_q x)} \geq C_2 \|f\|_{L^p(x^\alpha\, d_q x)},$$

holds. Here we assume that $f \neq 0$ if $p < 0$. Moreover, for $p \in \mathbb{R}\backslash\{0\}$, the constant C_2 is the best possible one.

Corollary 1.3. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(1 + \alpha_i)/p > 0$ for $i = 1, \ldots, n$. Assume that Φ is a nonnegative function and $f \in L^p(x^\alpha\, d_q x)$. If $1 \leq p < +\infty$, then the following inequality*

$$\|\mathbf{H}^*_n f\|_{L^p(x^\alpha\, d_q x)} \leq C_3 \|f\|_{L^p(x^\alpha\, d_q x)},$$

holds with

$$C_3 = (1 - q)^n \prod_{i=1}^{n} \frac{1}{1 - q^{(1+\alpha_i)/p}}.$$

If $p < 1$ $(p \neq 0)$, then we have the reverse inequality

$$\|\mathbf{H}^*_n f\|_{L^p(x^\alpha\, d_q x)} \geq C_3 \|f\|_{L^p(x^\alpha\, d_q x)}.$$

Here we assume that $f \neq 0$ if $p < 0$. Moreover, for $p \in \mathbb{R}\backslash\{0\}$, the constant C_3 is the best possible one.

It is interesting to see that the constant C_1 in Theorem 1.1 and the constant C_0 obtained by Wu and Chen [18] are in the same integral form, but with different measures, one is continuous and the other is discrete. More significantly, we are able to see that C_1 is also the sharp constant in the case of $p < 1$ $(p \neq 0)$ with a reverse inequality for the q-analogue multivariate Hausdorff operator $\mathbf{H}_\Phi$. With the same method, in the last section we will show that C_0 is also the best constant for the reverse inequality

$$\|H_\Phi f\|_{L^p(x^\alpha\, dx)} \geq C_0 \|f\|_{L^p(x^\alpha\, dx)},$$

in the case of $p < 1$ $(p \neq 0)$.

It should be pointed out that from Theorem 1.1 we can obtain $L^p(x^\alpha\, d_q x)$ boundedness for q-analogues of many well-known operators when we take different functions Φ. These operators include the Cesàro operator, the Hardy–Littlewood–Pólya operator, the Riemann–Liouville fractional derivatives, and the weighted Hardy operator, among many others.

2. Proof of Theorem 1.1

Proof. Using the definition given in (1.3),

$$\mathbf{H}_\Phi f(x) = (1 - q)^n \sum_{j_1=-\infty}^{+\infty} \cdots \sum_{j_n=-\infty}^{+\infty} \Phi\Big(\frac{x_1}{q^{j_1}}, \ldots, \frac{x_n}{q^{j_n}}\Big) f(q^{j_1}, \ldots, q^{j_n}),$$

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where $j_1, \dots j_n$ are integers. Then for $p \in \mathbb{R}\backslash\{0\}$, by changing variables $k_i = l_i - j_i$ for $1 \le i \le n$, we have

$$\begin{aligned}
\|\mathbf{H}_\Phi f\|_{L^p(x^\alpha\, d_q x)} &= \bigg(\int_G |\mathbf{H}_\Phi f(x)|^p x^\alpha\, d_q x\bigg)^{1/p}\\
&= (1-q)^{n(1+(1/p))}\bigg(\sum_{l_1=-\infty}^{+\infty}\cdots\sum_{l_n=-\infty}^{+\infty}\prod_{i=1}^n q^{l_i(1+\alpha_i)}\\
&\quad\times\bigg|\sum_{j_1=-\infty}^{+\infty}\cdots\sum_{j_n=-\infty}^{+\infty}\Phi\Big(\frac{q^{l_1}}{q^{j_1}},\dots,\frac{q^{l_n}}{q^{j_n}}\Big)f\Big(q^{j_1},\dots,q^{j_n}\Big)\bigg|^p\bigg)^{1/p}\\
&= (1-q)^{n(1+(1/p))}\bigg(\sum_{l_1=-\infty}^{+\infty}\cdots\sum_{l_n=-\infty}^{+\infty}\prod_{i=1}^n q^{l_i(1+\alpha_i)}\\
&\quad\times\bigg|\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\dots,q^{k_n})f(q^{l_1-k_1},\dots,q^{l_n-k_n})\bigg|^p\bigg)^{1/p}. \qquad (2.1)
\end{aligned}$$

We first study the case $1 \le p < \infty$. Assume that (1.5) holds. Using the above expression (2.1) and the Minkowski inequality,

$$\begin{aligned}
\|\mathbf{H}_\Phi f\|_{L^p(x^\alpha\, d_q x)} &\le (1-q)^{n(1+(1/p))}\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\dots,q^{k_n})\\
&\quad\times\bigg(\sum_{l_1=-\infty}^{+\infty}\cdots\sum_{l_n=-\infty}^{+\infty}\prod_{i=1}^n q^{l_i(1+\alpha_i)}|f(q^{l_1-k_1},\dots,q^{l_n-k_n})|^p\bigg)^{1/p}.
\end{aligned}$$

Changing variables $m_i = l_i - k_i$ for $1 \le i \le n$, the above estimate is

$$\begin{aligned}
&(1-q)^{n(1+(1/p))}\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\dots,q^{k_n})\prod_{i=1}^n q^{k_i(1+\alpha_i)/p}\\
&\quad\times\bigg(\sum_{m_1=-\infty}^{+\infty}\cdots\sum_{m_n=-\infty}^{+\infty}\prod_{i=1}^n q^{m_i(1+\alpha_i)}|f(q^{m_1},\dots,q^{m_n})|^p\bigg)^{1/p}\\
&= \int_G \Phi(t_1,\dots,t_n)\prod_{i=1}^n t_i^{(1+\alpha_i)/p-1}\, d_q t_1\cdots d_q t_n\, \|f\|_{L^p(x^\alpha\, d_q x)},
\end{aligned}$$

which implies that the inequality (1.4) holds with the constant (1.5).

We need to show that the constant (1.5) is the best one in (1.4). Suppose $N \in \mathbb{Z}^+$ and $0 < \theta < 1$. Let $y = (y_1, y_2, \dots, y_n) \in G$, and

$$f_{\theta,N}(y) = \prod_{j=1}^n y_j^{-(1+\alpha_j)/p}\chi_{[q^{N(1+\theta)},\ q^{-N(1+\theta)}]}(y_j).$$

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Denote $\sharp I$ by the number of integers in the interval I. A straightforward calculation shows that

$$\begin{aligned}\|f_{\theta,N}\|^p_{L^p(x^\alpha\,d_qx)} &= (1-q)^n \sum_{j_1=-\infty}^{+\infty}\cdots\sum_{j_n=-\infty}^{+\infty}\prod_{i=1}^{n}\chi_{[q^{N(1+\theta)},\,q^{-N(1+\theta)}]}(q^{j_i})\\ &= (1-q)^n(\sharp[-N(1+\theta),N(1+\theta)])^n\\ &= (1-q)^n(2[N(1+\theta)]+1)^n,\end{aligned}$$

where $[N(1+\theta)]$ denotes the integral part of the real number $N(1+\theta)$. Then,

$$\begin{aligned}&\|\mathbf{H}_\Phi f_{\theta,N}\|^p_{L^p(x^\alpha\,d_qx)}\\ &= \int_G |\mathbf{H}_\Phi f_{\theta,N}(x)|^p x^\alpha\,d_qx\\ &= (1-q)^{n(p+1)}\sum_{l_1=-\infty}^{+\infty}\cdots\sum_{l_n=-\infty}^{+\infty}\prod_{i=1}^{n}q^{l_i(1+\alpha_i)}\\ &\quad\times\bigg|\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\ldots,q^{k_n})f_{\theta,N}(q^{l_1-k_1},\ldots,q^{l_n-k_n})\bigg|^p\\ &= (1-q)^{n(p+1)}\sum_{l_1=-\infty}^{+\infty}\cdots\sum_{l_n=-\infty}^{+\infty}\bigg(\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\ldots,q^{k_n})\\ &\quad\times\prod_{i=1}^{n}(q^{k_i((1+\alpha_i)/p)}\chi_{[q^{N(1+\theta)},\,q^{-N(1+\theta)}]}(q^{l_i-k_i}))\bigg)^p\\ &\ge (1-q)^{n(p+1)}\sum_{l_1=-\infty}^{+\infty}\cdots\sum_{l_n=-\infty}^{+\infty}\prod_{j=1}^{n}\chi_{[q^N,\,q^{-N}]}(q^{l_j})\\ &\quad\times\bigg(\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\ldots,q^{k_n})\\ &\quad\times\prod_{i=1}^{n}(q^{k_i((1+\alpha_i)/p)}\chi_{[q^{N(1+\theta)},\,q^{-N(1+\theta)}]}(q^{l_i-k_i})\chi_{[q^{\theta N},\,q^{-\theta N}]}(q^{k_i}))\bigg)^p\\ &\ge (1-q)^{n(p+1)}\sum_{l_1=-N}^{N}\cdots\sum_{l_n=-N}^{N}\bigg(\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\ldots,q^{k_n})\\ &\quad\times\prod_{i=1}^{n}(q^{k_i(1+\alpha_i)/p}\chi_{[q^{\theta N},\,q^{-\theta N}]}(q^{k_i}))\bigg)^p\\ &= (1-q)^{n(p+1)}(2N+1)^n\bigg(\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\ldots,q^{k_n})\\ &\quad\times\prod_{i=1}^{n}(q^{k_i(1+\alpha_i)/p}\chi_{[q^{\theta N},\,q^{-\theta N}]}(q^{k_i}))\bigg)^p.\end{aligned}$$

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Thus,

$$\frac{\|\mathbf{H}_\Phi f_{\theta,N}\|^p_{L^p(x^\alpha\, d_qx)}}{\|f_{\theta,N}\|^p_{L^p(x^\alpha\, d_qx)}} \geq \frac{(2N+1)^n(1-q)^{np}}{(2[N(1+\theta)]+1)^n}\Big(\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\ldots,q^{k_n})$$
$$\times\prod_{i=1}^n(q^{k_i(1+\alpha_i)/p}\chi_{[q^{\theta N},\, q^{-\theta N}]}(q^{k_i}))\Big)^p$$
$$\geq \frac{(2N+1)^n(1-q)^{np}}{(2N(1+\theta)+1)^n}\Big(\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\ldots,q^{k_n})$$
$$\times\prod_{i=1}^n(q^{k_i(1+\alpha_i)/p}\chi_{[q^{\theta N},\, q^{-\theta N}]}(q^{k_i}))\Big)^p.$$

Now we fix the θ and let $N\to+\infty$. This yields the following result.

$$\lim_{N\to+\infty}\frac{\|\mathbf{H}_\Phi f_{\theta,N}\|^p_{L^p(x^\alpha\, d_qx)}}{\|f_{\theta,N}\|^p_{L^p(x^\alpha\, d_qx)}}$$
$$\geq\Big(\frac{2}{2(1+\theta)}\Big)^n\Big((1-q)^n\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\ldots,q^{k_n})\prod_{i=1}^n q^{k_i(1+\alpha_i)/p}\Big)^p.$$

By letting $\theta\to0^+$, we conclude that

$$\lim_{\theta\to0^+}\lim_{N\to+\infty}\frac{\|\mathbf{H}_\Phi f_{\theta,N}\|_{L^p(x^\alpha\, d_qx)}}{\|f_{\theta,N}\|_{L^p(x^\alpha\, d_qx)}}\geq(1-q)^n\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\ldots,q^{k_n})\prod_{i=1}^n q^{k_i(1+\alpha_i)/p}.$$

Hence, we finish the proof of the case $1\leq p<+\infty$.

When $p<1$ and $p\neq0$, we use the Minkowski inequality to conclude that

$$\|\mathbf{H}_\Phi f\|_{L^p(x^\alpha\, d_qx)}\geq(1-q)^{n(1+1/p)}\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_n=-\infty}^{+\infty}\Phi(q^{k_1},\ldots,q^{k_n})$$
$$\times\Big(\sum_{l_1=-\infty}^{+\infty}\cdots\sum_{l_n=-\infty}^{+\infty}\prod_{i=1}^n q^{l_i(1+\alpha_i)}|f(q^{l_1-k_1},\ldots,q^{l_n-k_n})|^p\Big)^{1/p}$$
$$=\int_G\Phi(t_1,\ldots,t_n)\prod_{i=1}^n t_i^{(1+\alpha_i)/p-1}\,d_qt_1\cdots d_qt_n\,\|f\|_{L^p(x^\alpha\, d_qx)}.$$

To show that the constant C_1 is sharp, we need to choose two classes of suitable functions according to the values of p. Thus, we divide p into two cases: $0<p<1$ and $p<0$.

(i) $0<p<1$. For $N\in\mathbb{Z}^+$, letting $y=(y_1,y_2,\ldots,y_n)\in G$, we take

$$f_N(y)=\prod_{j=1}^n y_j^{-(1+\alpha_j)/p}\chi_{[q^N,\, q^{-N}]}(y_j).$$

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A direct calculation shows that

$$\int_G |f_N(y)|^p y^\alpha \, d_q y = (1-q)^n (2N+1)^n.$$

Then, also

$$\begin{aligned}
&\|\mathbf{H}_\Phi f_N\|^p_{L^p(x^\alpha\, d_q x)} \\
&\quad = \int_G |\mathbf{H}_\Phi f_N(x)|^p x^\alpha \, d_q x \\
&\quad = (1-q)^{n(p+1)} \sum_{l_1=-\infty}^{+\infty} \cdots \sum_{l_n=-\infty}^{+\infty} \prod_{i=1}^n q^{l_i(1+\alpha_i)} \\
&\qquad \times \bigg(\sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_n=-\infty}^{+\infty} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^n (q^{(k_i-l_i)(1+\alpha_i)/p} \chi_{[q^N,\, q^{-N}]}(q^{l_i-k_i})) \bigg)^p \\
&\quad = (1-q)^{n(p+1)} \sum_{l_1=-\infty}^{+\infty} \cdots \sum_{l_n=-\infty}^{+\infty} \\
&\qquad \bigg(\sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_n=-\infty}^{+\infty} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^n (q^{k_i(1+\alpha_i)/p} \chi_{[q^N,\, q^{-N}]}(q^{l_i-k_i})) \bigg)^p \\
&\quad = (1-q)^{n(p+1)} \sum_{l_1=-\infty}^{+\infty} \cdots \sum_{l_n=-\infty}^{+\infty} \bigg(\sum_{k_1=l_1-N}^{l_1+N} \cdots \sum_{k_n=l_n-N}^{l_n+N} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^n q^{k_i(1+\alpha_i)/p} \bigg)^p.
\end{aligned}$$

It follows from Hölder's inequality that

$$\begin{aligned}
&\sum_{l_1=-N}^{N} \cdots \sum_{l_n=-N}^{N} \bigg(\sum_{k_1=l_1-N}^{l_1+N} \cdots \sum_{k_n=l_n-N}^{l_n+N} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^n q^{k_i(1+\alpha_i)/p} \bigg)^p \\
&\quad \le \bigg(\sum_{l_1=-N}^{N} \cdots \sum_{l_n=-N}^{N} \bigg(\sum_{k_1=l_1-N}^{l_1+N} \cdots \sum_{k_n=l_n-N}^{l_n+N} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^n q^{k_i(1+\alpha_i)/p} \bigg) \bigg)^p \\
&\qquad \times \bigg(\sum_{l_1=-N}^{N} \cdots \sum_{l_n=-N}^{N} 1 \bigg)^{1-p} \\
&\quad \le (2N+1)^n \bigg(\sum_{k_1=-2N}^{2N} \cdots \sum_{k_n=-2N}^{2N} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^n q^{k_i(1+\alpha_i)/p} \bigg)^p.
\end{aligned}$$

We now complete the proof of the case $0 < p < 1$, since

$$\begin{aligned}
\lim_{N\to+\infty} \frac{\|\mathbf{H}_\Phi f_N\|_{L^p(x^\alpha\, d_q x)}}{\|f_N\|_{L^p(x^\alpha\, d_q x)}} &\le \lim_{N\to+\infty} (1-q)^n \sum_{k_1=-2N}^{2N} \cdots \sum_{k_n=-2N}^{2N} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^n q^{k_i(1+\alpha_i)/p} \\
&= (1-q)^n \sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_n=-\infty}^{\infty} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^n q^{k_i(1+\alpha_i)/p}.
\end{aligned}$$

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(ii) $p < 0$. Assume $y = (y_1, \ldots, y_n) \in G$. For $\varepsilon > 0$, let

$$f_\varepsilon(y) = \prod_{j=1}^{n} y_j^{-(1+\alpha_j+\varepsilon)/p} \chi_{[1,+\infty)}(y_j).$$

Then,

$$\int_G |f_\varepsilon(y)|^p y^\alpha \, d_q y = \Big(\frac{1-q}{1-q^\varepsilon}\Big)^n,$$

and

$$\begin{aligned}
\|\mathbf{H}_\Phi f_\varepsilon\|^p_{L^p(x^\alpha\, d_q x)} &= (1-q)^{n(p+1)} \sum_{l_1=-\infty}^{+\infty} \cdots \sum_{l_n=-\infty}^{+\infty} \prod_{i=1}^{n} q^{l_i(1+\alpha_i)} \\
&\quad \times \Big| \sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_n=-\infty}^{+\infty} \Phi(q^{k_1}, \ldots, q^{k_n}) f_\varepsilon(q^{l_1-k_1}, \ldots, q^{l_n-k_n}) \Big|^p \\
&= (1-q)^{n(p+1)} \sum_{l_1=-\infty}^{+\infty} \cdots \sum_{l_n=-\infty}^{+\infty} \prod_{i=1}^{n} q^{-l_i\varepsilon} \\
&\quad \times \Big(\sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_n=-\infty}^{+\infty} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^{n} (q^{k_i(1+\alpha_i+\varepsilon)/p} \chi_{[1,+\infty)}(q^{l_i-k_i})) \Big)^p \\
&\geq (1-q)^{n(p+1)} \sum_{l_1=-\infty}^{0} \cdots \sum_{l_n=-\infty}^{0} \prod_{i=1}^{n} q^{-l_i\varepsilon} \\
&\quad \times \Big(\sum_{k_1=l_1}^{+\infty} \cdots \sum_{k_n=l_n}^{+\infty} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^{n} q^{k_i(1+\alpha_i+\varepsilon)/p} \Big)^p \\
&\geq (1-q)^{n(p+1)} \sum_{l_1=-\infty}^{0} \cdots \sum_{l_n=-\infty}^{0} \prod_{i=1}^{n} q^{-l_i\varepsilon} \\
&\quad \times \Big(\sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_n=-\infty}^{+\infty} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^{n} q^{k_i(1+\alpha_i+\varepsilon)/p} \Big)^p \\
&= \Big(\frac{1-q}{1-q^\varepsilon}\Big)^n \Big((1-q)^n \sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_n=-\infty}^{+\infty} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^{n} q^{k_i(1+\alpha_i+\varepsilon)/p} \Big)^p,
\end{aligned}$$

where we have used $p < 0$ and $\Phi > 0$. Letting $\varepsilon \to 0^+$, then

$$\lim_{\varepsilon \to 0^+} \frac{\|\mathbf{H}_\Phi f_\varepsilon\|_{L^p(x^\alpha\, d_q x)}}{\|f_\varepsilon\|_{L^p(x^\alpha\, d_q x)}} \leq (1-q)^n \sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_n=-\infty}^{\infty} \Phi(q^{k_1}, \ldots, q^{k_n}) \prod_{i=1}^{n} q^{k_i(1+\alpha_i)/p}.$$

Combining all the estimates, we show that the constant C_1 is sharp and therefore finish the proof of the theorem. □

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3. A final remark

In this section, we can modify the previous argument to yield the best constant for the Hausdorff operator H_Φ given by (1.1) in the L^p spaces with $p < 1$ $(p \neq 0)$, ignoring whether these spaces make sense. The following result can be regarded as fixing the gap present in Wu and Chen's result, found in [18]. Using similar notation as before, we let $t = (t_1, t_2, \ldots, t_n) \in G$ and $dt = dt_1\, dt_2 \ldots dt_n$.

THEOREM 3.1. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Assume that Φ is a nonnegative function and $f \in L^p(x^\alpha\, dx)$. If $p < 1$ $(p \neq 0)$, then we have the reverse inequality*

$$\|H_\Phi f\|_{L^p(x^\alpha\, dx)} \geq C_0 \|f\|_{L^p(x^\alpha\, dx)},$$

provided that

$$C_0 = \int_G \Phi(t) \prod_{i=1}^{n} t_i^{(1+\alpha_i)/p-1}\, dt < +\infty. \tag{3.1}$$

Here we assume $f \neq 0$ and $\Phi > 0$ if $p < 0$. Moreover, for $p < 1$ $(p \neq 0)$, the constant C_0 is the best possible one.

PROOF. By changing variables,

$$H_\Phi f(x) = \int_G \frac{\Phi(\frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n})}{t_1 \cdots t_n} f(t_1, \ldots, t_n)\, dt = \int_G \frac{\Phi(t_1, \ldots, t_n)}{t_1 \cdots t_n} f\Big(\frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n}\Big)\, dt. \tag{3.2}$$

Assume that (3.1) holds. Using the equality (3.2) and the Minkowski inequality for the case $p < 1$ $(p \neq 0)$,

$$\begin{aligned}
\|H_\Phi f\|_{L^p(x^\alpha\, dx)} &= \Big(\int_G |H_\Phi f(x)|^p x^\alpha\, dx\Big)^{1/p} \\
&= \Big(\int_G \Big(\int_G \frac{\Phi(t)}{t_1 \cdots t_n} f\Big(\frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n}\Big)\, dt\Big)^p x^\alpha\, dx\Big)^{1/p} \\
&\geq \int_G \frac{\Phi(t)}{t_1 \cdots t_n} \Big(\int_G f^p\Big(\frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n}\Big) x^\alpha\, dx\Big)^{1/p} dt \\
&= \int_G \Phi(t) \prod_{i=1}^{n} t_i^{(1+\alpha_i)/p-1}\, dt\, \|f\|_{L^p(x^\alpha\, dx)}.
\end{aligned}$$

The next step is to show that the constant C_0 is sharp. Similar to the proof of Theorem 1.1, we need to construct two classes of functions with different p.

Case 1. $0 < p < 1$. For any fixed real number r satisfying $0 < r < 1$, letting $N \in \mathbb{Z}^+$, we take

$$f_N(x) = \prod_{i=1}^{n} x_i^{-(1+\alpha_i)/p} \chi_{[r^N, r^{-N}]}(x_i), \quad \text{for } x = (x_1, \ldots, x_n) \in G.$$

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Let us observe that

$$\|f_N\|^p_{L^p(x^\alpha\,dx)} = (2N)^n(\ln(r^{-1}))^n.$$

Also,

$$\begin{aligned}\|H_\Phi f_N\|^p_{L^p(x^\alpha\,dx)} &= \bigg(\int_G\bigg(\int_G \frac{\Phi(t)}{t_1\cdots t_n} f_N\Big(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\Big)\,dt\bigg)^p x^\alpha\,dx\bigg)^{1/p}\\ &= \bigg(\int_G\bigg(\int_G \frac{\Phi(t)}{t_1\cdots t_n}\prod_{i=1}^n\Big(\frac{t_i}{x_i}\Big)^{(1+\alpha_i)/p}\chi_{[r^N,r^{-N}]}\Big(\frac{x_i}{t_i}\Big)\,dt\bigg)^p x^\alpha\,dx\bigg)^{1/p}\\ &= \bigg(\int_G\bigg(\int_{x_1r^N}^{x_1r^{-N}}\cdots\int_{x_nr^N}^{x_nr^{-N}}\Phi(t)\prod_{i=1}^n t_i^{(1+\alpha_i)/p-1}\,dt\bigg)^p\prod_{i=1}^n x_i^{-1}\,dx\bigg)^{1/p}.\end{aligned}$$

Considering the last term, we see that

$$\begin{aligned}&\bigg(\int_{r^N}^{r^{-N}}\cdots\int_{r^N}^{r^{-N}}\bigg(\int_{x_1r^N}^{x_1r^{-N}}\cdots\int_{x_nr^N}^{x_nr^{-N}}\Phi(t)\prod_{i=1}^n t_i^{(1+\alpha_i)/p-1}\,dt\bigg)^p\prod_{i=1}^n x_i^{-1}\,dx\bigg)^{1/p}\\ &\le\bigg(\int_{r^N}^{r^{-N}}\cdots\int_{r^N}^{r^{-N}}\bigg(\int_{r^{2N}}^{r^{-2N}}\cdots\int_{r^{2N}}^{r^{-2N}}\Phi(t)\prod_{i=1}^n t_i^{(1+\alpha_i)/p-1}\,dt\bigg)^p\prod_{i=1}^n x_i^{-1}\,dx\bigg)^{1/p}\\ &=\int_{r^{2N}}^{r^{-2N}}\cdots\int_{r^{2N}}^{r^{-2N}}\Phi(t)\prod_{i=1}^n t_i^{(1+\alpha_i)/p-1}\,dt\,\|f_N\|_{L^p(x^\alpha\,dx)}.\end{aligned}$$

Letting $N\to+\infty$, gives

$$\begin{aligned}\lim_{N\to+\infty}\frac{\|H_\Phi f_N\|_{L^p(x^\alpha\,dx)}}{\|f_N\|_{L^p(x^\alpha\,dx)}} &\le \lim_{N\to+\infty}\int_{r^{2N}}^{r^{-2N}}\cdots\int_{r^{2N}}^{r^{-2N}}\Phi(t)\prod_{i=1}^n t_i^{(1+\alpha_i)/p-1}\,dt\\ &= \int_G\Phi(t)\prod_{i=1}^n t_i^{(1+\alpha_i)/p-1}\,dt.\end{aligned}$$

Case 2. $p<0$. For $\varepsilon>0$, we let

$$f_\varepsilon(x)=\prod_{i=1}^n x_i^{-(1+\alpha_i+\varepsilon)/p}\chi_{[1,+\infty)}(x_i),\qquad \text{for } x=(x_1,\ldots,x_n)\in G.$$

It is easy to see that

$$\|f_\varepsilon\|^p_{L^p(x^\alpha\,dx)}=\varepsilon^{-n}.$$

Also

$$\begin{aligned}\|H_\Phi f_\varepsilon\|^p_{L^p(x^\alpha\,d_qx)} &= \int_G\bigg(\int_G\frac{\Phi(t)}{t_1\cdots t_n}f_\varepsilon\Big(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\Big)\,dt\bigg)^p x^\alpha\,dx\\ &= \int_G\bigg(\int_G\frac{\Phi(t)}{t_1\cdots t_n}\prod_{i=1}^n\Big(\frac{t_i}{x_i}\Big)^{(1+\alpha_i+\varepsilon)/p}\chi_{[1,+\infty)}\Big(\frac{x_i}{t_i}\Big)\,dt\bigg)^p x^\alpha\,dx\end{aligned}$$

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$$= \int_G \Big(\int_0^{x_1} \cdots \int_0^{x_n} \Phi(t) \prod_{i=1}^n t_i^{(1+\alpha_i+\varepsilon)/p-1}\, dt \Big)^p \prod_{i=1}^n x_i^{-1-\varepsilon}\, dx$$

$$\geq \int_1^{+\infty} \cdots \int_1^{+\infty} \Big(\int_0^{x_1} \cdots \int_0^{x_n} \Phi(t) \prod_{i=1}^n t_i^{(1+\alpha_i+\varepsilon)/p-1}\, dt \Big)^p \prod_{i=1}^n x_i^{-1-\varepsilon}\, dx.$$

Since $\Phi(t) > 0$ and $p < 0$, the last term is greater than or equal to

$$\int_1^{+\infty} \cdots \int_1^{+\infty} \Big(\int_0^{+\infty} \cdots \int_0^{+\infty} \Phi(t) \prod_{i=1}^n t_i^{(1+\alpha_i+\varepsilon)/p-1}\, dt \Big)^p \prod_{i=1}^n x_i^{-1-\varepsilon}\, dx$$

$$= \varepsilon^{-n} \Big(\int_G \Phi(t) \prod_{i=1}^n t_i^{(1+\alpha_i+\varepsilon)/p-1}\, dt \Big)^p.$$

Letting $\varepsilon \to 0^+$, we see that

$$\lim_{\varepsilon\to 0^+} \frac{\|H_\Phi f_\varepsilon\|_{L^p(x^\alpha\, dx)}}{\|f_\varepsilon\|_{L^p(x^\alpha\, dx)}} \leq \lim_{\varepsilon\to 0^+} \int_G \Phi(t) \prod_{i=1}^n t_i^{(1+\alpha_i+\varepsilon)/p-1}\, dt = \int_G \Phi(t) \prod_{i=1}^n t_i^{(1+\alpha_i)/p-1}\, dt.$$

This proves the theorem. □

Acknowledgements

The authors are very grateful to Silei Wang and to the referee for their helpful comments.

References

[1] G. A. Anastassiou, 'Taylor Widder representation formulae and Ostrowski, Grüss, integral means and Csiszar type inequalities', *Comput. Math. Appl.* **54** (2007), 9–23.

[2] M. H. Annaby and Z. S. Mansour, *q-Fractional Calculus and Equations* (Springer, Heidelberg–New York, 2012).

[3] A. O. Baiarystanov, L. E. Persson, S. Shaimardan and A. Temirkhanova, 'Some new Hardy-type inequalities in q-analysis', *J. Math. Inequal.* **10** (2016), 761–781.

[4] G. Bangerezako, 'Variational calculus on q-nonuniform lattices', *J. Math. Anal. Appl.* **306** (2005), 161–179.

[5] J. Chen, D. Fan and S. Wang, 'Hausdorff operators on Euclidean space', *Appl. Math. J. Chinese Univ. Ser.* B **28** (2013), 548–564.

[6] T. Ernst, *A Comprehensive Treatment of q-Calculus* (Birkhäuser/Springer Basel AG, Basel, 2012).

[7] H. Exton, *q-hypergeometric Functions and Applications*, Ellis Horwood Series: Mathematics and its Applications (Ellis Horwood, Chichester, UK, 1983).

[8] J. Guo and F. Zhao, 'Some q-inequalities for Hausdorff operators', *Front. Math. China* **12** (2017), 879–889.

[9] F. H. Jackson, 'On q-definite integrals', *Quart. J. Pure Appl. Math.* **41** (1910), 193–203.

[10] F. H. Jackson, 'q-difference equations', *Amer. J. Math.* **32** (1910), 305–314.

[11] V. Kac and P. Cheung, *Quantum Calculus* (Springer-Verlag, New York, 2002).

[12] E. Liflyand, 'Hausdorff operators on Hardy spaces', *Eurasian Math. J.* **4** (2013), 101–141.

[13] L. Maligranda, R. Oinarov and L. E. Persson, 'On Hardy q-inequalities', *Czechoslovak Math. J.* **64** (2014), 659–682.

[14] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis* (Kluwer Academic, 1993).
[15] Y. Miao and F. Qi, 'Several q-integral inequalities', *J. Math. Inequal.* **3** (2009), 115–121.
[16] B. G. Pachpatte, 'On multivariable Hardy type inequalities', *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **38** (1992), 355–361.
[17] W. T. Sulaiman, 'New types of q-integral inequalities', *Adv. Pure Appl. Math.* **1** (2011), 77–80.
[18] X. Wu and J. Chen, 'Best constants for Hausdorff operators on n-dimensional product spaces', *Sci. China Math.* **57** (2014), 569–578.
[19] S. Wu and L. Debnath, 'Inequalities for convex sequences and their applications', *Comput. Math. Appl.* **54** (2007), 525–534.

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